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EXISTENCE OF MONOTONE SOLUTIONS FOR NONLINEAR PERTURBED DIFFERENTIAL INCLUSIONS

(monotone solution/admissible set valued map)

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ABSTRACT

Let X be a separable Banach space whose dual is uniformly convex, $A : D(A) \subset X \to 2^X$ an m-dissipative operator generating a compact semigroup $S(t): \overline{D(A)} \to \overline{D(A)}$, $t \geq 0$, D a locally closed set in $\overline{D(A)}$ and $F : D \to 2^X$ a nonempty, closed convex and bounded valued mapping which is strongly-weakly upper-semicontinuous and locally bounded on D. Let «$\preceq$» be a preorder on D, relation characterized by the set-valued map $P : D \to 2^D$, defined by $P(\xi) = \{\eta \in D;\ \xi \preceq \eta\}$ whose graph is closed in $D \times D$. Whe denote by $u(\cdot, 0, \xi, y + p)$ the unique mild solution of

$$u'(t) \in Au(t) + y + p,$$

satisfying $u(0, 0, \xi, y + p) = \xi$ and we prove:

Theorem. *Under the general assumptions above a sufficient condition in order that for each $\xi \in D$ there exists at least one mild solution u of*

$$u'(t) \in Au(t) + F(u(t))$$

satisfying $u(0) = \xi$ and $u(s) \preceq u(t)$ for each $s < t$ is the «bounded w-hypermonotonicity condition» below:

(Mw$\mathcal{HMC}$) There exists a locally bounded function $\mathcal{M}$: $D \to R_+^*$ satisfying: for each $\xi \in D$ there exists $y \in F(\xi)$ such that for each $\delta > 0$ each weak neighborhood V of 0 there exist $t \in (0, \delta]$ and $p \in V$ with $\|p\| \leq \mathcal{M}(\xi)$ and $u(t, 0, \xi, y + p) \in P(\xi)$.

1. INTRODUCTION

Let us consider the problem:

$$\frac{du}{dt}(t) \in Au(t) + F(u(t))\ t \geq 0 \qquad (\mathcal{DI})$$

with

$$u(0) = \xi \qquad (IC)$$

where $A : D(A) \subset X \to 2^X$ is the infinitesimal generator of a semigroup, D is a nonempty subset in $D(A)$ and $F : D \to 2^X$ a nonempty, closed, convex and bounded valued mapping. Let «$\preceq$» be a preorder on D defined by means of the set-valued mapping $P : D \to 2^D$, given by

$$P(\xi) = \{\eta \in D;\ \xi \preceq \eta\}$$

for each $\xi \in D$. Our purpose is to find a sufficient condition in order that for each $\xi \in D$ the Cauchy problem $(\mathcal{DI})$ and (IC) have monotone mild solutions, i.e. mild solutions satisfying

$$u(s) \preceq u(t) \text{ for each } t > s.$$

If the differential inclusion $(\mathcal{DI})$ has such a solution, we say that the set-valued map P is *admissible* with respect to $(\mathcal{DI})$.

The problem has been studied by many authors. We begin with a short review of the main contributions in this area and we will try to make a clasification according the general frame used. So, we will consider the case in which $A = 0$, the case in which A is the infinitesimal generator of a C_0-semigroup of contractions and we will also specify when X is finite or infinite dimensional.

We start by recalling the pioneering work of Aubin, Cellina and Nohel [2], where $A = 0$, the preorder enjoied an additional convexity property and X is a finite dimensional Banach space. This assumption has been discarted in the, by now classical, paper of Haddad [11], also for X finite dimensional. Aubin-Cellina [1] have considered the case when $A = 0$ and F is upper hemicontinuous (see [1], Definition 1, pag. 59) with compact values and P is lower-semicontinuous (see [1], Definition 6, pag. 45) with closed graph, this time in a Hilbert space. The necessary and sufficient condition for existence of monotone trajectories given there, described in our terms is:

For each $\xi \in D$ *there exist* $y \in F(\xi)$, *a sequence* $(t_n)_n$ *decreasing to* 0 *and a sequence* $(p_n)_n$ *strongly convergent to* 0 *satisfying*

$$\xi + t_n (y + p_n) \in P(\xi)$$

for each $n \in \mathbb{N}^*$. For the proof, see [1], Theorem 3, pag. 182.

The case in which A is the generator of a compact differentiable C_0-semigroup and P is lower-semicontinuous multivalued map with closed graph was analyzed by Shi Shuzhong [18]. Under this circumstances, the necessary and sufficiency for the admissibility of P is:

For each $\xi \in D$ *there exist* $y \in F(\xi)$, *a sequence* $(t_n)_n$ *decreasing to* 0, *a sequence* $(p_n)_n$ *strongly convergent to* 0 *satisfying*

$$S(t_n)\xi + t_n(y + p_n) \in P(\xi),$$

for each $n \in \mathbb{N}^*$.

We remember that the simplest necessary condition in order that P is admissible with respect to $(\mathcal{DI})$ is the viability of D with respect to $(\mathcal{DI})$. We recall that D is a *viable domain* with respect to $(\mathcal{DI})$ if for each $\xi \in D$ there exists at least one local mild solution of $(\mathcal{DI})$ and satisfying (IC).

Starting from the necessary and sufficient conditions in order that a given subset D of a Banach space X be a viable domain for a semilinear differential inclusion (A is the infinitesimal generator of a C_0-semigroup of contractions) obtained by Cârja and Vrabie [6] combined with an extension for the infinite dimensional Banach spaces of Proposition 9 from Cârja and Ursescu [5], enabled us in [8] to prove some necessary and sufficient conditions in order that for each $\xi \in D$ the semilinear differential inclusion $(\mathcal{DI})$ have monotone solutions. Our technique makes possible the renunciation to the lower semicontinuity of P. The main result given there is:

Theorem. *Let* X *be a reflexive and separable Banach space,* D *a nonempty, locally weakly closed set in* X *and* «$\preceq$» *a preorder on* D *characterized by the set-valued mapping* $P : D \to 2^D$, $P(\xi) = \{\eta \in D;\ \xi \preceq \eta\}$ *whose graph is weakly × weakly sequentially closed in* $D \times D$. *Let* $A : D(A) \subset X \to X$ *be the infinitesimal generator of a* C_0-*semigroup* $S(t) : X \to X$, $t \geq 0$ *and* $F : D \to 2^X$ *a nonempty, closed, convex and bounded valued mapping which is weakly-weakly upper-semicontinuous. Then a necessary and sufficient condition in order that P be admissible with respect to* $(\mathcal{DI})$ *is the «bounded w-monotonicity» condition:*

(BwℳC) There exists a locally bounded function $\mathcal{M} : D \to R_*^+$ *such that for each* $\xi \in D$ *there exists* $y \in F(\xi)$ *such that for each* $\delta > 0$ *and each weak neighborhood* V *of* 0, *there exist* $t \in (0, \delta)$ *and* $p \in V$ *with* $\|p\| \leq \mathcal{M}(\xi)$ *and satisfying*

$$S(t)\xi + t(y + p) \in P(\xi)$$

See Chis-Ster [8] for the proof.

We have also studied the case in which A is the generator of a compact C_0-semigroup of contractions and the next result holds:

Theorem. *Let* X *be a reflexive Banach space,* D *a nonempty, locally closed set in* X *and* «$\preceq$» *a preorder on* D *characterized by the set-valued mapping* $P : D \to 2^D$, $P(\xi) = \{\eta \in D;\ \xi \preceq \eta\}$ *whose graph is closed in* $D \times D$. *Let* $A : D(A) \subset X \to X$ *be the infinitesimal generator of a compact* C_0-*semigroup of contractions* $S(t) : X \to X$, $t \geq 0$ *and* $F : D \to 2^X$ *a nonempty, closed, convex valued mapping which is strongly-weakly upper-semicontinuous and locally bounded. Then a sufficient condition in order that* P *is admissible with respect to* $(\mathcal{DI})$ *is the monotonicity condition* $(\mathcal{M}C)$.

(ℳC) for each $\xi \in D$ *there exists* $y \in F(\xi)$ *such that for each* $\delta > 0$ *there exist* $t \in (0, \delta]$ *and* $p \in X$ *with* $\|p\| < \delta$ *satisfying*

$$S(t)\xi + t(y + p) \in P(\xi)$$

For more details, see Chis-Ster [9].

Concerning the nonlinear case, a very recent result of Cârja and Vrabie [7] regarding the viability of a nonempty set with respect to $(\mathcal{DI})$ inspired us a sufficient condition for the admissibility of P. The technique used in the proof is the same as in Chis-Ster [8], [9], the Proposition 9 from Cârja and Ursescu remaining valid even in the case in which A is nonlinear, but with the additional assumption of uniformly convexity of the dual of X. The result proved by Cârja and Vrabie, on which is based our work, is:

Theorem 1.1. *Let* X *be a separable Banach space whose dual is uniformly convex,* $A : D(A) \subset X \to 2^X$ *an m-dissipative operator, generator of a compact semigroup* $S(t) : \overline{D(A)} \to \overline{D(A)}$, $t \geq 0$ D *a locally closed subset in* $\overline{D(A)}$ *and* $F : D \to 2^X$ *a nonempty, closed, convex and bounded valued mapping which is strongly-weakly upper semicontinuous and locally bounded. Under the general assumptions above a sufficient condition in order for each* $\xi \in D$ *there exists at least one mild solution* u *of* $(\mathcal{DI})$ *and satisfying* (IC) *is the «bounded hypertangency condition» below.*

(BwℋTC) There exists a locally bounded function $\mathcal{M}$: $D \to R_*^+$ *such that for each* $\xi \in D$ *there exists* $y \in F(\xi)$

such that for each $\delta > 0$ *and each weak neighborhood* V *of* 0, *there exist* $t \in (0, \delta]$ *and* $p \in V$ *with* $\|p\| \leq \mathcal{M}(\xi)$ *and satisfying*

$$u(t, 0, \xi, y + p) \in D.$$

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2. PRELIMINARIES

We assume familiarity with basic concepts and results concerning multivalued mappings and nonlinear evolution equations driven by *m*-dissipative operators and we refer to Aubin-Cellina [1], Barbu [3], Deimling [10], Vrabie [19] for more details.

We consider the differential inclusion

$$\frac{du}{dt}(t) \in Au(t) + F(u(t)) \; t \geq 0 \qquad (\mathcal{DI})$$

and the initial condition

$$u(0) = \xi \in D \qquad (IC)$$

where $A : \overline{D(A)} \subset X \to 2^X$ is an m-dissipative operator and D is a nonempty set in $\overline{D(A)}$ on which we have defined the preorder «$\preceq$» characterized by the set valued map:

$$P : D \to 2^D, \; P(\xi) = \{\eta \in D; \; \xi \preceq \eta\}.$$

Remark 2.1. The set valued map $P : D \to 2^D$ satisfies the conditions:

$$\xi \in P(\xi) \text{ for each } \xi \in D$$

and

$$P(\eta) \subset P(\xi) \text{ for each } \xi \in D \text{ and for each } \eta \in P(\xi).$$

Definition 2.1. The function $u : [0, T] \to D$ is a *mild solution* for $(\mathcal{DI})$ and (IC) if there exists $f \in L^1([0, T]; X)$, with $f(t) \in F(u(t))$ a.e. for $t \in [0, T]$ and such that u is a mild solution for the differential inclusion

$$\frac{du}{dt}(t) \in Au(t) + f(t) \; t \geq 0 \qquad (2.1)$$

In all that follows we denote by $u(\cdot, 0, \xi, f)$ the unique mild solution of (2.1) satisfying $u(0, 0, \xi, f) = \xi$ and by $S(t) : \overline{D(A)} \to \overline{D(A)}$, $t \geq 0$ the semigroup of nonexpansive mappings generated by A, i. e. $S(t)\xi = u(t, 0, \xi, 0)$ for each $t \geq 0$ and $\xi \in \overline{D(A)}$.

Definition 2.2. The semigroup $\overline{D(A)} \to \overline{D(A)}$, $t \geq 0$ is compact if for each $t > 0$ $S(t)$ is a compact operator.

Definition 2.3. Let X be a Banach space. A subset K in $L^1(0, T; X)$ is called *uniformly integrable* if given $\varepsilon > 0$ there exists $\delta(\varepsilon)$ such that

$$\int_E \|f(t)\| \, dt \leq \varepsilon$$

for each measurable subset E in $[0, T]$ whose Lebesgue measure is less than $\delta(\varepsilon)$, uniformly for $f \in K$.

Remark 2.2. If a subset K in $L^1(0, T; X)$ is bounded in $L^p(0, T; X)$ for some $p > 1$, then it is uniformly integrable.

Let K a subset in $L^1(0, T; X)$ and let us denote by $M(K)$ the set of all mild solutions of the problem (2.1) and (IC) with $f \in K$. In what follows, we will use the next consequence of a fundamental compactness result in $C([0, T]; X)$ of Baras (See Vrabie [19], Theorem 2.3.3, pag. 47.)

Proposition 2.1. *Let* X *a real Banach space whose dual is uniformly convex. Let* $A : D(A) \subset X \to 2^X$ *be an m-dissipative operator, the generator of a compact semigroup and let* $\xi \in \overline{D(A)}$ *a fixed element. Then the solution mapping* $M : L^1(0, T; X) \to C([0, T]; X)$ *defined by* $M(f) := u$ *for each* $f \in L^1(0, T; X)$, *where* u *is the unique mild solution of* (2.1), *is sequentially continuous from* $L^1(0, T; X)$ *endowed with its weak topology into* $C([0, T]; X)$ *endowed with its strong topology.*

See Vrabie [19], Corollary 2.3.1, pag. 49.

Definition 2.4. The function $u : [0, T] \to X$ is a *monotone solution* for $(\mathcal{DI})$ and (IC) with respect to the preorder «$\preceq$» if it is a mild solution in the sense of Definition 2.1 and in addition satisfies:

$$u(s) \preceq u(t)$$

for each $s, t \in [0, T]$ with $s \leq t$ or equivalently,

$$u(t) \in P(u(s))$$

for each $s \in [0, T]$ and each $t \in [s, T]$.

Definition 2.5. The nonempty set $D \subset X$ is a *viable domain* for $(\mathcal{DI})$ if for every $\xi \in D$ there exists at least one local mild solution $u : [0, T] \to D$ the sense of Definition 2.1.

Definition 2.6. $P : D \to 2^D$ is *admissible* with respect to $(\mathcal{DI})$ if for every $\xi \in D$ there exists at least one monotone solution $u : [0, T] \to X$ in the sense of Definition 2.4.

Definition 2.7. Let X be a Banach space, w the weak topology on X, $A : D(A) \subset X \to 2^X$ be an m-dissipative operator. Let D a nonempty set in X.

We say that $y \in X$ is *w-A hypertangent* to D at $\xi \in D$ if for each $\delta > 0$ and each weak neighborhood V of 0 there exist $t \in (0, \delta)$ and $p \in V$ such that

$$u(t, 0, \xi, y + p) \in D.$$

The set of all w-A hypertangent elements to D at $\xi \in D$ is denoted by $(w)\mathcal{HT}_D^A(\xi)$,

Next, let us define $\gamma : D \times X \to \overline{R}_+$ by

$$\gamma(\xi, y) = \begin{cases} +\infty & \text{if } y \in X \setminus (w)\mathcal{HT}_D^A(\xi) \\ \liminf\limits_{\substack{(t,p)\to(0,0) \\ u(t,0,\xi,y+p)\in D}} \|p\| & \text{if } y \in (w)\mathcal{HT}_D^A(\xi), \end{cases}$$

where the convergence $p \to 0$ is considered in the weak topology.

Now, let us define $\Gamma : D \to \overline{R}_+$ by

$$\Gamma(\xi) = \inf_{y\in F(\xi)} \gamma(\xi, y).$$

Remark 2.3. We note that $\gamma(\xi, y)$ is finite whenever $y \in (w)\mathcal{HT}_D^A(\xi)$.

Definition 2.8. The nonempty set D satisfies:

(i) *w-hypertangency condition* ($w\mathcal{HTC}$) with respect to ($\mathcal{DI}$) if

$$F(\xi) \cap (w)\mathcal{HT}_D^A(\xi) \neq \emptyset$$

for each $\xi \in D$.

(ii) *bounded w-hypertangency condition* ($Bw\mathcal{HTC}$) with respect to ($\mathcal{DI}$) if it satisfies the ($w\mathcal{HTC}$) with respect to ($\mathcal{DI}$) and the function Γ, defined as above, is locally bounded on D.

Definition 2.9. The set-valued mapping $P : D \to 2^D$ satisfies:

(i) *w-hypermonotonicity condition* ($w\mathcal{HMC}$) with respect to ($\mathcal{DI}$) if

$$F(\xi) \cap (w)\mathcal{HT}_{P(\xi)}^A(\xi) \neq \emptyset$$

for each $\xi \in D$.

(ii) *bounded w-hypermonotonicity condition* ($Bw\mathcal{HMC}$) with respect to ($\mathcal{DI}$) if it satisfies the ($w\mathcal{HMC}$) with respect to ($\mathcal{DI}$) and the function Γ, defined as above, is locally bounded on D.

Proposition 2.2. *A set-valued mapping $P : D \to 2^D$ satisfies the bounded w-hypermonotonicity condition ($Bw\mathcal{HMC}$) with respect to ($\mathcal{DI}$) if and only if it satisfies*

($Mw\mathcal{HMC}$) There exists a locally bounded function $\mathcal{M} : D \to R_+^*$ such that for each $\xi \in D$, there exists $y \in F(\xi)$ with the property that for each $\delta > 0$ and each weak neighborhood V of 0 there exist $t \in (0, \delta)$ and $p \in V$, $\|p\| \leq \mathcal{M}(\xi)$ and satisfying

$$u(t, 0, \xi, y + p) \in P(\xi).$$

The proof follows exactly the same lines as that of Cârja and Vrabie [7], Proposition 2.1.

Definition 2.10. The set valued map $F : D \to 2^X$ with nonempty values is *strongly-weakly (weakly-weakly) upper-semicontinuous* if for every $u \in D$ and each neighborhood V of $F(u)$ in the weak topology, there exists a neighborhood W of u in the strong (weak) topology such that $F(v) \subset V$ for every $v \in W$.

3. THE MAIN RESULT

We may now proceed to the statements of our main result.

Theorem 3.1. *Let X be a separable Banach space whose dual is uniformly convex, $A : D(A) \subset X \to 2^X$ an m-dissipative operator, generator of a nonlinear compact semigroup of contractions $S(t) : \overline{D(A)} \to \overline{D(A)}$, $t \geq 0$, D a nonempty locally closed subset in $\overline{D(A)}$, $F : D \to 2^X$ a nonempty, closed, convex and bounded valued mapping, which is strongly-weakly upper semicontinuous on D and locally bounded. Let «$\preceq$» a preorder on D, relation characterized by the set-valued mapping $P : D \to 2^D$, $P(\xi) = \{\eta \in D;\ \xi \preceq \eta\}$, whose graph is strongly-strongly closed in $D \times D$. Then a sufficient condition in order that P is admissible with respect to ($\mathcal{DI}$) is ($Mw\mathcal{HMC}$).*

Now we present a proposition which illustrated our tehnique what we have been talking about in the previous section.

Proposition 3.1. *Let X be a separable Banach space whose dual is uniformly convex, $A : D(A) \subset X \to 2^X$ an m-dissipative operator, generator of a nonlinear compact semigroup of contractions $S(t) : \overline{D(A)} \to \overline{D(A)}$, $t \geq 0$, D a nonempty locally closed subset in $\overline{D(A)}$, $F : D \to 2^X$ a nonempty, closed, convex and bounded valued mapping, which is strongly-weakly upper semicontinuous and locally bounded on D. Let «$\preceq$» a preorder on D, relation characterized by the set-valued mapping $P : D \to 2^D$, $P(\xi) = \{\eta \in D;\ \xi \preceq \eta\}$, whose graph is strongly-strongly in $D \times D$.*

Then P is admissible with respect to ($\mathcal{DI}$) if and only if for every $\xi \in D$, $P(\xi)$ is a viable domain for ($\mathcal{DI}$).

Remark 3.1. The necessity is obvious. For the sufficiency, we need some facts about noncontinuable mild solutions of ($\mathcal{DI}$) and (IC). Since, by hypotheses, for each $\xi \in D$ there exists at least one local mild solution of ($\mathcal{DI}$) and (IC), reasoning as in Vrabie [19], Theorem 3.2.1, p. 92, we may prove that for each $\xi \in D$ there exists at least one mild noncontinuable solution $u_\xi : [0, T(u_\xi)) \to D$. Let us denote the set of all noncontinuable solutions of ($\mathcal{DI}$) and (IC) by $S_{nc}(\xi)$. The proof of the next lemma, with no alterations for the nonlinear case, may be found in Cârja and Vrabie. See [6], Lemma 3.1.

Lemma 3.1. *Let X be a reflexive Banach space, $A : D(A) \subset X \to 2^X$ the generator of a nonlinear semigroup of contractions $S(t): \overline{D(A)} \to \overline{D(A)}$, $t \geq 0$, D a nonempty locally closed set in $\overline{D(A)}$, $F : D \to 2^X$ a nonempty set-valued map which is locally bounded. If D is a viable domain for ($\mathcal{DI}$) then for each $\xi \in D$ there exist T_ξ, $r > 0$ and $M > 0$ such that, for each $u_\xi \in S_{nc}(\xi)$, $T_\xi \leq T(u_\xi)$,*

$$u_\xi(t) \in B(0, r)$$

for each $t \in [0, T_\xi]$ and

$$\|y\| \leq M$$

for each $t \in [0, T_\xi]$ and each $y \in F(u_\xi(t))$.

In order to prove the sufficiency of Proposition 3.1 we need a technical lemma which is interesting by itself.

Lemma 3.2. *Let X be a Banach space, $A : D(A) \subset X \to 2^X$ an m-dissipative operator D a nonempty, locally closed set in $\overline{D(A)}$ and «$\preceq$» a preorder on D characterized by the set-valued mapping $P : D \to 2^D$, $P(\xi) = \{\eta \in D;\ \xi \preceq \eta\}$. Let $F : D \to 2^X$ a nonempty valued mapping which is locally bounded. Assume that, for each $\xi \in D$, $P(\xi)$ is a viable domain with respect to ($\mathcal{DI}$). Then, for each $\xi \in D$ there exists $T > 0$ such that, for each net Δ : $0 = t_1 < t_2 < \ldots < t_n = T$, there exists at least one mild solution $u : [0, T] \to D$ of ($\mathcal{DI}$) and (IC) which satisfies:*

$$u([s, T]) \subset P(u(s))$$

for each $s \in \Delta$.

See Chis-Ster [8], Lemma 3.2.

Proof of the sufficiency of Proposition 3.1. Let us consider $\xi \in D$ and $T = T_\xi > 0$ as given by Lemma 3.2 and the sequence $(\Delta_n)_{n\in I\!N^*}$ of nets of [0, T] defined by

$$\Delta_n : 0 = t_1 < t_2 < t_3 < \ldots < t_n = T,\ t_i = \frac{iT}{2^n},\ i \in \{0, \ldots, 2^n\}.$$

Clearly $0 \in \Delta_n$, $\Delta_n \subseteq \Delta_{n+1}$ for each $n \in I\!N^*$ and $\Delta = \cup_{n\in I\!N^*}\Delta_n$ is dense in [0, T]. From Lemma 3.2 it follows that there exists a sequence of mild solutions $u_n : [0, T] \to X$ of ($\mathcal{DI}$) and (IC) satisfying

$$u_n([s, T]) \subseteq P(u_n(s))$$

for each $n \in I\!N^*$ and for each $s \in \Delta_n$.

We will show next that the set $\{u_n;\ n \in I\!N^*\}$ is relatively sequentially compact in $C([0, T]; X)$. Since u_n is a mild solution of ($\mathcal{DI}$) and (IC), there exists $f_n \in L^1(0, T; X)$, $f_n(t) \in F(u_n(t))$ a.e. for $t \in [0, T]$ such that

$$\frac{du_n}{dt}(t) \in Au_n(t) + f_n(u_n(t))\ t \geq 0 \tag{3.1}$$

and

$$u_n(0) = \xi. \tag{3.2}$$

Let us denote by

$$K = \{f_n \in L^1(0, T; X);\ f_n(t) \in F(u_n(t)) \text{ a.e. } t \in [0, T];$$
$$u_n : [0, T] \to X;\ u_n \in M(K)\}$$

where

$$M(K) = \{u_n;\ u_n : [0, T] \to X \text{ is a mild noncontinuable}$$
$$\text{solution for (3.1) and (3.2), } n \in I\!N^*\},$$

Choosing T, r and M given by Lemma 3.1, it follows that the hypotheses of Proposition 2.1 are satisfied and therefore we may assume without loss of generality that.

$$u_n \to u$$

in $C([0, T]; X)$. Since $\|f_n(t)\| \leq M$ for every $t \in [0, T]$ and X is reflexive it follows that K is weakly relatively compact in $L^p(0, T; X)$ for each $p \geq 1$. Recalling Proposition 2.1, we have that $u : [0, T] \to X$ defined before is the mild solution (2.1) and (IC) with $f \in L^1(0, T; X)$ defined up there. Using the same argument as in Vrabie ([19], Theorem 3.1.2, pag. 88), we have that $f(t) \in F(u(t))$ a.e. $t \in [0, T]$. This means that u is a mild solution of ($\mathcal{DI}$) and (IC).

It remains to show that

$$u([s, T]) \subset P(u(s))$$

for each $s \in [0, T]$. For this, see Chis-Ster [9], Proposition 3.1.

Proof of the Theorem 3.1. By the virtue of Proposition 3.1, it is sufficient to show that $P(\xi)$ is a viable domain for ($\mathcal{DI}$), for each $\xi \in D$. From the hypotheses, since $\mathcal{M} : D \to R_+^*$ is locally bounded, it follows that, for each $\xi \in D$, $\mathcal{M}$

: $P(\xi) \to R_+^*$ is locally bounded too. Let $\xi \in D$ be arbitrary and let $\eta \in P(\xi) \subset D$. From the $(Bw\mathcal{HMC})$ combined with Remark 2.1, there exists $y \in F(\eta)$ such that for each $\delta > 0$ and each weak neighborhood V of 0, there exist $t \in (0, \delta]$ and $p \in V$ with $\|p\| \leq \mathcal{M}(\eta)$ and satisfying

$$u(t, 0, \eta, y + p) \in P(\eta) \subset P(\xi).$$

Now, we have obtained the condition $(Bw\mathcal{HTC})$ from Theorem 1.1 for $P(\xi)$. It follows that $P(\xi)$ is a viable domain for $(\mathcal{DI})$, for each $\xi \in D$. An appeal to Proposition 3.1 shows that P is admissible with respect to $(\mathcal{DI})$ and this completes the proof of the sufficiency.

Remark 3.2. If we consider the strong topology on X, denoted by s, we can define in the same way $(s)\mathcal{HT}_D^A(\xi)$ and it's easy to see that $(s)\mathcal{HT}_D^A(\xi) \subset (w)\mathcal{HT}_D^A(\xi)$ and $(s\mathcal{HMC})$ implies $(Bw\mathcal{HMC})$.

Now, we have the a consequence of Theorem 3.1:

Corollary 3.1. *Let X be a separable Banach space whose dual is uniformly convex, $A : D(A) \subset X \to 2^X$ an m-dissipative operator, generator of a nonlinear compact semigroup of contractions $S(t): \overline{D(A)} \to \overline{D(A)}$, $t \geq 0$ D a nonempty locally closed subset in $\overline{D(A)}$, $F : D \to 2^X$ a nonempty, closed, convex and bounded valued mapping, which is strongly-weakly upper semicontinuous on D and locally bounded. Let «$\preceq$» a preorder on D, relation characterized by the set-valued mapping $P : D \to 2^D$, $P(\xi) = \{\eta \in D;\ \xi \preceq \eta\}$, whose graph is closed in $D \times D$. Then a sufficient condition in order that P is admissible with respect to $(\mathcal{DI})$ is $(s\mathcal{HMC})$.*

Concerning the finite dimensional case, we have:

Corollary 3.2. *Let X be a finite dimensional Banach space whose dual is uniformly convex, $A : D(A) \subset X \to 2^X$ an m-dissipative operator, generator of a nonlinear semigroup of contractions $S(t) : X \to X$, $t \geq 0$, D a nonempty locally compact subset in X, $F : D \to 2^X$ a nonempty, compact and convex valued mapping, which is upper semicontinuous on D. Let «$\preceq$» a preorder on D, relation characterized by the set-valued mapping $P : D \to 2^D$, $P(\xi) = \{\eta \in D;\ \xi \preceq \eta\}$, whose graph is closed in D _ D. Then a sufficient condition in order that P is admissible with respect to $(\mathcal{DI})$ is $(s\mathcal{HMC})$.*

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