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Matemáticas

A NOTION OF RENORMALIZED SOLUTION FOR THE STUDY OF A GENERAL CONSERVATION LAW

(degenerate quasilinear parabolic equations)

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ABSTRACT

We consider the nonlinear parabolic equation

$$\frac{\partial}{\partial t}\beta(u) - div\big(A(.,\beta(u))\nabla\Phi(u)\big) = 0,$$

associated with Neumann conditions on the boundary, when the function Φ degenerates for the extremal values of the variable u. We whow the existence of a weak solution and, using a notion of renormalized solution, we prove a comparison principle without any assumption on the local behaviour of nonlinearities involved in the problem. When β is supposed to be injective we derive a uniqueness result.

1. INTRODUCTION

We present a problem stemming from the modelization of a fluid flow through a porous medium, where the diffusion effect takes into account the instantaneous state of the system. The physical context considered in the setting of the problem leads to the following class of parabolic degenerate equations,

$$(\mathcal{P})\begin{cases}\frac{\partial}{\partial t}\beta(u) - div\big(A(.,\beta(u))\nabla\Phi(u)\big) = 0,\\ \beta(u)(0) = \beta(u_0).\end{cases}$$

where,

i) β is a nondecreasing function (β may be constant between critical values, for instance when we study a system submitted to high pressures).

ii) The diffusion operator degenerates when the unknown reaches its extremal values (0 and 1 by a normalization process).

iii) When the fluid is confined within an impervious enclosure, the equation is completed by prescribing a Neumann condition on the boundary,

$$\big(A(.,\beta(u))\nabla\Phi(u)\big)\cdot\vec{n} = 0.$$

Under the general conditions described above, our analysis, based on a notion of renormalized solution, generalizes the results given by A. Plouvier [16], A. Plouvier et G. Gagneux [17] (when $\beta = Id$ and Φ only degenerates for one extremal value).

2. GENERAL ASSUMPTIONS

Ω denotes an open bounded set of $\mathbb{R}^N$ with Lipschitz boundary, T a strictly positive real number, and $Q = \Omega \times]0, T[$.

Functional spaces. The Hilbert spaces $H = L^2(\Omega)$ will be equipped with their inner scalar product:

$$\forall u \in L^2(\Omega),\ \forall v \in L^2(\Omega),\ (u, v) = \int_\Omega uv dx.$$

$$\forall u \in H^1(\Omega),\ \forall v \in H^1(\Omega),\ ((u, v))_V = (u, v) + ((u, v)),$$

where we define,

$$\forall u \in H^1(\Omega),\ \forall v \in H^1(\Omega),\ ((u, v)) = \int_\Omega \nabla u.\ \nabla v dx.$$

The duality bracket $\langle .,. \rangle_{V',V}$ will be briefly denoted $\langle .,. \rangle$.

Identifying H to its dual space leads to the classical embeddings,

$$V \to H \to V'$$

Nonlinearities. We consider a function β satisfying,

($H1$) $\{\beta \in W^{1,\infty}(]0, 1[), \beta(0) = 0, \beta' \geq 0\ \pounds^1 - a.e.\ in\ ]0, 1[\}$.

As mentioned in the introduction, the function β may not be injective.

For (i, j) in $(1, ..., N)^2$, a_{ij} is a Caratheodory type function,

$$a_{ij} : \Omega \times \mathbb{R}^+ \to \mathbb{R}$$
$$(x,\lambda) \mapsto a_{ij}(x,\lambda)$$

and we assume that,

$$(H2)\begin{cases} i)\ \exists l > 0,\ \|a_{ij}\|_{L^\infty(\Omega\times\mathbb{R}^+)} \le l \text{ for any } (i,j) \text{ in } (1,...,N)^2, \\ ii)\ \exists \alpha > 0, \forall\lambda \in \mathbb{R}^+, \forall\xi \in \mathbb{R}_N,\ a_{ij}(.,\lambda)\xi\cdot\xi \ge \alpha|\xi|^2\ £^N - a.e.\ in\ \Omega, \\ iii)\ \text{for any } (i,j) \text{ in } (1,...,N)^2, \text{ and for almost every } x \text{ in } \Omega, \text{ the} \\ \text{function } \lambda \to a_{ij}(x,\lambda) \text{ is Lipschitz continuous.} \end{cases}$$

Furthermore, Φ satisfies,

$$(H3)\begin{cases} i)\ \Phi \in C^1([0,1]), \Phi(0) = \Phi'(0) = \Phi'(1) = 0\ and\ \forall r \in]0,1[, \Phi'(r) > 0. \\ ii)\ \Phi^{-1} \in C^{0,\theta}([0,\Phi(1)]),\ with\ \theta \in]0,1[, \\ i.e.\ \Phi^{-1} \text{ is } \theta-\text{Hölder continuous on } [0,\Phi(1)]. \end{cases}$$

Initial datum:

$(H0)$ $\{u_0 \in L^\infty(\Omega),\ 0 \le u_0 \le 1,\ £^N - a.e.\ on\ \Omega\}$.

3. EXISTENCE OF A WEAK SOLUTION

A viscosity method, associated with compactness and monotonicity properties (see A. Plouvier [16], M. Artola [1] in the case where $\Phi = Id$) leads to the following existence result:

Proposition. *Under assumptions (H0), ..., (H3), there exists at least one function u satisfying,*

- *Regularity properties:*

$$(R)\begin{cases} u \in L^\infty(Q),\ 0 \le u \le 1, & £^{N+1} - a.e.\ in\ Q, \\ \Phi(u) \in L^2(0,T;V). & \beta(u)(0) = \beta(u_0),\ £^N - a.e.\ on\ \Omega, \\ \frac{\partial}{\partial t}\beta(u) \in L^2(0,T;V'), & \lim_{t\to 0^+} \beta(u)(t) = \beta(u_0)\ in\ L^1(\Omega), \end{cases}$$

- *Variational formulation:*

$$(V)\begin{cases} £^1 - a.e.\ in\ ]0,T[,\ \forall v \in V, \\ \left\langle \frac{\partial}{\partial t}\beta(u), v \right\rangle + \int_\Omega \{A(.,\beta(u))\nabla\Phi(u)\}\cdot\nabla v dx = 0. \end{cases}$$

Such a function will be called a weak solution of $(\mathcal{P})$.

Remark. The property (R) justifies the existence of a trace $\beta(u)(0)$ because the solution $\beta(u)$ is continuous on $[0, T]$ with values in V', and scalar continuous with values in $L^2(\Omega)$, according to J.L. Lions and E. Magenes [14] vol. 1, p. 297.

Proof of the proposition. The proof will be performed in three steps:

Step 1: The regularized problem $(\mathcal{P}_\varepsilon)$:

We consider the problem $(\mathcal{P}_\varepsilon)$ obtained by changing $\beta \to \beta\varepsilon$ and $\Phi \to \Phi\varepsilon$ in the initial problem $(\mathcal{P})$ according to the following definition,

$$\beta\varepsilon = \beta + \varepsilon Id\ and\ \Phi\varepsilon = \Phi + \varepsilon Id.$$

By setting $v_\varepsilon = \beta\varepsilon(u_\varepsilon)$ for $\varepsilon \in]0, 1]$, we can refer to A. Plouvier [16], in order to insure the existence and uniqueness of a function u_ε satisfying,

$$u_\varepsilon \in L^\infty(Q),\ 0 \le u_\varepsilon \le 1,\ £^{N+1} - a.e.\ in\ Q,$$

$$\beta_\varepsilon(u_\varepsilon) \in L^2(0,T;V)\ and\ \frac{\partial}{\partial t}\beta_\varepsilon(u_\varepsilon) \in L^2(0,T;V')$$

and which verifies, $£^1 - a.e.\ in\ ]0, T[$ the variational equation,

$$\forall v \in V, \left\langle \frac{\partial}{\partial t}\beta_\varepsilon(u_\varepsilon), v \right\rangle + \int_\Omega \{A(.,\beta_\varepsilon(u_\varepsilon))\nabla\Phi_\varepsilon(u_\varepsilon)\}\cdot\nabla v dx = 0$$

associated with the initial condition,

$$\beta_\varepsilon(u_\varepsilon)(0) = \beta_\varepsilon(u_0),\ £^N - a.e.\ on\ \Omega.$$

Step 2: A priori estimates. By choosing $v = u_\varepsilon$ in the previous variational formulation, and by a F. Mignot and A. Bamberger's lemma (see appendix) we get the following *a priori* estimates:

$\Phi(u_\varepsilon)$ is bounded in the space $L^2(0, T; V)$,

$\sqrt{\varepsilon u_\varepsilon}$ is bounded in the space $L^2(0, T; V)$,

$\frac{\partial}{\partial t}\beta_\varepsilon(u_\varepsilon)$ is bounded in the space $L^2(0, T; V')$.

Step 3: Passing to the limit when $\varepsilon \to 0^+$:

1. Convergence of $\Phi(\upsilon_\varepsilon)$:

There exist $\chi_1 \in L^2(0, T; V)$ and a subsequence (still denoted by $\Phi(u_\varepsilon)$) such that,

$$\Phi(u_\varepsilon) \to \chi_1\ in\ L^2(0, T; V).$$

A classical result on closed convex sets insures that the limit χ_1 satisfies the inequalities $0 \le \chi_1 \le \Phi(1)$, $£^{N+1} - a.e.\ in\ Q$. As Φ^{-1} is continuous, the function $u = \Phi^{-1}(\chi_1)$ *is* $£^{N+1}$ – measurable and $0 \le u \le 1$ $£^{N+1} - a.e.\ in\ Q$. We will keep in mind that,

$$\Phi(u) \in L^2(0, T; V),\ \Phi(u_\varepsilon) \to \Phi(u)\ in\ L^2(0, T; V).$$

2. Convergence of β_ε (u_ε):

We first note that the function $r \to \beta\varepsilon \circ \Phi^{-1}(r)$ is θ-Hölder continuous, and we define the space:

$$\mathcal{H} = \left\{ v \in L^{2/\theta}\left(0,T;W^{\theta,2/\theta}(\Omega)\right), \frac{\partial v}{\partial t} \in L^2(0,T;V') \right\}.$$

By taking into account the E. Gagliardo's characterization of Sobolev spaces (*cf.* J.L. Lions and E. Magenes [14], t. 1, p. 108), and the previous *a priori* estimates, we observe that,

β_ε (u_ε) *is* bounded in the space $\mathcal{H}$.

Following J.L. Lions (*cf.* [13] p. 142), it comes that the imbedding of $\mathcal{H}$ in $L^{2/\theta}$ (Q) is compact (as the imbedding of $W\theta^{,2/}\theta$ (Ω) compact). Hence, there exists χ_2 in $\mathcal{H}$ satisfying $\beta\varepsilon$ $(u_\varepsilon) \to \chi_2$ weakly in $\mathcal{H}$ and strongly in $L^{2/\theta}$ (Q). A Minty type monotonicity argument (*cf.* [13] p. 157) easily leads to $\chi_2 = \beta$ (u). E will keep in mind that,

$$\frac{\partial}{\partial t}\beta_\varepsilon(u_\varepsilon) \to \frac{\partial}{\partial t}\beta(u) \text{ in } L^2(0,T;V') \text{ and } \beta_\varepsilon(u_\varepsilon) \to \beta(u) \ \ \pounds^{N+1} - a.e. \text{ in } Q.$$

Conclusion

The above analysis shows that u satisfies the regularity property (R); furthermore, passing to the limit when ε tends towards 0^+, we immediately deduce the variational equality (V).

4. RENORMALIZED FORMULATION OF THE PROBLEM

Under restrictive assumptions on the functions involved in the equation (the mapping $r \mapsto \|\cdot\|$ $\|A$ $(.,\beta \circ \Phi^{-1}(r))\|$ is α-Hölder continuous with $\alpha > 1/2$), a classical method developed for the treatment of first order nonlinear hyperbolic problems (see for instance J. Carrillo [8], G. Gagneux et M. Madaune-Tort [12] p. 115-120, A. Plouvier [16]), leads to a comparison principle for weak solutions of $(\mathcal{P})$.

The main goal of this paper is to perform such a result without any condition on the local behaviour of nonlinear terms. In order to overcome this difficulty, we will check, following A. Plouvier et G. Gagneux [17] (according to the ideas presented by D. Blanchard, H. Redwane [6], F. Murat [15], L. Boccardo, D. Giachetti, J.I. Díaz, F. Murat [7], R.J. Di Perna et P.L. Lions [11]) that any weak solution is automatically a renormalized one (with a suitable definition), and working with this new formulation, we will derive the expected comparison result.

Let us consider for $n \in I\!N^*$, the truncation functions S_n and Σ_n defined by,

S_n $(r) = \min$ $\{(2nr - 1)^+ , 1\}$ *and* Σ_n $(r) = \min$ $\{(-2nr + 2n - 1)^+ , 1\}$

We can give the following definition:

Definition. *We say that u is a renormalized solution of* $(\mathcal{P})$ *if the conditions below are fulfilled,*

- *Regularity properties:*

$$(R)\begin{cases} u \in L^\infty(Q), 0 \le u \le 1, \ \pounds^{N+1} - a.e. \text{ in } Q, & \\ \Phi(u) \in L^2(0,T;V), & \dfrac{\partial}{\partial t}\beta(u) \in L^2(0,T;V'), \\ \beta(u)(0) = \beta(u_0) \ \pounds^N - a.e. \text{ on } \Omega, & \lim\limits_{t \to 0^+} \beta(u)(t) = \beta(u_0) \text{ in } L^1(\Omega), \end{cases}$$

- *Variational equality:*

$$(V^*)\begin{cases} \text{for any } \eta \in \left]0,\frac{1}{2}\right[\text{ and for any } T_\eta \text{ in } W^{1,\infty}(]0,1[), \\ \text{constant on the sets } [0,\eta] \text{ and } [1-\eta,1], \\ \text{for any } w \in L^\infty(\Omega) \cap H^1(\Omega) \text{ and } \pounds^1 - a.e. \text{ in }]0,T[, \\ \left\langle \dfrac{\partial}{\partial t}\beta(u), T_\eta(u)w \right\rangle + \displaystyle\int_\Omega \{A(.,\beta(u))\nabla\Phi(u)\} \cdot \{w\nabla T_\eta(u) + T_\eta(u)\nabla w\} dx = 0 \end{cases}$$

- *Parabolic-hyperbolic adjusting conditions:*

$$(L_0)\left\{ \lim_{n \to +\infty} \int_Q \{A(.,\beta(u))\nabla\Phi(u)\} \cdot \nabla S_n(u) dx dt = 0 \right.$$

Remark. In a first step, the definition of a renormalized solution only takes into account the parabolicity area of the phenomenon, namely the measurable set u^{-1} $([1/2n, 1 - 1/2n])$ (we recall that the diffusion operator only degenerates for the extremal values 0 and 1). In a second step we reach the hyperbolicity area u^{-1} $(\{0, 1\})$ by means of the adjusting conditions (L_0) and (L_1).

We give now the main theorem of this section:

Theorem. *u is a renormalized solution of the problem* $(\mathcal{P})$ *if and only if u is a weak solution of the problem* $(\mathcal{P})$.

In other words, the weak and renormalized formulation are equivalent.

Proof of the theorem:

First step. Let us consider a weak solution of $(\mathcal{P})$ and derive the properties appearing in the above definition.

Variational equality (V^*). For $\eta \in \left]0,\frac{1}{2}\right[$, let us choose a generic function T_η in $W^{1,\infty}$ $(]0, T[)$, constant on $[0, \eta]$ and $[1 - \eta, 1]$. We start proving that,

$$\pounds^1 - a.e. \text{ in }]0, T[, \ T_\eta(u) \in H^1(\Omega) \cap L^\infty(\Omega).$$

One can easily check that, for $\pounds^1$ – almost every t in $]0, T[$,

$$T_\eta(u)=T_\eta \circ \tilde{\Phi}^{-1}(\min\{\max\Phi(\eta), \Phi(u)\}, \Phi(1-\eta)\}),$$
$$\pounds^N\text{–a.e. in } \Omega$$

where $\tilde{\Phi}^{-1}$ is the restriction of Φ^{-1} to the set $[\Phi(\eta), \Phi(1-\eta)]$. As Φ' is bounded from below on $[\eta, 1-\eta]$ by a strictly positive real number, $\tilde{\Phi}^{-1}$ and $T_\eta \circ \tilde{\Phi}^{-1}$ are shown to be Lipschitz continuous on $[\Phi(\eta), \Phi(1-\eta)]$. Consequently, G. Stampacchia's lemma gives,

$$\pounds^1 - a.e. \text{ in }]0, T[, \; T_\eta(u) \in H^1(\Omega)$$

and hence the result.

Let w be an element of $H^1(\Omega) \cap L^\infty(\Omega)$.

A classical result in the Banach algebra $H^1(\Omega) \cap L^\infty(\Omega)$ (see H. Brezis [5]) insures that,

$$T_\eta(u)w \in H^1(\Omega) \cap L^\infty(\Omega),$$

$$and\ \forall i \in \{1,\dots,N\},\ \frac{\partial(T_\eta(u)w)}{\partial x_i} = T_\eta(u)\frac{\partial w}{\partial x_i} + w\frac{\partial T_\eta(u)}{\partial x_i} \text{ in } L^2(\Omega).$$

By taking $v = T\eta(u)\, w$ in the variational formulation (V), the previous rule immediately gives (V^*).

Parabolic hyperbolic adjusting relations. The above analysis allows us to choose $v = S_n(u)$ in the variational equality (V); after integrating on $]0, T[$, we get,

$$\int_Q \{A(.,\beta(u))\nabla\Phi(u)\}\nabla S_n(u)\,dx\,dt = -\int_0^T \left\langle \frac{\partial}{\partial t}\beta(u), S_n(u)\right\rangle dt.$$

As the function $r \to S_n(r)$ is increasing and continuous, the extension of A. Bamberger and F. Mignot's lemma (see appendix) gives the following identity:

$$\int_0^T \left\langle \frac{\partial}{\partial t}\beta(u), S_n(u)\right\rangle dt = -\int_\Omega \left(\int_{\beta(u)(0)}^{\beta(u)(T)} S_n \circ \gamma(r)\,dr \right) dx,$$

where γ denotes any quasi-inverse of β.

By dominated convergence, we easily check that,

$$\lim_{n\to+\infty} \int_0^T \left\langle \frac{\partial}{\partial t}\beta(u), S_n(u)\right\rangle dt = -\int_\Omega (\beta(u)(T) - \beta(u)(0))\,dx.$$

Furthermore, the mass conservation law, described by considering $v = 1_\Omega$ in the variational formulation (V) yields,

$$\int_{\Omega} \cdots (\beta(u)(T) - \beta(u)(0))\,dx = 0.$$

The latter identity leads to (L_0); the proof of (L_1) is analogous to the previous one (by taking $v = \Sigma_n(u)$ in the weak formulation).

Second step. If u is a renormalized solution of (P) we can choose any test-function of the form $u \in H^1(\Omega) \cap L^\infty(\Omega)$ because $u = 1(u)\, u$ (where $u \to 1(u)$ denotes the constant function equal to 1, which satisfies the condition specified in (V^*)). We immediately deduce that u is a weak solution of $(\mathcal{P})$.

5. COMPARISON PRINCIPLE

We are now able to state a comparison principle for weak-renormalized solutions of the problem.

Theorem. *Let u and $\hat{u}$ be solutions of $(\mathcal{P})$ associated with the initial data u_0 and $\hat{u}_0$ respectively. Then, for $\pounds^1$-almost every t in $]0, T[$, the following inequality holds:*

$$\int_\Omega (\beta(u)(t) - \beta(\hat{u})(t))^+\,dx \le \int_\Omega (\beta(u_0) - \beta(\hat{u}_0))^+\,dx$$

Proof of the theorem. For $\varepsilon \in]0, 1]$, let us define the function p_ε:

$$\forall r \in [0,1],\ p_\varepsilon(r) = min\left(\frac{r^+}{\varepsilon}, 1\right).$$

For $n \in I\!N^*$, we consider the truncated function Φ_n defined by,

$$\forall r \in [0, 1],\ \Phi_n(r) = \min\{\max\{\Phi(1/2n), \Phi(r)\}, \Phi(1-1/2n)\},$$

and we denote, for the sake of simplicity, $T_n = S_n\Sigma_n$.

For $\pounds^2$– almost every (t, s) fixed in $]0, T[^2$, let us choose in the Banach algebra $L^\infty(\Omega) \cap H^1(\Omega)$, the test-function $v = T_n(u)\, p_\varepsilon(\Phi_n(u) - \Phi_n(\hat{u}))$ in the weak formulation associated with u, and $v = T_n(\hat{u})\, p_\varepsilon(\Phi_n(u) - \Phi_n(\hat{u}))$ in the weak formulation associated with $\hat{u}$. In other words, we take $T_\eta = T_n$ and $w = p_\varepsilon(\Phi_n(u) - \Phi_n(\hat{u}))$ in the renormalized formulation of $(\mathcal{P})$.

We introduce the sequence,

$$\xi_\delta(t,s) = \xi\left(\frac{t+s}{2}\right)\rho_\delta\left(\frac{t-s}{2}\right),$$

where $\xi \in \mathcal{D}(]0, T[)$, $\xi \ge 0$, and $\rho\delta \in \mathcal{D}(]0, T[)$, $\rho_\delta \ge 0$, supp $(\rho\delta) \subset]-\delta, \delta[$, and $\int_{I\!R} \rho\delta\,dt = 1$.

We suppose that δ is sufficiently small, so that $\xi_\delta \in \mathcal{D}(]0, T[^2)$.

If we consider the difference between the variational equality associated with u and $\hat{u}$ we get, after integrating on $]0, T[^2$,

$$0=\int_0^T\int_0^T\left\langle\frac{\partial}{\partial t}\beta(u),T_n(u)w\right\rangle\xi_\delta\,dtds$$
$$-\int_0^T\int_0^T\left\langle\frac{\partial}{\partial s}\beta(\hat u),T_n(\hat u)w\right\rangle\xi_\delta\,dtds$$
$$+\int_{]0,T[^2\times\Omega}\{A(x,\beta(\hat u))\nabla\Phi(u)\}\nabla T_n(u)\cdot w\,dx\,\xi_\delta\,dtds$$
$$-\int_{]0,T[^2\times\Omega}\{A(x,\beta(\hat u))\nabla\Phi(\hat u)\}\nabla T_n(\hat u)\cdot w\,dx\,\xi_\delta\,dtds$$
$$+\int_{]0,T[^2\times\Omega}\{T_n(u)A(x,\beta(u))\nabla\Phi(u)-T_n(\hat u)A(x,\beta(\hat u))\nabla\Phi(\hat u)\}\cdot\nabla w\,dx\,\xi_\delta\,dtds.$$

For the sake of brevity we will write,

$$I^1_{\varepsilon n}-I^2_{\varepsilon n}+I^3_{\varepsilon n}-I^4_{\varepsilon n}+I^5_{\varepsilon n}=0.$$

Our goal is to study each term of the above identity, according to the following strategy:

Step 1: when ε goes to 0^+, n being fixed in $I\!N^*$.

Step 2: when n goes to $+\infty$.

Conclusion. We deduce the comparison property by passing to the limit when δ goes to 0^+ in the inequality obtained after the previous steps.

Convergence of $I^1_{\varepsilon n}$ **:**

We have,

$$I^1_{\varepsilon n}=\int_0^T\int_0^T\left\langle\frac{\partial}{\partial t}\beta(u),T_n(u)p_\varepsilon(\Phi_n(\hat u))\right\rangle\xi_\delta\,dtds.$$

We note, for $\pounds^{N+1}$ – almost every (s, x) fixed in Q, that the function

$$F:r\to\left(T_n\circ\Phi^{-1}\right)(r)p_\varepsilon\left[\left(\Phi_n\circ\Phi^{-1}\right)(r)-\Phi_n(\hat u)\right],r\in[0,1],$$

is Lipschitz continuous on [0, 1] (because T_n and Φ_n are constant on $\left[0,\frac{1}{2n,}\right]$ and $\left[1-\frac{1}{2n},1\right]$ whereas Φ^{-1} is Lipschitz continuous on the compact set $\left[\Phi\left(\frac{1}{2n,}\right),\Phi\left(1-\frac{1}{2n}\right)\right]$. The extension of F. Mignot and A. Bamberger's lemma (see appendix, corollary) leads to the equality, for $\pounds^1$ – almost every s in]0, T[,

$$\int_0^T\left\langle\frac{\partial}{\partial t}\beta(u),T_n(u)p_\varepsilon(\Phi_n(u)-\Phi_n(\hat u))\right\rangle\xi_\delta dt=$$
$$-\int_0^T\frac{\partial\xi_\delta}{\partial t}\left(\int_\Omega\left\{\int_{\beta(\hat u)}^{\beta(u)}T_n(\gamma(r))p_\varepsilon(\Phi_n(\gamma(r))-\Phi_n(\hat u))dr\right\}dx\right)dt,$$

where γ denotes any quasi-inverse of β.

The deminated convergence theorem gives,

$$\lim_{n\to+\infty}\lim_{\varepsilon\to0^+}I^1_{\varepsilon n}=-\int_{]0,T[^2\times\Omega}\frac{\partial\xi_\delta}{\partial t}\left\{(\beta(u)-\beta(\hat u))^+\right\}dxdtds.$$

Convergence of $I^2_{\varepsilon n}$ **:**

We prove in the same way the convergence of $I^2_{\varepsilon n}$, towards the real number,

$$I^2=\int_{]0,T[^2\times\Omega}\frac{\partial\xi_\delta}{\partial s}\left\{(\beta(u)-\beta(\hat u))^+\right\}dxdtds.$$

Convergence of $I^3_{\varepsilon n}$ **:**

We have,

$$I^3_{\varepsilon n}\quad\int_{]0,T[^2\times\Omega}\{A(x,\beta(u))\nabla\Phi(u)\}\nabla T_n(u)\cdot w\,dx\,\xi_\delta\,dtds$$

where w denotes the function $p_\varepsilon\,(\Phi_n\,(u)-\Phi_n\,(\hat u))$.

As $T_n(u) = S_n(u)\,\Sigma_n(u)$, the usual calculus rule in the Banach algebra $L^\infty(\Omega)\cap H^1(\Omega)$ gives, $\pounds^{N+1}$ – *a.e. in Q,*

$$\nabla T_n(u)=\Sigma_n(u)\ \nabla\Sigma_n(u)+\Sigma_n(u)\ \nabla\Sigma_n(u).$$

Consequently,

$$I^3_{\varepsilon n}=\int_{]0,T[^2\times\Omega}\{A(x,\beta(u))\nabla\Phi(u)\}\nabla\Sigma_n(u)\cdot S_n(u)w\,dx\,\xi_\delta\,dtds$$
$$+\int_{]0,T[^2\times\Omega}\{A(x,\beta(u))\nabla\Phi(u)\}\nabla S_n(u)\cdot\Sigma_n(u)w\,dx\,\xi_\delta\,dtds.$$

We will note, for the sake of brevity,

$$I^3_{\varepsilon n}=I^{3*}_{\varepsilon n}+I^{3**}_{\varepsilon n}.$$

Convergence of $I^{3*}_{\varepsilon n}$ **:**

Step 1: (ε → 0^+)

As $0\le p_\varepsilon\le1$ and as the sequence $(p_\varepsilon)_\varepsilon$ simply converges towards the function sg_0^+, the dominated convergence theorem yields,

$$\lim_{\varepsilon\to0^+}I^{3*}_{\varepsilon n}=I^{3*}_n=$$
$$\int_{]0,T[^2\times\Omega}\{A(x,\beta(u))\nabla\Phi(u)\}\nabla\Sigma_n(u)\cdot S_n(u)sg_0^+(\Phi_n(u)-\Phi_n(\hat u))\xi_\delta\,dxdtds$$

Step 2: ($n\to+\infty$)

We note that the function $r\to(\Sigma_n\circ\Phi^{-1})(r)$ is Lipschitz continuous on [0, Φ (1)] (because Σ_n is constant on [0, 1 – 1/n] and [1 – 1/2n, 1] , whereas Φ^{-1} is Lipschitz continuous on the compact set [Φ (1 – 1/n), Φ (1 – 1/2n)]). Following G. Stampacchia gives $\pounds^{N+1}$ – *a.e. in Q,*

$$\nabla\Sigma_n(u)=\nabla\left[(\Sigma_n\circ\Phi^{-1})(\Phi(u))\right]$$

$= (\Sigma_n \circ \Phi^{-1})' (\Phi (u)). \nabla \Phi (u).$

By noting that $(\Sigma_n \circ \Phi^{-1})$ is decreasing on $[0, \Phi (1)]$, the coercivity assumption (*H*2) *ii*) gives,

$$\{A (., \beta (u)) \nabla \Phi (u)\} \nabla \Sigma_n (u) \leq 0, \text{£}^{N+1} - a.e. \text{ in } Q.$$

Therefore we find,

$$T\|\xi_\delta\|_\infty \int_Q \{A(x,\beta(u))\nabla\Phi(u)\}\nabla\Sigma_n(u)\, dxdtds \leq I_{\varepsilon n}^{3*} \leq 0,$$

and we conclude by means of (L_1) that,

$$\lim_{n\to+\infty} I_{\varepsilon n}^{3*} = 0.$$

An analogous proof (wich uses the monotonicity of $r \to (S_n \circ \Phi^{-1}) (r)$ and the identity (L_0)), leads to,

$$\lim_{n\to+\infty} \lim_{\varepsilon\to 0^+} I_{\varepsilon n}^{3**} = 0.$$

Finally, we conclude that,

$$\lim_{n\to+\infty} \lim_{\varepsilon\to 0^+} I_{\varepsilon n}^{3**} = 0.$$

Convergence of $I_{\varepsilon n}^4$ **:**

We show in the same way that,

$$\lim_{n\to+\infty} \lim_{\varepsilon\to 0^+} I_{\varepsilon n}^4 = 0.$$

Convergence of $I_{\varepsilon n}^5$ **:**

We have,

$$I_{\varepsilon n}^5 = \int_{]0,T[^2\times\Omega} \left\{T_n(u)A(x,\beta(u))\nabla\Phi(u) - T_n(\hat{u})A(x,\beta(\hat{u}))\nabla\Phi(\hat{u})\right\}\cdot\nabla w\, \xi_\delta dxdtds,$$

where w denotes the function $p_\varepsilon (\Phi_n (u) - \Phi_n (\hat{u}))$.

We compute, following G. Stampacchia,

$$\text{£}^{N+2} - a.e. \text{ in }]0,T[^2\times\Omega,\ \nabla w = \frac{1}{\varepsilon}\nabla(\Phi_n(u)-\Phi_n(\hat{u}))\cdot\chi_{[0<\Phi_n(u)-\Phi_n(\hat{u})<\varepsilon]}.$$

Let us consider the £$^{N+2}$ -measurable sets,

$$K\varepsilon = [0 < \Phi_n (u) - \Phi_n (\hat{u}) < \varepsilon] \cap [u < 1 - 1/2n] \cap [\hat{u} > 1/2n]$$

$$J\varepsilon = [0 < \Phi_n (u) - \Phi_n (\hat{u}) < \varepsilon] \cap ([u \geq 1 - 1/2n] \cup [\hat{u} \leq 1/2n])$$

and introduce the decomposition,

$$I_{\varepsilon n}^5 = \frac{1}{\varepsilon}\int_{J_\varepsilon} \{T_n(u)A(x,\beta(u))\nabla\Phi(u) - T_n(\hat{u})A(x,\beta(\hat{u}))\nabla\Phi(\hat{u})\}\cdot\nabla(\Phi_n(u)-\Phi_n(\hat{u}))\xi_\delta dxdtds$$
$$+\frac{1}{\varepsilon}\int_{K_\varepsilon} \{T_n(\hat{u})A(x,\beta(\hat{u}))\}\nabla(\Phi(u)-\Phi(\hat{u}))\cdot\nabla(\Phi_n(u)-\Phi_n(\hat{u}))\, \xi_\delta\, dxdtds$$
$$+\frac{1}{\varepsilon}\int_{K_\varepsilon} \{T_n(u)A(x,\beta(u)) - T_n(\hat{u})A(x,\beta(\hat{u}))\}\nabla\Phi(u)\cdot\nabla(\Phi_n(u)-\Phi_n(\hat{u}))\, \xi_\delta\, dxdtds.$$

We will write, for the sake of simplicity,

$$I_{\varepsilon n}^5 = I_{\varepsilon n}^{5*} + I_{\varepsilon n}^{5**} + I_{\varepsilon n}^{5***}$$

Convergence of $I_{\varepsilon n}^{5*}$ **:**

We define, for £$^{N+2}$ – almost every (t, s, x) in $]0, T[^2 \times \Omega$ the real number,

$$f_{\varepsilon n}(t,s,x) = \frac{1}{\varepsilon}\{T_n(u)A(x,\beta(u))\nabla\Phi(u) -$$
$$T_n(\hat{u})A(x,\beta(\hat{u}))\nabla\Phi(\hat{u})\}\cdot\nabla(\Phi_n(u)-\Phi_n(\hat{u}))\xi_\delta,$$

and write that,

$$I_{\varepsilon n}^{5*} = \int_{J_\varepsilon} f_{\varepsilon n}(t,s,r)\, dxdtds.$$

The definition of J_ε associated with the coercivity condition (*H*2) *ii*) implies that

$$f\varepsilon_n \geq 0,\ \text{£}^{N+2} - a.e. \text{ in } J\varepsilon.$$

Finally we conclude that,

$$I_{\varepsilon n}^{5*} \geq 0.$$

Convergence of $I_{\varepsilon n}^{5**}$ **:**

We first show the identity,

$K\varepsilon = [0 < \Phi (u) - \Phi (\hat{u}) < \varepsilon] \cap [1/2n < u < 1 - 1/2n] \cap [1/2n < \hat{u} < 1 - 1/2n]$.

Indeed, if (t, s, x) is given in $]0, T[^2 \times \Omega$, we note that:

The inequalities $\Phi_n (\hat{u}) (s, x) < \Phi_n (u) (t, x)$ *and* $u (t, x) < 1 - 1/2n$ automatically imply that $\hat{u} (s, x) < 1 - 1/2n$.

The inequalities $\Phi_n (\hat{u}) (s, x) < \Phi_n (u) (t, x)$ *and* $\hat{u} (s, x) > 1/2n$ automatically imply that $u (t, x) > 1/2n$.

The expected result then follows.

Now, we can remark that,

$$\nabla \Phi_n (u) = \nabla \Phi (u) \ et\ \nabla \Phi_n (\hat{u}) = \nabla \Phi (\hat{u}),\ \text{£}^{N+2} - a.e. \text{ in } K\varepsilon,$$

and consequently,

$$I_{\varepsilon n}^{5**} = \frac{1}{\varepsilon}\int_{K_\varepsilon} \{T_n(\hat{u})A(x,\beta(\hat{u}))\}\nabla(\Phi(u)-\Phi(\hat{u}))\cdot\nabla(\Phi(u)-\Phi(\hat{u}))\xi_\delta \, dxdtds.$$

According to the coercivity condition (*H*2) *ii*) we conclude that,

$$I_{\varepsilon n}^{5**} \geq 0.$$

Convergence of $I_{\varepsilon n}^{5***}$:

As mentioned before,

$K\varepsilon = [0 < \Phi(u) - \Phi(\hat{u}) < \varepsilon] \cap [1/2n < u < 1 - 1/2n] \cap [1/2n < \hat{u} < 1 - 1/2n]$.

Then we can write,

$$I_{\varepsilon n}^{5***} = \frac{1}{\varepsilon}\int_{K_\varepsilon} \{T_n(u)A(x,\beta(u)) - T_n(\hat{u})A(x,\beta(\hat{u}))\}\nabla\Phi(u)\cdot\nabla(\Phi(\hat{u})-\Phi(\hat{u}))\xi_\delta \, dxdtds,$$

If $\|.\|_2$ denotes the matrix norm subordinate to the euclidian norm of $\mathbb{R}^N$, we can estimate, $£^{N+2}$ – almost everywhere in $K\varepsilon$,

$$\frac{1}{\varepsilon}\left|\{T_n(u)A(x,\beta(u)) - T_n(\hat{u})A(x,\beta(\hat{u}))\}\nabla\Phi(u)\cdot\nabla(\Phi(u)-\Phi(\hat{u}))\right|$$
$$\leq \frac{1}{\varepsilon}\left\|\{T_n(u)A(x,\beta(u)) - T_n(\hat{u})A(x,\beta(\hat{u}))\}\right\|_2 \cdot |\nabla\Phi(u)| \cdot |\nabla(\Phi(u)-\Phi(\hat{u}))|$$

For any (i, j) given in $(1, ..., N)^2$, the function,

$$r \rightarrow (T_n \circ \Phi^{-1})(r) \cdot a_{ij}(x,(\beta\circ\Phi^{-1})(r)),$$

is Lipschitz continuous on $[\Phi(1/2n), \Phi(1 - 1/2n)]$, uniformly with respect to x (c_n will denote the associated Lipschitz constant).

Moreover we know that in the space $\mathcal{M}_N(\mathbb{R})$, the norm $\|.\|_2$ introduced above is dominated by the matrix euclidian norm (see for instance P.G. Ciarlet [9], p. 20); we deduce that, – *a.e. in* $K\varepsilon$,

$$\|\{T_n(u)A(x, \beta(u)) - T_n(\hat{u})A(x, \beta(\hat{u}))\}\|_2 \leq Nc_n |\Phi(u) - \Phi(\hat{u})|,$$

and we easily estimate,

$$\left|\frac{1}{\varepsilon}\int_{K_\varepsilon} \{T_n(u)A(x,\beta(u)) - T_n(\hat{u})A(x,\beta(\hat{u}))\}\nabla\Phi(u)\cdot\nabla(\Phi(u)-\Phi(\hat{u}))\xi_\delta dxdtds\right|$$
$$\leq Nc_n\int_{K_\varepsilon} |\nabla\Phi(u)|\cdot|\nabla(\Phi(u)-\Phi(\hat{u}))|\xi_\delta dxdtds$$

Concerning the right-hand side of the inequality, we note that,

$$£^{N+2} - a.e.\ in]0,T[^2 \times \Omega, \lim_{\varepsilon\to 0} \chi_{K_\varepsilon} = 0.$$

Therefore, by dominated convergence (for n fixed in IN^*) we get,

$$\lim_{\varepsilon\to 0^+} Nc_n \int_{K_\varepsilon} |\nabla\Phi(u)|\cdot|\nabla(\Phi_n(u)-\Phi_n(\hat{u}))|\xi_\delta dxdtds = 0,$$

so that,

$$\lim_{\varepsilon\to 0^+} I_{\varepsilon n}^{5***} = 0.$$

We successively deduce,

$$\liminf_{\varepsilon\to 0^+} I_{\varepsilon n}^5 \geq \liminf_{\varepsilon\to 0^+} I_{\varepsilon n}^{5*} + \liminf_{\varepsilon\to 0^+} I_{\varepsilon n}^{5**} + \lim_{\varepsilon\to 0^+} I_{\varepsilon n}^{5***} \geq 0$$

and conclude that,

$$\liminf_{n\to+\infty}\left(\liminf_{\varepsilon\to 0^+} I_{\varepsilon n}^5\right) \geq 0.$$

CONCLUSION

By passing to the lower limit in the initial equality,

$$I_{\varepsilon n}^1 - I_{\varepsilon n}^2 + I_{\varepsilon n}^3 - I_{\varepsilon n}^4 + I_{\varepsilon n}^5 = 0,$$

the previous analysis gives,

$I^1 - I^2 \leq 0$,

that is,

$$-\int_{]o,T[^2\times\Omega}\left(\frac{\partial\xi_\delta}{\partial t} + \frac{\partial\xi_\delta}{\partial s}\right)(\beta(u)-\beta(\hat{u}))^+ dxdtds \leq 0.$$

By using a well-known argument based on the notion of Lebesgue's point, we can pass to the limit in the above inequality when δ goes to 0^+, and write,

$$\forall \xi \mathcal{D}^+(]0,T[), -\int_{]0,T[\times\Omega} \xi'(t)(\beta(u)-\beta(\hat{u}))^+ dxdt \leq 0$$

In other words, we have

$$\frac{\partial}{\partial t}\int_\Omega (\beta(u)-\beta(\hat{u}))^+ dx \leq 0 \ in\ \mathcal{D}'(]0,T[).$$

Let us consider the real measurable positive function z defined by,

$$z(t)=\int_{\Omega}\left(\beta(u)-\beta(\hat{u})\right)^{+}dx \; for \; t\in[0,T].$$

As $\frac{dz}{dt}\leq 0$ *in* $\mathcal{D}'$ (]0, T[), we state, according to L. Schwartz [18], p. 29 et p. 53-54, that $\frac{dz}{dt}$ is a negative measure and z is almost everywhere equal to a bounded function $\bar{z}$ which decreases on]0, T[. The L^1 (Ω)-continuity of $t \rightarrow \beta(u)(t)$ at 0 finally implies that $\bar{z}(t) \leq z(0)$ for any t in]0, T[. This achieves the proof of the comparison principle.

6. A UNIQUENESS RESULT

The comparison principle described in the preceding section easily leads to the following theorem:

Theorem. *Let u and $\hat{u}$ be solutions of ($\mathcal{P}$) associated with the same initial datum u_0. Then we have the identity,*

$$\beta(u) = \beta(\hat{u}), \; £^{N+1} - a.e. \; in \; Q.$$

As a particular case, when β is supposed to be injective, the problem ($\mathcal{P}$) admits a unique weak solution.

Remark. When β is constant on a subdomain of]0, 1[, and the initial datum is correctly chosen, we easily prove that the uniqueness result does not hold.

APPENDIX: EXTENSION OF F. MIGNOT AND A. BAMBERGER'S LEMMA

By using a double convexity inequality, A. Bamberger (who followed an idea of F. Mignot) performed an «integration by parts» formula for nonlinear terms like $\left\langle \frac{\partial}{\partial t}\beta(u),\Phi(u)\right\rangle$, where β is a stricly monotone continuous function (see also A. Bamberger [3], [4], S.N. Antontsev and J.I. Díaz [2], J.I. Díaz and F. De Thelin [10], G. Gagneux et M. Madaune-Tort [12] p. 31). Here we give (without proof) an extension of the result when β is not supposed to be injective.

For the sake of brevity, we will say that F satisfies (H) if the following condition is fulfilled,

$$(H)\left\{F:\mathrm{IR}\rightarrow\mathrm{IR}\begin{array}{l} is\ increasing\ continuous\\ and\ onto.\end{array}\right\}$$

Note that the technical condition F (IR) = IR is not restrictive. Indeed, we often consider a function F defined on a bounded interval, and the condition is then immediately satisfied for a suitable extrapolation $\tilde{F}$.

1. Notion of quasi-inverse

Let $\beta : \mathrm{IR} \rightarrow \mathrm{IR}$ be a function satisfying (H).

Definition. We will say that γ is a quasi-inverse of β if γ is a function whose graph is contained in the graph of β^{-1}.

Properties. We easily show that,

i) Any quasi-inverse γ of β is strictly increasing on IR and then continuous up to an enumerable set.

ii) Two quasi-inverses γ and γ′ are equal up to an enumerable set.

2. F. Mignot and A. Bamberger's lemma

We first set the following general assumptions:

i) Let $Q = \Omega$ _]0, T[, where Ω is an open bounded set of IR^N and T be a strictly positive real number. We consider a Hilbert space V wich gives rise to the classical scheme,

$$V \rightarrow L^2(\Omega) \rightarrow V'$$

ii) Let β and Φ be functions satisfying (H).

iii) We consider a function u defined almost everywhere in Q, such that

$$\beta(u) \in L^2(Q),$$

$$\frac{\partial}{\partial t}\beta(u)L^2(0,T;V'),$$

$$\Phi(u)\in L^2(0,T;V).$$

Then the following integration rule holds,

$$\int_0^T\left\langle\frac{\partial}{\partial t}\beta(u),\Phi(u)\right\rangle\xi(t)dt=-\int_0^T\xi'(t)\cdot\int_{\Omega}\left\{\int_0^{\beta(u)(t,x)}\Phi(\gamma(r))dr\right\}dxdt,$$

for $\xi \in C^1$ ([0, T]) such that $\xi(0) = \xi(T) = 0$, and where γ denotes any quasi-inverse of β.

Consequently we can write in L^1 (]0, T[) (where the derivative is taken in the sense of distributions):

$$\left\langle\frac{\partial}{\partial t}\beta(u),\Phi(u)\right\rangle=\frac{\partial}{\partial t}\int_{\Omega}\left\{\int_0^{\beta(u)(t,x)}\Phi(\gamma(r))dr\right\}dx.$$

To end up this work, let us mention the following corollary (often used in the paper), wich does not require a monotonicity condition on the function involved in the duality brackets.

3. Corollary

We need the assumptions described in section 2), in the particular case where $V = H^1(\Omega)$, and we consider a general function F in $W^{1,\infty}$ ($), without restriction with respect to monotonicity.

Then the following integration rule holds,

$$\int_0^T \left\langle \frac{\partial}{\partial t}\beta(u),(F\circ\Phi)(u)\right\rangle \xi(t)dt = -\int_0^T \xi'(t)\cdot\int_\Omega \left\{ \int_0^{\beta(u)(t,x)} (F\circ\Phi)(\gamma(r))dr \right\} dxdt,$$

for $\xi \in C^1([0, T])$ such that $\xi(0) = \xi(T) = 0$, and where γ denotes any quasi-inverse of β.

Consequently we can write in $L^1(]0, T[)$ (where the derivative is taken in the sense of distributions):

$$\left\langle \frac{\partial}{\partial t}\beta(u),(F\circ\Phi)(u)\right\rangle = \frac{\partial}{\partial t}\int_\Omega \left\{ \int_0^{\beta(u)(t,x)} (F\circ\Phi)(\gamma(r))dr \right\} dx.$$

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