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A BATOR'S QUESTION ON DUAL BANACH SPACES

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ABSTRACT

We obtain a characterisation of the nonseparability of the dual of a separable Banach space X by the existence of an operator T from X into C(Δ), being Δ the Cantor ternary set, giving an answer to a question proposed by E.M. Bator in 1992.

RESUMEN

Obtenemos una caracterización de la no separabilidad del dual de un espacio de Banach separable X mediante la existencia de cierto operador T de X en C(Δ), siendo Δ el conjunto ternario de Cantor, dando una respuesta a la pregunta propuesta por E.M. Bator en 1992.

1. INTRODUCTION

It is said that x is a condensation point of the topological space X if every neighbourhood of x is uncountable. If all the points of X are condensation points we can determine two non void disjoint balls B_{11} and B_{12} of radius less than 1, two non void disjoint balls B_{21} and B_{22} (B_{23} and B_{24}) of radius less than 1/2 contained in B_{11} (B_{12}, respectively) and so on. Then we have that

$$\Delta = \left(\overline{B}_{11} \cup \overline{B}_{12}\right) \cap \left(\overline{B}_{21} \cup \overline{B}_{22} \cup \overline{B}_{23} \cup \overline{B}_{24}\right) \cap \ldots$$

is homeomorphic to the Cantor ternary set with dyadic subsets $\overline{B}_{11} \cap \Delta$, $\overline{B}_{12} \cap \Delta, \ldots$

If the topological space X has an uncountable quantity of points and verifies the second axiom of numerability, then the union Z of open countable subsets is a countable set, because a countable family of these open sets cover Z. Then every point of Y = X-Z is a condensation point of Y. In particular, if A is an uncountable subset of a compact and metrizable topological space, $\overline{A}$ contains a copy of the Cantor ternary set.

Then, if X is a separable Banach space such that its dual X* is not separable, we can find a Cantor ternary set in the weak* dual unit ball. By making an appropriate use of the Hahn-Banach theorem C. Stegall [4] and E.M. Bator [1] found the Cantor ternary set in such a way that the characteristic functions of the Cantor dyadic subsets can be uniformly approximated by elements of X.

In fact, by the nonseparability of the unit sphere S_{X^*} given $\mu > 0$ we can determine by transfinite induction $A = \left\{x_\alpha^* : \alpha < \omega_1\right\} \subset S_{X^*}$ and $\left\{x_\alpha^{**} : \alpha < \omega_1\right\} \subset X^{**}$ such that $x_\alpha^{**}\left(x_\alpha^*\right) = 1$, $\left\|x_\alpha^{**}\right\| \leq 1 + \mu$ and $x_\beta^{**}\left(x_\alpha^*\right) = 0$ when $\alpha < \beta < \omega_1$. This can be done since once determined $\left\{x_\alpha^* : \alpha < \beta\right\}$ and $\left\{x_\alpha^{**} : \alpha < \beta\right\}$ the closed linear hull of $\left\{x_\alpha^* : \alpha < \beta\right\}$ is separable, and then there is a x_β^{**} in X** such that $x_\beta^{**}\left(x_\alpha^*\right) = 0$ if $\alpha < \beta$ and $\left\|x_\beta^{**}\right\| = 1 + \mu$. The distance from the origin to the hyperplane $x_\beta^{**}\left(x^*\right) = 1$ is $1/(1 + \mu) < 1$, implying that the intersection of this hyperplane and S_{X^*} is not void. Taking x_β^* equal to a point of this intersection we finish the induction.

We can suppose that every point of A is a weak*-condensation point, deleting a countable family if it were necessary.

Let $\delta > 0$. Given x_α^* and x_β^* with $\alpha < \beta$, we know that there exists x_β^{**} with $\left\|x_\beta^{**}\right\| < 1 + \eta$ such that

$$x_\beta^{**}\left(x_\alpha^*\right) = 0 \quad x_\beta^{**}\left(x_\beta^*\right) = 1$$

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By the weak* density of B_X in $B_{X^{**}}$ we can find x_β with $\|x_\beta\| < 1 + \eta$ and such that

$$\left|x_\beta\left(x_\alpha^*\right)\right| < \delta = 0 \quad \left|x_\beta\left(x_\beta^*\right) - 1\right| < \delta$$

The preceding two inequalities enable us to determine two weak* neighbourhoods V_1^* and V_2^* of the points x_α^* and x_β^* such that

$$\left|x_\beta\left(x^*\right)\right| < \delta = 0 \text{ for } x^* \in V_1^* \text{ and } \left|x_\beta\left(x^*\right) - 1\right| < \delta \text{ for } x^* \in V_2^*$$

Now we take x_γ^* in V_1^* such that $\beta < \gamma$. If we apply the preceding reasoning to the points x_β^* and x_γ^* we can find some x_γ with $\|x_\gamma\| < 1 + \eta$ and two weak*-neighbourhoods $W_{11}^*(\subset V_1^*)$ and $W_{12}^*(\subset V_2^*)$of the points x_γ^* and x_β^* such that

$$\left|x_\gamma\left(x^*\right) - 1\right| < \delta \text{ for } x^* \in W_{11}^* \text{ and } \left|x_\gamma\left(x^*\right)\right| < \delta \text{ for } x^* \in W_{12}^*$$

holding

$$\left|x_\beta\left(x^*\right)\right| < \delta \text{ for } x^* \in W_{11}^* \text{ and } \left|x_\beta\left(x^*\right) - 1\right| < \delta \text{ for } x^* \in W_{12}^*$$

Then the difference between $x_{11} = x_\gamma$ and $x_{12} = x_\beta$ acting on the weak* closure of $W_{11}^* \cup W_{12}^*$ and the characteristic functions corresponding to the weak* closure of W_{11}^* and W_{12}^* is δ. By an obvious dicotomic induction process there follows the following Stegal theorem (4):

Let X be a separable Banach space such that X is nonseparable. Then for every $\varepsilon > 0$, there exists a subset Δ of B* which is homeomorphic to the Cantor set, along with subsets $\{C_{ni}\}_{n=1}^{\infty}\,{}_{i=1}^{2^n}$ of Δ weak* homeomorphic to the dyadic intervals, and a sequence $\{x_{ni}\}_{n=1}^{\infty}\,{}_{i=1}^{2^n}$ in X such that $\|x_{ni}\| < 1 + \varepsilon$ for all n, i and*

$$\left|x_{ni}\left(x^*\right) - \chi_{C_{ni}}\left(x^*\right)\right| \le \varepsilon 2^{-n} \quad \text{for all} \quad x^* \in \Delta$$

$\chi_{C_{ni}}$ being the characteristic function on the set C_{ni}.

Stegall's result is equivalent to the nonseparability of X*. In fact, given x* in Δ, let $\{i_n\}_{n=1}^{\infty}$ be the unique sequence such that $x^* \in C_{ni_n}$. Then from $\left|x^*\left(x_{ni_n}\right) - 1\right| \le \varepsilon 2^{-n}$ it follows that if x** is a weak* cluster point of the sequence $\{x_n\}_{n=1}^{\infty}$ then we have x** (x*) = 1. If y* ∈ Δ − {x*} there is some n_0 such that $y^* \notin C_{ni_n}$ for n > n_0, and then we have $\left|y^*\left(x_{ni_n}\right)\right| \le \varepsilon 2^{-n}$ for n > n_0, implying x** (y*) = 0. Therefore Δ is weak discrete, thus norm discrete, and consequently X* is nonseparable.

2. BATOR'S PROBLEM

From Stegall's result it follows that the natural evaluation map T : X → C(Δ) given by T(x)(x*) = x*(x) has dense range. Bator (1, example 5) shows that the existence of a continuous linear map T from a separable Banach space X onto a dense subspace of the space of real continuous functions defined on the Cantor ternary set Δ does not characterise separable spaces with nonseparable duals, because the range of the mapping T from 1^2 into C(Δ) given by $T(\{\alpha_n\}) = \sum_{n=1}^{\infty} \frac{1}{n} \alpha_n t^n$ is dense, since it contains the polynomials, and $(1^2)^* = 1^2$ is separable.

Bator (1. Page 85) asks for what property of a continuous linear map T from a separable Banach space X into the space C(Δ) of the real functions defined on the Cantor ternary set Δ would be able to characterise separable Banach spaces with nonseparable dual. A very interesting result in this direction had been obtained previously by Pelczynsky-Hagler theorem (2, 3) that states that 1^1 embeds in a separable Banach space X if, and only if, there exists a continuous linear surjection from X into C(Δ).

The following result gives an answer to Bator question.

Proposition 1. *Let X be a separable Banach space. X* is nonseparable if, and only if, given $0 < \varepsilon < \frac{1}{2}$ there is a continuous linear mapping* T : X → C(Δ) *with dense range such that* $T((1 + \varepsilon)B_X) + \varepsilon B_{C(\Delta)}$ *contains the characteristics functions* $\chi_{C_{ni}}$, $1 \le i \le 2^n$, $1 \le n < \infty$, *of the dyadic intervals of* Δ.

Proof. If X* is nonseparable then, following with the notation given in the preceding Stegall theorem, we have that the sequence $\{x_{ni}\}_{n=1}^{\infty}\,{}_{i=1}^{2^n}$ belongs to $(1 + \varepsilon)B_X$ and $\left|x_{ni}\left(x^*\right) - \chi_{C_{ni}}\left(x^*\right)\right| \le \varepsilon 2^{-n} < \varepsilon$ for every x* ∈ Δ, which means that if T is the natural evaluation map (T(x)(x*) = x*(x)) then $\chi_{C_{ni}} - T(x_{ni}) \in \varepsilon B_{C(\Delta)}$.

Conversely, let us suppose that there is a continuous linear mapping T : X → C(Δ) with dense range such that $T((1 + \varepsilon)B_X + \varepsilon B_{C(\Delta)})$ contains the characteristic functions $\chi_{C_{ni}}$, $1 \le i \le 2^n$, $1 \le n < \infty$, of the dyadic intervals of Δ.

As the range of T is dense we have that T* is one-to-one. As usual, we identify Δ with a weak* compact subset of the unit sphere of C(Δ)*. Then T* (Δ) is an uncountable weak* compact subset of X* and we are going to prove that it is norm discrete, implying the statement.

By hypothesis given $0 < \varepsilon < \frac{1}{2}$ and C_{ni} there is $x_{ni} \in (1 + \varepsilon)B_X$ such that

$$\left\| Tx_{ni} - \chi_{C_{ni}} \right\| \le \varepsilon$$

and, therefore, for each $\mu \in \Delta$ we have

$$\left| (Tx_{ni})(\mu) - \chi_{C_{ni}}(\mu) \right| \le \varepsilon \qquad (1)$$

Therefore, given two different points δ and δ' in Δ we may find C_{ni} such that $\delta \in C_{ni}$ and $\delta' \notin C_{ni}$. Then, replacing μ by δ and δ' in (1), we have:

$$\left| (Tx_{ni})(\delta) - 1 \right| \le \varepsilon$$

and

$$\left| (Tx_{ni})(\delta') - 0 \right| \le \varepsilon$$

From these two inequalities it follows:

$$\left| \langle x_{ni}, T^*\delta - T^*\delta' \rangle \right| = \left| \langle Tx_{ni},\ \delta - \delta' \rangle \right| = \left| (Tx_{ni})(\delta) - (Tx_{ni})(\delta') \right| \ge 1 - 2\varepsilon$$

and, from $\|x_{ni}\| \le 1 + \varepsilon$ we deduce that

$$\left\| T^*\delta - T^*\delta' \right\| \ge \frac{1 - 2\varepsilon}{1 + \varepsilon}$$

which shows that $T^*(\Delta)$ is norm discrete.

Which the same technique the following proposition may be proved:

Proposition 2. *Let X be a separable Banach space. X* is nonseparable if, and only if, there is a continuous linear mapping* $T : X \to C(\Delta)$ *with dense range, two positive numbers m and* δ *and a natural number* n_o *such that* $T(mB_X) + \delta B_{C(\Delta)}$ *contains the characteristic functions* $\chi_{C_{ni}}$, $1 \le i \le 2^n$, $n_0 \le n < \infty$, *of the dyadic intervals of* Δ *corresponding to the steps* $n_o + 1$, $n_o + 2, \ldots$ ..

REFERENCES

1. Bator, E.M. (1992) A basic construction in duals of separable Banach spaces. Rocky Mt J. Math. **22** (1), 81-92.
2. Hagler, J. (1973) Some more Banach spaces which contains l^1. Studia Math, **46**, 35-42.
3. Pelczynsky, A. (1968) On Banach spaces containing L_1. Studia Math, **30,** 231-246.
4. Stegall, C. (1973) Banach spaces whose duals contain $l^1(\Gamma)$ with applications to the study of dual $L^1(\mu)$ spaces. T.A. Math. Soc. **206,** 213-223.