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SEMIDIRECT PRODUCTS OF LOCALLY CONVEX ALGEBRAS AND THE THREE-SPACE-PROBLEM

(locally convex algebras/locally m-convex algebras/semi-direct product/three-space-problem))

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ABSTRACT

Motivated by the notion of a topological group, which is the semidirect product of a normal subgroup and a subgroup (see[4]), which turned out to be a rich source for examples and counterexamples, we introduce an analogous notion of a locally convex algebra, which is the semidirect product of an ideal and a subalgebra. (Cf. a more specialized notion for operator algebras in [3; Ch. 13]). We present a general method of constructing such semidirect products, which contains the direct products and the adjunction of a unit element as special cases. As an application we obtain an example of an algebra. *A provided* with a Banach space topology $\mathcal{T}$ and containing an ideal C such that both $(C, \mathcal{T} \cap C)$ and $(A/_C, \mathcal{T}/_C)$ are Banach algebras but $(A, \mathcal{T})$ is not.

NOTATIONS

By an algebra we will always mean an associative algebra over the field $IK \in \{IR, \mathbb{C}\}$. A character on an algebra A is by definition a linear multiplicative functional different from zero, and $\sigma(A)$ denotes the set of all characters on A. Given $x \in A$, $\sigma_A(x)$ denotes the spectrum of x with respect to A. If A contains a unit element e, then $G(A)$ denotes the set of invertible elements in A. An element $x \in A$ is called quasiinvertible in A, if there is (a so-called quasiinverse element) $y \in$ A satisfying $xy = yx = x + y$, and the quasiinverse element y is uniquely determined. An algebra provided with a locally convex topology such that multiplication is jointly continuous, is called a locally convex algebra; and a locally convex algebra is called locally m-convex if its 0-nbhd-filter has a basis consisting of sets U satisfying $U^2 \subset U$.

Definition 1. *Let A be an algebra containing an ideal C and a subalgebra B such that $A = C + B$ and $C \cap B = \{0\}$. Then we call A the semidirect product of C and B and use the notation* $A = C \otimes_S B$.

Remark. Let $A = C \otimes_S B$. Then clearly the quotient algebra $A/_C$ is canonically isomorphic to B. Let $q : A \to B$ denote the corresponding linear multiplicative surjection with $ker q = C$. For the characters on A we have the representation.

$$\sigma(A) = \sigma(C) \cup (\sigma(B) \circ q).$$

More precisely, if $\psi \in \sigma(C)$, there is a unique extension of ψ to a character on A (see [2; theorem C.1]) and if $\varphi \in \sigma(B)$ then $\varphi \circ q \in \sigma$(A). Conversely, let $\mathcal{X} \in \sigma(A)$. If $\mathcal{X}|_C \neq 0$, then $\mathcal{X}|_C \in \sigma(C)$ and $\mathcal{X}$ is its unique extension to a character on A; if $\mathcal{X}|_C = 0$, there is $\psi \in \sigma(B)$ satisfying $\mathcal{X} = \psi \circ q$.

Also the quasiinvertibility of elements c + b $\in A$, where $b \in B$, $c \in C$, can be characterized. In fact, formal adjunction of a unit element to A yields $A_e = (C \times \{0\}) \otimes_S B_e$ which we may abbreviate to $A_e = C \otimes_S B_e$. Let $c \in C$ and $b \in B$ be given. Then $c + b$ is quassinvertible in A if and only if b is quassinvertible in B and $c(e - b)^{-1}$ is quasiinvertible in C. In fact, if $e - (c + b) = -c + (e - b) \in G(A_e)$, then $e - b \in G(B_e) \subset G(A_e)$, hence $-c(e - b)^{-1} + e \in G(A_e)$ and $c(e - b)^{-1}$ is quasiinvertible in A_e and hence in C, as C is an ideal in A_e. Conversely, if $e - b \in G(B_e) \subset G(A_e)$ and $e - c(e - b)^{-1} \in G(A_e)$, then $e - (b + c) \in G(A_e)$. Thus $b + c$ is quasiinvertible in A_e and hence also in A.

Now, consequently, a complex number $\lambda \in IK \setminus \{0\}$ belongs to $\sigma_A(c + b)$ if and only if either $\lambda \in \sigma_B(b)$ or $\frac{1}{\lambda}b$ is quasiinvertible in B with quasiinverse element $d \in$ B and $\lambda \in \sigma_C(c - cd)$.

Definition 2. *Let $(A, \mathcal{T})$ be a locally convex algebra and let $C \subset A$ be an ideal, $B \subset A$ a subalgebra such that $A = C \otimes_S B$. Then we call $(A, \mathcal{T})$ the topological semidirect product of C and B, if the canonical linear bijection*

$$(C, \mathcal{T} \cap C) \times (B, \mathcal{T} \cap B) \to (A, \mathcal{T}), (c, b) \to c + b$$

is a homeomorphism. In that case the quotient algebra $(A/_C, \mathcal{T}/_C)$ is canonically topologically isomorphic to the algebra $(B, \mathcal{T} \cap B)$.

Proposition. *Let $(A, \mathcal{T})$ be a locally convex algebra, which is the topological semidirect product of an ideal C and a subalgebra B such that both $(C, \mathcal{T} \cap C)$ and $(B, \mathcal{T} \cap B)$ are locally m-convex. Then $(A, \mathcal{T})$ is also locally m-convex.*

Proof. Let U be an absolutely convex 0-nbhd in $(A, \mathcal{T})$; we may assume that $(U \cap C)^2 \subset U \cap C$. As $(A, \mathcal{T})$ is a locally convex algebra, there are absolutely convex 0-nbhds V and W in $(C, \mathcal{T} \cap C)$ and $(B, \mathcal{T} \cap B)$, respectively; such that

$$(\Gamma(V \cup W))^2 \cup (\Gamma(V \cup W))^3 \subset U, \quad V^2 \subset V \subset U, \quad W^2 \subset W \subset U,$$

(Γ denoting the absolutely convex hull). We will show that $(\Gamma(V \cup W))^k \subset U$ for all $k \in I\!N$, which proves that $\tilde{U} := \Gamma\left(\cup_{k \in I\!N} (V \cup W)^k\right)$ is a 0-nbhd in $(A, \mathcal{T})$ contained in U which satisfies $\tilde{U}^2 \subset \tilde{U}$. For the proof we first observe that $(\Gamma X)(\Gamma Y) \subset \Gamma(XY)$ for arbitrary subsets X, Y in A. Thus we must only prove that $(V \cup W)^k \subset U$ for all $k \in I\!N$. Now $(V \cup W)^k$ is the union of $V^k (\subset V \subset U)$, $W^k (\subset W \subset U)$ and of finite products of sets of the form VW, WVW, WV, VWV. These latter sets are all contained in $U \cap C$ and $(U \cap C)^l \subset U$ for all $l \in I\!N$.

Our next aim is to present a method to construct algebras which are semidirect products.

Proposition. *Let C and B be algebras and assume that there is a linear multiplicative map*

$$l : B \to L(C) := \{f : C \to C \text{ linear}\}, b \to l_b$$

and a linear antimultiplicative map

$$r : B \to L(C), b \to r_b$$

such that $r_b \circ l_{\tilde{b}} = l_{\tilde{b}} \circ r_b$ for all $b, \tilde{b} \in B$ and such that $l_b(ac) = l_b(a)c$, $r_b(ac) = ar_b(c)$, $al_b(c) = r_b(a)c$ for all $b \in B$, $a, c \in C$. Then the multiplication.

$$(C \times B) \times (C \times B) \to (C \times B), \ ((c_1, b_1), (c_2, b_2)) \to (c_1 c_2 + l_{b_1}(c_2) + r_{b_2}(c_1), \ b_1 b_2)$$

makes $A := C \times B$ (provided with componentwise addition and scalar multiplication) an associative algebra, such that A is the semidirect product of the ideal $C \times \{0\}$ and the subalgebra $\{0\} \times B$. We will use the notation $A = C \times_S B$. Moreover, let $\mathcal{T}$ and $\mathcal{S}$ be locally convex topologies on C and B, respectively, such that $(C, \mathcal{T})$ and $(B, \mathcal{S})$ are locally convex algebras. Then A provided with the product topology $\mathcal{T} \times \mathcal{S}$ is a locally convex algebra if and only if the two bilinear maps

$$\varphi : (B, \mathcal{S}) \times (C, \mathcal{T}) \to (C, \mathcal{T}), (b, c) \to l_b(c)$$

and

$$\psi : (C, \mathcal{T}) \times (B, \mathcal{S}) \to (C, \mathcal{T}), (c, b) \to r_b(c)$$

are continuous.

Proof. One directly computes that the above multiplication is associative and distributive. For the last assertion, $(A, \mathcal{T} \times \mathcal{S})$ is a locally convex alegra if and only if for all 0-nbhds U in $(C, \mathcal{T})$ and V in $(B, \mathcal{S})$ there are 0-nbhds $\tilde{U}$ in $(C, \mathcal{T})$ and $\tilde{V}$ in $(B, \mathcal{S})$ such that $\tilde{U}^2 + \varphi(\tilde{V} \times \tilde{U}) + \psi(\tilde{U} \times \tilde{V}) \subset U$ and $\tilde{V}^2 \subset V$, wich is obviously equivalent to the continuity of φ and ψ at (0, 0).

Examples.

1. Let C be an algebra, $B := \mathbb{C}$ and for all $\lambda \in \mathbb{C}$ let $l_\lambda := r_\lambda : C \to C, c \to \lambda c$. Then $C \times_S B$ coincides with C_e (formal adjunction of a unit element). If $(C, \mathcal{T})$ is a locally convex algebra, then $(C \times_S B, \mathcal{T} \times \mathcal{T}_{|\cdot|})$ and $(C, \mathcal{T})_e$ coincide.

2. Let B, C be locally convex algebras and for all $b \in B$ let $l_b := r_b :=$ 0-map. Then $C \times_S B$ is equal to the direct product $C \times B$ with componentwise multiplication.

3. Let D be an algebra and le $B, C \subset D$ be subalgebras such that $BC \cup CB \subset C$. For each $b \in B$ let $l_b : c \to bc$ and $r_b : c \to cb$. Then l, r satisfy the requirements of the proposition, hecen $A = C \times_S B$ is a well-defined algebra. Thus a semidirect product $A = C \times_S B$ need not be commutative, even if C and B are commutative. Take e.g.

$$B := \left\{ \begin{pmatrix} a & \circ \\ \circ & b \end{pmatrix} : a, b \in \mathbb{C} \right\}, C := \left\{ \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} : c \in \mathbb{C} \right\} \subset M_2(\mathbb{C}).$$

4. Let C be an algebra with unit element e, and let $\mathcal{T}$, $\mathcal{S}$ be locally convex topologies on C such that $(C, \mathcal{T})$ and $(C, \mathcal{S})$ are locally convex algebras. Form $A := C \times_S C$ according to 3) by putting $B := C =: D$. Then the following are equivalent.

i) $(A, \mathcal{T} \times \mathcal{S})$ is a locally convex algebra.

ii) Multiplication in $(A, \mathcal{T} \times \mathcal{S})$ is separately continuous.

iii) $\mathcal{S} \supset \mathcal{T}$.

If in addition $(C, \mathcal{T})$, $(C, \mathcal{S})$ are locally m-convex, then i) - iii) are equivalent to

iv) $(A, \mathcal{T} \times \mathcal{S})$ is a locally m-convex algebra.

Proof. ii) $\to$ iii): As left multiplication with $(e, 0)$ on $(A, \mathcal{T} \times \mathcal{S})$ is continuous and as $(e, 0) \cdot (0, a) = (a, 0)$ for all $a \in C$, the identity map $(C, \mathcal{S}) \to (C, \mathcal{T})$ is continuous.

iii) $\to$ i): is true by the last assertion in the proposition, because multiplication as a map $(C, \mathcal{T}) \times (C, \mathcal{T}) \to (C, \mathcal{T})$

is continuous, thus also continuous on $(C, \mathcal{T}) \times (C, \mathcal{S})$ and on $(C, \mathcal{S}) \times (C, \mathcal{T})$.

The last part follows from the above proposition.

5. Let C be an algebra and let $\mathcal{T}$, $\mathcal{S}$ be Hausdorff locally convex topologies on C, such that $\mathcal{S} \underset{\neq}{\supset} \mathcal{T}$. Then $(A, \mathcal{R}) := (C \times_S C, \mathcal{T} \times \mathcal{S})$ is a locally convex algebra. As $\Delta := \{(-c, c) : c \in C\}$ is an ideal in A, we obtain that $(A, \mathcal{R})$ is the semidirect product of Δ and $\{0\} \times C$ which are both closed, but not the topological semidirect product of Δ and $\{0\} \times C$. On the other hand $(A, \mathcal{R})$ is topologically isomorphic as an algebra to the direct topological product of $(\Delta, \mathcal{R} \cap \Delta)$ and $(C \times \{0\}, \mathcal{R} \cap (C \times \{0\}))$.

Theorem. *There exists an algebra A provided with a Banach space topology $\mathcal{R}$, which contains an ideal C such that both $(C, \mathcal{R} \cap C)$ and $(A/_C, \mathcal{R}/_C)$ are Banach algebras and such that on $(A, \mathcal{R})$ multiplication is not continuous.*

Proof. Let X be a linear space of dimension $2^{\mathbb{N}}$, then by [1, 5, exercise 24], X is linearly isomorphic to l^1 and to l^2. Consequently, there exist two different Banach space topologies $\mathcal{T}$ and $\mathcal{S}$ on X. Providing X with zero-multiplication we obtain the algebra X_{nil}; formal adjunction of a unit element yields the algebra $C := (X_{nil})_e$, which carries the two different Banach algebra topologies $\tilde{\mathcal{S}}$ and $\tilde{\mathcal{T}}$ generated by S and T, respectively.

Now, by example 4, $A := C \times_S C$ provided with $\tilde{\mathcal{T}} \times \tilde{\mathcal{S}}$ satisfies the requirement, as $\mathcal{S} \not\supset \mathcal{T}$ and $(A, \tilde{\mathcal{T}} \times \tilde{\mathcal{S}})/(C \times \{0\})$ is isomorphic to the Banach algebra $(C, \tilde{\mathcal{S}})$.

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