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GENERALIZED RADON SPACES WHICH ARE NOT RADON

(τ-additivity/μ-compactness/Radon spaces)

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ABSTRACT

For any Borel measure μ in a topological space X we show that τ-additivity and μ-compactness are two concepts closely related (Theorem 2.1). Moreover, if $\mathcal{F}$ is the family of all closed subsets of X, we give two new examples of Radon spaces of type ($\mathcal{F}$) which are not Radon spaces.

RESUMEN

Para cualquier medida de Borel μ en un espacio topológico X probamos que los conceptos de τ-aditividad y μ-compacidad están estrechamente relacionados (Teorema 2.1). Además, si $\mathcal{F}$ es la familia de todos los subconjuntos cerrados de X, damos dos nuevos ejemplos de espacios de Radon de tipo ($\mathcal{F}$) que no son espacios de Radon.

1. INTRODUCTION AND PRELIMINARIES

Let X be a topological space not necessarily Hausdorff. We shall denote by $\mathcal{G}(X)$, $\mathcal{F}(X)$, $\mathcal{K}(X)$, $\mathcal{B}(X)$, respectively, the families of all open, closed, compact closed and Borel subsets of X. When no confusion can arise, we shall write $\mathcal{G}$, $\mathcal{F}$, $\mathcal{K}$ and $\mathcal{B}$ instead of $\mathcal{G}(X)$, $\mathcal{F}(X)$, $\mathcal{K}(X)$ and $\mathcal{B}(X)$, respectively.

A nonempty family $\mathcal{A}$ of subsets of X is called *directed upwards* if for each A, B in $\mathcal{A}$ there is C in $\mathcal{A}$ such that $A \cup B \subset C$. If $\mathcal{A}$ is directed upwards and $A_0 = \cup \mathcal{A}$, we write $\mathcal{A} \uparrow A_0$.

A *Borel measure* in X is a countably additive measure on $\mathcal{B}$. If μ is a Borel measure in X and $A \subset X$, we let

$$\mu^*(A) = \inf \{\mu(G): A \subset G \in \mathcal{G}\}$$

and

$$\mu_*(A) = \sup \{\mu(F): A \supset F \in \mathcal{F}\}.$$

Let μ be a Borel measure in X. A set $B \in \mathcal{B}$ is called

a) *μ-outer regular* if $\mu(B) = \mu^*(B)$;

b) *μ-inner regular* if $\mu(B) = \mu_*(B)$.

The measure μ is called

A) *locally finite* if each $x \in X$ has an open neighborhood V_x such that $\mu(V_x) < +\infty$;

B) *τ-additive* if $\sup \{\mu(G): G \in \mathcal{G}_0\} = \mu(G_0)$ for each $\mathcal{G}_0 \subset \mathcal{G}$ with $\mathcal{G}_0 \uparrow G_0$;

C) *inner regular* if each $B \in \mathcal{B}$ is μ-inner regular;

D) *outer regular* if each $B \in \mathcal{B}$ is μ-outer regular;

E) *regular* if it is outer and inner regular.

If X is a T_1 topological space, a Borel measure μ in X is called *diffused* when $\mu(\{x\}) = 0$ for each $x \in X$.

If X is a Hausdorff space, a *Radon measure* in X is a locally finite Borel measure μ in X such that

$$\mu(B) = \sup \{\mu(K): B \supset K \in \mathcal{K}\}$$

for each $B \in \mathcal{B}$.

The Radon measures were introduced in 1973 by L. Schwartz in [7]. In the same year, B. Rodríguez-Salinas introduces in [6] the Radon measures of type ($\mathcal{H}$) in arbitrary topological spaces, replacing the compact inner regularity of Radon measures by the μ-compact one.

Let μ be a Borel measure in X. A set $B \in \mathcal{B}$ is called *μ-compact* if for each $\varepsilon > 0$ and each open cover $\mathcal{G}_0$ of B there is a finite subfamily $\mathcal{G}_1$ of $\mathcal{G}_0$ such that $\mu(B - \cup \mathcal{G}_1) < \varepsilon$.

Let $\mathcal{H}$ be a family of closed subsets of X. A Borel measure μ in X is called a *Radon measure of type* ($\mathcal{H}$) if each $H \in \mathcal{H}$ is a μ-compact set with finite μ-measure and

$$\mu(B) = \sup \{\mu(H): B \supset H \in \mathcal{H}\}$$

for each $B \in \mathcal{B}$.

Clearly, each Radon measure μ in X is a Radon measure of type ($\mathcal{K}$). For a extensive treatment of Radon measures of type ($\mathcal{H}$) we refer to [5].

The space X is called a *Radon space* when each finite Borel measure in X is a Radon measure. For properties of Radon spaces we refer to ([7], Section 3, Chapter II) and ([2], Section 11).

For any family $\mathcal{H}$ of closed subsets of X, P. Jiménez Guerra and B. Rodríguez-Salinas introduce in [4] the Radon spaces of type ($\mathcal{H}$). The space X is called a *Radon space of type* ($\mathcal{H}$) when each finite Borel measure in X is a Radon measure of type ($\mathcal{H}$).

Clearly, a Radon space is a Radon space of type ($\mathcal{K}$).

For any Borel measure μ in X, we show in this paper that the concepts of μ-compactness and τ-additivity are closely related. Indeed, when $\mathcal{H} \subset \mathcal{B}$ and μ satisfy the two following properties

a) For each $H \in \mathcal{H}$ there is $G \in \mathcal{G}$ with $H \subset G$ and $\mu(G) < +\infty$,

b) $\mu(G) = \sup \{\mu(H): G \supset H \in \mathcal{H}\}$ for each $G \in \mathcal{G}$,

we establish that μ is τ-additive if and only if each $H \in \mathcal{H}$ is a μ-compact set (Theorem 2.1).

This result lets to give other equivalent definition of Radon measure of type ($\mathcal{H}$) which replaces the μ-compactness of each $H \in \mathcal{H}$ by the τ-additivity of μ, whenever $\mathcal{H}$ satisfies above property (a) (Corollary 2.2).

Moreover, we deduce that each regular, hereditarily Lindelöf space (in particular, each separable metric space) is a Radon space of type ($\mathcal{F}$) (Theorem 2.4 and Corollary 2.5), and we give two examples of Radon spaces of type ($\mathcal{F}$) which are not Radon spaces (Examples 3.1 and 3.2).

For the last example we shall recall some standar results.

If $Y \subset X$, then $\mathcal{B}(Y) = \{B \cap Y: B \in \mathcal{B}(X)\}$ ([2], Proposition 3.4).

Let μ be a Borel measure in X and let

$$\mu^{\oplus}(A) = \inf \{\mu(B): A \subset B \in \mathcal{B}(X)\}$$

for each $A \subset X$. Then the restriction μ_Y of $\mu^{\oplus}$ to $\mathcal{B}(Y)$ is a Borel measure in Y. We say that μ_Y is the *restriction* to Y of Borel measure μ. If μ is outer regular, then $\mu^{\oplus} = \mu^*$ and μ_Y is outer regular for each $Y \subset X$ ([2], Proposition 3.6).

2. THE RESULTS

Theorem 2.1. *Let $\mathcal{H} \subset \mathcal{B}$ and let μ be a Borel measure in X with the following properties:*

a) *For each $H \in \mathcal{H}$ there is $G \in \mathcal{G}$ such that $H \subset G$ and $\mu(G) < +\infty$;*

b) $\mu(G) = \sup \{\mu(H): G \supset H \in \mathcal{H}\}$ *for each* $G \in \mathcal{G}$.

Then μ is τ-additive if and only if each $H \in \mathcal{H}$ is μ-compact.

Proof. Assume that μ is τ-additive and let $H \in \mathcal{H}$. Let $\mathcal{A}$ be an open cover of H and let $\varepsilon > 0$. By (a), there is $G \in \mathcal{G}$ such that $H \subset G$ and $\mu(G) < +\infty$. Let $\mathcal{V}$ be the family of all finite unions of the sets $A \cap G$ where $A \in \mathcal{A}$ and let $V_o = \cup \mathcal{V}$. Then $\mathcal{V} \uparrow V_o$ and as μ is τ-additive, we have

$$\sup \{\mu(V): V \in \mathcal{V}\} = \mu(V_o) \leq \mu(G) < +\infty$$

hence there is $V \in \mathcal{V}$ such that $\mu(V_o - V) < \varepsilon$, and if $V = \bigcup_{i=1}^{n} (A_i \cap G)$ with $A_i \in \mathcal{A}$ for i = 1, 2,..., n, we have

$$\mu\left(H - \bigcup_{i=1}^{n} A_i\right) \leq \mu\left(V_o - \bigcup_{i=1}^{n} A_i\right) \leq \mu(V_o - V) < \varepsilon.$$

Thus H is μ-compact.

Conversely, assume that each $H \in \mathcal{H}$ is μ-compact and let $\mathcal{U} \subset \mathcal{G}$ such that $\mathcal{U} \uparrow U_o$. For each $H \in \mathcal{H}$ with $H \subset U_o$ and for each $\varepsilon > 0$ there is a finite subfamily $\mathcal{U}_1$ of $\mathcal{U}$ such that $\mu(H - \cup \mathcal{U}_1) < \varepsilon$, and there is $U_1 \in \mathcal{U}$ such that $\cup \mathcal{U}_1 \subset U_1$, hence

$$\mu(H) \leq \mu(\cup \mathcal{U}_1) + \mu(H - \cup \mathcal{U}_1)$$
$$\leq \mu(U_1) + \varepsilon$$
$$\leq \sup \{\mu(U): U \in \mathcal{U}\} + \varepsilon.$$

It follows that

$$\mu(U_o) \leq \sup \{\mu(U): U \in \mathcal{U}\}.$$

The reverse inequality is obvious. Therefore, μ is τ-additive.

Corollary 2.2. *Let $\mathcal{H} \subset \mathcal{F}$ and let μ be a Borel measure in X with the following properties:*

a) *For each $H \in \mathcal{H}$ there is $G \in \mathcal{G}$ such that $H \subset G$ and $\mu(G) < +\infty$;*

b) $\mu(B) = \sup \{\mu(H): B \supset H \in \mathcal{H}\}$ *for each* $B \in \mathcal{B}$.

Then μ is a Radon measure of type ($\mathcal{H}$) in X if and only if μ is τ-additive.

Proof. By (a), $\mu(H) < +\infty$ for each $H \in \mathcal{H}$ and by Theorem 2.1, each $H \in \mathcal{H}$ is μ-compact if and only if μ is τ-additive.

Corollary 2.3. *Let $\mathcal{H} \subset \mathcal{F}$ and let μ be a finite, τ-additive Borel measure in X. Then μ is a Radon measure of type ($\mathcal{H}$) in X if and only if*

$$\mu(B) = \sup \{\mu(H): B \supset H \in \mathcal{H}\}$$

for each $B \in \mathcal{B}$.

Proof. It follows from Corollary 2.2.

Lemma 2.4. *Let μ be a Borel measure in a hereditarily Lindelöf space X. Then μ is τ-additive.*

Proof. Let $\mathcal{G}_o \subset \mathcal{G}$ such that $\mathcal{G}_o \uparrow G_o$. Since G_o is Lindelöf, there is a sequence $(G_n) \subset \mathcal{G}_o$ such that $G_o = \bigcup_{n=1}^{\infty} G_n$. Set $U_n = \bigcup_{k=1}^{n} G_k$ for each $n \in \mathbb{N}$. Then (U_n) is increasing and for each $n \in \mathbb{N}$ there is $V_n \in \mathcal{G}_o$ such that $U_n \subset V_n$, hence

$$\mu(G_o) = \sup \{\mu(U_n): n \in \mathbb{N}\}$$

$$\leq \sup \{\mu(V_n): n \in \mathbb{N}\}$$

$$\leq \sup \{\mu(G): G \in \mathcal{G}_o\}.$$

The reverse inequality is obvious.

Theorem 2.5. *Each regular, hereditarily Lindelöf space X is a Radon space of type ($\mathcal{F}$).*

Proof. Let μ be a finite Borel measure in X. By above lemma, μ is τ-additive. Since X is regular, from ([2], Proposition 6.10) it follows that each $G \in \mathcal{G}$ is μ-inner regular, and from ([2], Proposition 6.2) we deduce that μ is regular. Consequently, μ is a Radon measure of type ($\mathcal{F}$) by Corollary 2.3.

Corollary 2.6. *Each separable metric space X is a Radon space of type ($\mathcal{F}$).*

3. EXAMPLES

We shall give two examples of Radon spaces of type ($\mathcal{F}$) which are not Radon spaces.

Example 3.1. The *Sorgenfrey interval* is the space $X = [0,1)$ with the topology generated by the family of all intervals $[a, b) \subset X$. The space X is regular and hereditarily Lindelöf (see, e. g. [1]) and from Theorem 2.5, it follows that X is a Radon space of type ($\mathcal{F}$).

On the other hand, $\mathcal{B}(X) = \mathcal{B}([0,1))$ where $\mathcal{B}([0,1))$ denotes the σ-algebra of the Borel subsets of [0,1) for the Euclidean topology. Let λ be the Lebesgue measure in X. Since λ is diffused and the compact subsets of X are at most countable ([1], 3.1 (b)), we have $\lambda(K) = 0$ for each $K \in \mathcal{K}(X)$, whereas $\lambda(X) = 1$. Thus λ is not a Radon measure in X. Therefore, X is not a Radon space.

Example 3.2. There are sets $Y \subset [0, 1]$ such that both Y and $[0, 1] - Y$ contain only countable compact subsets (see [3], 10.2). Such sets are called *Berstein sets.*

Let Y be a Berstein set with the Euclidean topology. Then Y is a separable metric space, hence it is a Radon space of type ($\mathcal{F}$) by Corollary 2.6.

Let λ be the Lebesgue measure in [0,1]. Since $\mathcal{F}([0,1]) = \mathcal{K}([0,1])$, we have

$$\lambda_*(Y) = \sup \{\lambda(K): Y \supset K \in \mathcal{K}([0,1])\} = 0.$$

On the other hand, If $Y \subset G \in \mathcal{G}([0,1])$ then $[0,1] - G$ is a compact set contained in $[0,1] - Y$, hence $[0,1] - G$ is countable and since λ is diffused, $\lambda([0,1]) - G) = 0$ and so, $\lambda(G) = 1$. Consequently,

$$\lambda^*(Y) = \inf \{\lambda(G): Y \subset G \in \mathcal{G}([0,1])\} = 1.$$

Let $\mu = \lambda_Y$ the restriction to Y of λ. Since μ is diffused, $\mu(K) = 0$ for each $K \in \mathcal{K}(Y)$. Since μ is outer regular, $\mu(Y) = \lambda^*(Y) = 1$. Thus μ is not a Radon measure in Y. Therefore, Y is not a Radon space.

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