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PERIODIC AND ALMOST-PERIODIC SOLUTIONS OF SEMILINEAR EQUATIONS IN BANACH SPACES

(semigroup/periodic or almost-periodic/semilinear equations)

M. BAHAJ AND E. HANEBALY

Université Mohammed V, Department de Mathemátiques, Faculté des Sciences, B.P. 1014 - Rabat-Marroc

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ABSTRACT

In this paper existence of periodic and almost-periodic mild solutions of a class of semilinear equation is established under suitable assumptions.

1. INTRODUCTION

Let X be a real Banach space, with norm $|\cdot|$, and we consider the following non linear ordinary differential equation:

$$x'(t) = Ax(t) + f(t, x(t)) \tag{1}$$

where $f(t, x) : \mathbb{R} \times X \to X$ is almost-periodic in t, uniformly for x in compact subsets of X, and A is the infinitesimal generator of a C_0-semigroup $S(t)$ satisfying $|S(t)|_{\mathcal{L}(X)} \le Me^{\beta t}$, where β is a negative number.

In this paper we are mainly interested in finding almost-periodic mild solution over $\mathbb{R}$ of (1). Existence and uniqueness of an almost-periodic mild solution of the inhomogeneous equation:

$$x'(t) = Ax(t) + g(t) \tag{2}$$

where g is almost-periodic function from $\mathbb{R}$ into X were proved in [8].

In the case when f is uniformly Lipschitz continuous with a Lipschitz constant small enough, existence and uniqueness of an almost-periodic mild solution over $\mathbb{R}$ of (1) were proved by Zaidman in [9].

Our work extends Zaidman's results of [8, 9].

Our objective in section 2 is to establish sufficient condicions on $f(t, x)$ that ensure the existence of the almost-periodic mild solution (1).

Our main theorem here generalize Zaidman's results obtained in [9], the proof presented applies the Krasnoselskii fixed point.

In section 3, we continue our study of the semilinear equation (1).

If the Lipschitz continuity of f in x or the compactness in x are dropped, then as is well known, the existence of a mild solution of (1) is no more guaranteed even $A = 0$.

In order to assume the existence of almost-periodic mild solution in this case, we have to impose further conditions on the operator A. If A is the infinitesimal generator of a compact C_0-semigroup $S(t)$ verifying $|S(t)|_{\mathcal{L}(X)} \le e^{\beta t}$, we have stronger results, we will see in this case the existence of almost-periodic mild solution is established.

Section 4 is finally devoted to the study of the semilinear equation of evolution (1); where D is a closed, bounded and convex subset of X, and $f : \mathbb{R} \times D \to X$ is continuous and ω-periodic. We consider various conditions on D, and f, assuring the existence of ω-periodic mild solutions of (1). Becker in [2] has studied the existence of solutions in a weak sense of

$$\begin{cases} x' + (A + B(t, x))\, x = f(t, x) \\ x(0) = x(\omega) \end{cases} \tag{3}$$

where A is the generator of a semigroup of compact type on a Hilbert space H, $B(t, x)$ is a bounded linear operator and $f(t, x)$ a function with values in H. He established that the uniqueness of the linearized version for B and f belonging to certain sets implies the existence of a weak solution of (3) using perturbation and convergence for equations in compact type. Our main result is established on a Banach space using a completely different method.

We start with the following definitions.

Definition 1.1. [1] *(Bochner's criterion)*

The continuous function $x : \mathbb{R} \to X$ is almost periodic over $\mathbb{R}$ if and only if for some sequence $(h_n)_{n\in\mathbb{N}}$ there exists a subsequence of $(h_n)_{n\in\mathbb{N}}$, (say $(h_n)_{n\in\mathbb{N}}$ again) such that the sequence of functions $x(.+h_n)$ converges to a function $\bar{x}(\cdot)$ uniformly on $\mathbb{R}$ as $n \to +\infty$,

$$\left(i.e.\ \lim_{n\to+\infty} \sup_{t\in\mathbb{R}} \left| x(t+h_n) - \bar{x}(t) \right| = 0 \right).$$

Definition 1.2. [9] *The continuous function: $x : \mathbb{R} \to X$ is a mild solution over $\mathbb{R}$ of the semilinear equation*

$$x'(t) = Ax(t) + f(t, x(t))$$

if the functional relation

$$x(t) = S(t-a)x(a) + \int_a^t S(t-a) f(\sigma, x(\sigma))\, d\sigma$$

is satisfied for all $a \in \mathbb{R}$, $\forall t \geq a$.

2. A NON LINEAR ABSTRACT DIFFERENTIAL EQUATION WITH ALMOST-PERIODIC SOLUTION

Our main assumptions concerning the function f in this section will be:

$f(t, x) = f_1(t, x) + f_2(t, x)$, where f_1 and f_2 are continuous from $\mathbb{R} \times X$ into X and satisfy (H1) there is a number $L > 0$ such that $|f_1(t, x) - f_1(t, y)| \leq L|x-y|$ for all $x, y \to X$ and $t \in \mathbb{R}$.

(H2) f_2 ($\mathbb{R} \times D$) is precompact for each bounded subset D of X, and locally uniformly continuous, that is, for each r, $\varepsilon > 0$, there is a $\delta(r, \varepsilon)$ such that $|f_2(t, x) - f_2(t, y)| \leq \varepsilon$ whenever $t \in \mathbb{R}$ and $x, y \in X$ with $|x| \leq r$, $|y| \leq r$ and $|x-y| \leq \delta$.

(H3) $\lim_{|x|\to+\infty} \frac{f_2(t,x)}{|x|} = 0$, uniformly in $t \in \mathbb{R}$.

Our aim here is to establish the following.

Theorem 2.1. *In addition to (H1), (H2) and (H3), we assume that A generates a semigroup $\{S(t)\}$ satisfying $|S(t)|_{L(X)} \leq Me^{\beta t}$, where β is a negative number*
$f_1(t, x) : \mathbb{R} \times X \to X$ is almost-periodic in t, uniformly for x in compact subsets of X,
$f_2(t, x) : \mathbb{R} \times X \to X$ is almost-periodic in t, uniformly for x in compact subsets of X,
and for L sufficiently small enough Lipschiz constant, then, equation (1) has at least one almost-periodic solution.

For the proof of this theorem we need preliminary lemmas.

Lemma 2.2. [8, 9] *Let $g : \mathbb{R} \to X$ is almost-periodic, then there exists one and only one almost-periodic mild solution over $\mathbb{R}$ of the differential equation (2), given by:*

$$x(t) = \int_{-\infty}^t S(t-\sigma)\, g(\sigma) d\sigma$$

Lemma 2.3. *If $a(t)$ belongs to $L^1(\mathbb{R}^+)$, $a(t) > 0$ for all $t \in \mathbb{R}^+$, and $a(t) f(t)$ belongs to $L^1(\mathbb{R}^+, X)$, then*

$$\left(\int_0^{+\infty} a(s) f(s) ds \right) \in \left(\int_0^{+\infty} a(s) ds \right) \overline{co}(f(s),\ 0 \leq s < +\infty)$$

Proof. *Set $s_i^n = \frac{i}{n}$ for $i = 0, 1, ..., n^2$.*

we have:

$$\int_0^{+\infty} a(s) f(s)\, ds = \lim_{n\to+\infty} \sum_{i=1}^{n^2} a(s_i^n) f(s_i^n)(s_i^n - s_{i-1}^n)$$

$$= \lim_{n\to+\infty} \sum_{i=1}^{n^2} a(s_i^n)(s_i^n - s_{i-1}^n) \left(\frac{1}{\sum_{i=1}^{n^2} a(s_i^n)(s_i^n - s_{i-1}^n)} \sum_{i=1}^{n^2} a(s_i^n) f(s_i^n)(s_i^n - s_{i-1}^n) \right)$$

using the convexity

$$\frac{1}{\sum_{i=1}^{n^2} a(s_i^n)(s_i^n - s_{i-1}^n)} \sum_{i=1}^{n^2} a(s_i^n) f(s_i^n)(s_i^n - s_{i-1}^n) \in \overline{co}(f(s), 0 \leq s < +\infty),$$

$\forall n \in IN^*$.

So:

$$\int_0^{+\infty} a(s) f(s) ds \in \left(\int_0^{+\infty} a(s) ds \right) \overline{co}(f(s),\ 0 \leq s < +\infty)$$

Proof of theorem 2.1.: We define.

AP(X) = {$\varphi : \mathbb{R} \to X$; φ is almost-periodic}, with the usual supremum norm over $\mathbb{R}$ which we denote by $|\cdot|_\infty$.

On *AP (X)* we define a mapping T into itself by:

$$T : AP(X) \to AP(X)$$

$$\varphi \to T\varphi = u$$

which u is the unique mild almost-periodic solution over $\mathbb{R}$ of the differential equation:

$$x'(t) = Ax(t) + f(t,\ \varphi(t))$$

(see lemma (2.2)).

Observe that, we can express

$$T\varphi = T_1\varphi + T_2\varphi \text{ for all } \varphi \in AP(X)$$

where

$$T_1\varphi(t) = \int_{-\infty}^{t} S(t-\sigma)f_1(\sigma,\varphi(\sigma))d\sigma$$

$$T_2\varphi(t) = \int_{-\infty}^{t} S(t-\sigma)f_2(\sigma,\varphi(\sigma))d\sigma$$

First, we prove the existence of a closed, bounded convex subset of *AP(X)* invariant for *T*. From hypothesis (H3), we have:

let $\varepsilon > 0$ there is r such that: $|f_2(t,x)| \le \varepsilon|x|$ for all $t \in \mathbb{R}$ and $x \in X$ with $|x| \ge r$.

Setting: $M(\varepsilon) = \sup_{\substack{t\in IR \\ |x|<r}} |f_2(t, x)|$.

Therefore:

(4) $|f_2(t, x)| \le M(\varepsilon) + \varepsilon|x|$ for all $(t, x) \in \mathbb{R} \times X$

$$|T\varphi(t)| \le \int_{-\infty}^{t} |S(t-\sigma)|_{\mathcal{L}(X)}\left(|f_1(\sigma, \varphi(\sigma))d\sigma - f_1(\sigma,0)|d\sigma\right) +$$
$$\int_{-\infty}^{t} |S(t-\sigma)|_{\mathcal{L}(X)}\left(|f_2(\sigma, \varphi(\sigma))| + |f_1(\sigma,0)|\right)$$

in view of the assumption (H1) and from (4), we have

$$|T\varphi|_\infty \le \frac{M}{|\beta|}\left(L|\varphi|_\infty + M(\varepsilon) + \varepsilon|\varphi|_\infty + \sup_{t\in IR}|f_1(t,0)|\right)$$

for ε, L smalls enough, let R be such that

$$\frac{M}{|\beta|}\left((L+\varepsilon)R + C\right) \le R$$

with $C = M(\varepsilon) + \sup_{t\in IR}|f_1(t,0)|$.

We may conclude that T maps the ball of radius R centred at 0 of $AP(X)(\overline{B}_\infty(0, R))$ into itself.

To complete the proof of theorem (2.1) it remains to show that T_2 is completely continuous in *AP(X)*.

The local uniform continuity of f_2 is easily seen to imply T_2 is continuous on bounded subsets of *AP(X)*.

Now we show that T_2 is compact (we can prove without loss of generality that $T_2(B_\infty(0, R))$ is relatively compact).

$$T_2\varphi(t) = \int_{-\infty}^{t} S(t-\sigma)f_2(\sigma,\varphi(\sigma))d\sigma$$

Introduce the new time

$$T_2\varphi(t) = \int_0^{+\infty} S(\sigma)f_2(t-\sigma, \varphi(t-\sigma))d\sigma$$

Let $\gamma > 0$ be such that $0 < \gamma < |\beta|$

$$T_2\varphi(t) = \int_0^{+\infty} e^{-\gamma\sigma}\left(e^{\gamma\sigma}S(\sigma)f_2(t-\sigma, \varphi(t-\sigma))d\sigma\right)$$

by lemma (2.3)

$$T_2\varphi(t) \in \left(\int_0^{+\infty} e^{-\gamma\sigma}d\sigma\right)\overline{co}\left(e^{\gamma\sigma}S(\sigma)f_2(t,x);\right.$$

$$\left.(\sigma,x) \in [t,+\infty[\times \overline{B}(0, R)\right) \text{ for all } |\varphi|_\infty \le R$$

if we denote $K = f_2(\mathbb{R}, \overline{B}(0,R))$, by $(H2)\ \overline{K}$ is compact, then

$$\overline{co}\left(e^{\gamma\sigma}S(\sigma)\,K;\ \sigma \ge 0\right)$$

is relatively compact too. In fact, it guarantees that $e^{\gamma\sigma}S(\sigma)\,x$ converges (uniformly in $x \in K$) to zero as $\sigma \to +\infty$. Therefore T_2 is compact.

We will now conclude that:

$$T = T_1 + T_2 : \overline{B}_\infty(0, R) \to \overline{B}_\infty(0, R)$$

T_1 is a strict contraction for L small see proof of theorem [9], T_2 is completely continuous.

Therefore by Krasnoselskii fixed point T has a fixed point.

Remark 2.1. *More generally the assumption (H3) in theorem 2.1 can be replaced by «let D be a closed, convex and bounded, f :* $\mathbb{R} \times D \to X$ *be continuous and bounded; suppose*

$$\liminf_{h\to 0^+} h^{-1}\, d\left(S(h)x + hf(t,x)\right), D) = 0,$$

for all $(t, x) \in \mathbb{R} \times X$»

3. ALMOST-PERIODIC SOLUTIONS OF SEMILINEAR EQUATIONS WITH COMPACT SEMIGROUPUS

We continue our study of the semi-linear equation:

(5) $x'(t) = Ax(t) + f(t, x(t))$

The main result of this section is the following existence of almost-periodic mild solution

Theorem 3.1. *Let A be the infinitesimal generator of a compact semigroup S(t), t ≥ 0 satisfying* $|S(t)|_{\mathcal{L}(X)} \leq e^{\beta t} (\beta < 0)$.

Let f(t, x) be almost-periodic in t uniformly for x in compact subsets of X, and assume that $\lim_{|x| \to +\infty} \frac{|f(t,x)|}{|x|} = 0$ *uniformly in t.*

The equation (5) has at least one almost-periodic mild solution.

In the proof of this theorem we will need the following result.

Lemma 3.2. [5] *Let* $\{T_n, T_n : D \subset X \to X, n = 1, 2, \ldots\}$ *be a family of compact (possible non linear) operators from the (non empty) subset D of X into X. Let* $T : D \to X$ *be defined by* $Tx = \lim_{n \to +\infty} T_n x$. *If* $T_n x \to Tx$ *as* $n \to +\infty$ *uniformly on bounded subsets of D, then T is compact.*

Proof of theorem 3.1. Under the same notations as in the proof of theorem 2.1.

$$T : AP(X) \to AP(X), \ \varphi \to T\varphi = \int_{-\infty}^{t} S(t-\sigma) f(\sigma, \varphi(\sigma)) d\sigma$$

using the following representation

$$T\varphi(t) = \int_0^{+\infty} S(\sigma) f(t-\sigma, \ \varphi(t-\sigma)) \, d\sigma$$

Let ε > 0, consider the mapping:

$$T_\varepsilon : AP(X) \to AP(X), \ T_\varepsilon \varphi(t) = \int_\varepsilon^{+\infty} S(\sigma) f(t-\sigma, \ \varphi(t-\sigma)) \, d\sigma$$

In fact from assumption of our theorem, we have that for every ε > 0, there exists $M(\varepsilon)$ such that

$$(6) \quad |f(t,x)| \leq M(\varepsilon) + \varepsilon |x| \text{ for all } t \in \mathbb{R}$$

and we have $T_\varepsilon \varphi(t) = S(\varepsilon) \int_\varepsilon^{+\infty} S(\sigma - \varepsilon) f(t-\sigma, \ \varphi(t-\sigma))$, thus, we may conclude that T_ε is compact.

$$T\varphi(t) - T_\varepsilon \varphi(t) = \int_0^\varepsilon S(\sigma) f(t-\sigma, \varphi(t-\sigma)) \, d\sigma$$

by (6) there exists a constant C_R (depending only on R) such that:

$$|T_\varepsilon \varphi - T\varphi|_\infty \leq C_R \int_0^\varepsilon e^{\beta\sigma} d\sigma, \text{ for all } \varphi \in AP(X) \text{ such that } |\varphi|_\infty \leq R$$

This yields $T_\varepsilon \varphi \to T\varphi$ as $\varepsilon \to 0$, uniformly on bounded subsets of *AP(X)*, and therefore *T* is compact as a uniform limit of compact operators (lemma 3.2).

Using the same as in the proof of theorem 2.1, we can construct an invariant bounded closed convex subset of *AP(X)* for *T*.

By Schauder's theorem, there exists $\varphi \in AP(X)$ such that $T\varphi = \varphi$.

The proof is complete.

4. EXISTENCE OF PERIODIC SOLUTIONS OF SEMILINEAR EVOLUTION EQUATIONS

Consider the following semilinear initial value problem

$$(7) \quad \begin{cases} x'(t) = Ax(t) + f(t, x(t)), \ t > 0 \\ x(0) = x_0 \end{cases}$$

where *A* is the infinitesimal generator of a C_0-semigroup *S(t)*, *t* ≥ 0.

We shall assume throughout that *f* satisfies

(H) $f : \mathbb{R} \times X \to X$ is continuous and $\lim_{|x| \to +\infty} \frac{|f(t,x)|}{|x|} = 0$ uniformly in *t* on bounded intervals.

Proposition 4.1. [6] *In addition to (H), we assume that A generates a compact semigroup S(t). Then for every* $x_0 \in X$, *the initial value problem (7) has at least one mild solution defined on* [0, +∞[.

Our aim here.

Theorem 4.2. *In addition to (H), we assume that A generates a compact semigroup, S(t), verifying* $|S(t)|_{\mathcal{L}(X)} \leq e^{\beta t}$, $\forall t > 0$, *and* $1 \in \rho(S(\omega))$ *(where* $\rho(S(\omega))$ *is the resolvent set of* $S(\omega)$*).*

Moreover, if $t \to f(t, x)$ *is ω-periodic, uniformly in x, then the equation (5) has at least one periodic mild solution.*

Proof. By proposition 4.1 the initial value problem (7) has a global mild solution $x(t) \in C([0, +\infty]; X)$. *x(t) is ω-*periodic mild solution of the equation.

$$x'(t) = Ax(t) + f(t, x(t))$$

if and only if: $x(0) = x(\omega)$,
and hence

$$(I - S(\omega)) x_0 = \int_0^\omega (\omega - \sigma) f(\sigma, x(\sigma)) \, d\sigma$$

Since $1 \in \rho(S(\omega))$, it follows that:

$$(8) \quad x_0 = (I - S(\omega))^{-1} \int_0^\omega S(\omega - \sigma) f(\sigma, x(\sigma)) \, d\sigma$$

Now, we consider the mapping from the Banach space $C([0, \omega]; X)$ endowed with the uniform norm over $[0, \omega]$ which we denote by $|\cdot|_\infty$ into itself defined as follows:

$$Tx(t) = S(t)(I - S(\omega))^{-1} \int_0^\omega S(\omega - \sigma) f(\sigma, x(\sigma))\, d\sigma + \int_0^t S(t - \sigma) f(\sigma, x(\sigma))\, d\sigma$$

Using the same argument as in the proof of theorem 2.1 we can construct a ball of radius R centred at 0 of $C([0, \omega]; X)$ invariant for T.

We first observe that it is equicontinuous subset of $C([0, \omega]; X)$.

Let $t, \bar{t} \in [0, \omega]$, $t < \bar{t}$, we have

$$Tx(t) - Tx(\bar{t}) = (S(t) - S(\bar{t}))\left((I - S(\omega))^{-1} \int_0^\omega S(\omega - \sigma) f(\sigma, x(\sigma))\, d\sigma\right) +$$
$$\int_0^t (S(\bar{t} - \sigma) - S(t - \sigma)) f(\sigma, x(\sigma))\, d\sigma + \int_t^{\bar{t}} S(\bar{t} - \sigma) f(\sigma, x(\sigma))\, d\sigma.$$

Setting:

$M = \sup_{\substack{t \in [0,\omega] \\ |x| \le R}} |f(t,x)|$ (finite, fue to hypothesis (H)), let $|S(t)|_{\mathcal{L}(X)} \le N$,

for $t \in [0, \omega]$.

$$|Tx(t) - Tx(\bar{t})| \le \omega M N \left|(I - S(\omega))^{-1}\right|_{\mathcal{L}(X)} |S(\bar{t}) - S(t)|_{\mathcal{L}(X)} +$$
$$M \int_0^\omega |S(\bar{t} - \sigma) - S(t - \sigma)|_{\mathcal{L}(X)}\, d\sigma + MN(\bar{t} - t).$$

Let $\varepsilon > 0$ choose $\delta > 0$ such that: $\delta MN \le \frac{\varepsilon}{3}$.

If $\bar{t} - t < \delta$, from [6, theorem 3.2.p.48] we can also assume that δ is sufficiently small so that:

$$\omega MN \left|(I - S(\omega))^{-1}\right|_{\mathcal{L}(X)} |S(\bar{t}) - S(t)|_{\mathcal{L}(X)} \le \frac{\varepsilon}{3}$$

and

$$\omega M |S(\bar{t} - \sigma) - S(t - \sigma)|_{\mathcal{L}(X)} \le \frac{\varepsilon}{3}, \text{ for all } \sigma \in [0, \omega]$$

and hence

$$|Tx(\bar{t}) - Tx(t)| \le \varepsilon, \text{ whenever } \bar{t} - t \le \delta.$$

To see that the set $\{Tx,\ |x|_\infty \le R\}$ has a compact closure.

Let $\alpha(.)$ denote the measure of noncompactness on X see [3, 4].

Now we shall prove that $\alpha(Tx(t);\ |x|_\infty \le R) = 0$ for all $t \in [0, \omega]$.

We consider two situations.

1. Fix $t > 0$, and let $0 < \varepsilon < t$, in this case we can write

$$Tx(t) = S(\varepsilon)Tx(t - \varepsilon) + \int_{t-\varepsilon}^t S(t - \sigma) f(\sigma, x(\sigma))\, d\sigma$$

using a property of the measure of noncompactness, we obtain:

$$\alpha(T(x(t));\ |x|_\infty \le R) \le \alpha(S(\varepsilon)Tx(t - \varepsilon); |x|_\infty \le R) +$$
$$+ \alpha\left(\int_{t-\varepsilon}^t S(t - \varepsilon) f(\sigma, x(\sigma))\, d\sigma; |x|_\infty \le R\right)$$

Since the map $S(\varepsilon)$ is compact it follows that:

$$\alpha(Tx(t); |x|_\infty \le R) \le \alpha\left(\int_{t-\varepsilon}^t S(t - \sigma) f(\sigma, x(\sigma))\, d\sigma; |x|_\infty \le R\right)$$

hence there is a constant C(depending only on R) such that:

$$\alpha\left(\int_{t-\varepsilon}^t S(t - \sigma) f(\sigma, x(\sigma))\, d\sigma; |x|_\infty \le R\right) \le \varepsilon C$$

since this holds for each $0 < \varepsilon < t$, we may conclude that:

$$\alpha(Tx(t); |x|_\infty \le R) = 0$$

2. for $t = 0$,

$$\alpha\left((I - S(\omega))^{-1} \int_0^\omega S(\omega - \sigma) f(\sigma, x(\sigma))\, d\sigma;\ |x|_\infty \le R\right)$$
$$\le \left|(I - S(\omega))^{-1}\right|_{\mathcal{L}(X)} \alpha\left(\int_0^\omega S(\omega - \sigma) f(\sigma, x(\sigma))\, d\sigma;\ |x|_\infty \le R\right).$$

Let $0 < \varepsilon < \omega$, we can write

$$\int_0^\omega S(\omega - \sigma) f(\sigma, x(\sigma))\, d\sigma =$$
$$= S(\varepsilon) \int_0^{\omega - \varepsilon} S(\omega - \varepsilon - \sigma) f(\sigma, x(\sigma))\, d\sigma + \int_{\omega - \varepsilon}^\omega S(\omega - \sigma) f(\sigma, x(\sigma))\, d\sigma$$

we have there exists a $C(R)$ such that

$$\alpha\left(\int_0^\omega S(\omega - \sigma) f(\sigma, x(\sigma))\, d\sigma;\ |x|_\infty \le R\right) \le \varepsilon C(R)$$

this being true for each $\varepsilon > 0$, we may conclude that

$$\alpha(Tx(0);\ |x|_\infty \le R) = 0$$

we have from Ascoli's theorem that $T(\overline{B}_\infty(0, R))$ is relatively compact in $C([0, \omega]; X)$.

Therefore, the Schauder-Tychonoff implies that: $T\hat{x} = \hat{x}$, for some $\hat{x}$ in $C([0, \omega]; X)$. It is trivial that $\hat{x}$ is ω-periodic mild solution of semilinear evolution equation:

$$x'(t) = Ax(t) + f(t, x(t)) \quad t > 0$$

$$x(0) = x_0$$

where x_0 is defined by (8). This complete the proof.

We close this section with a simple generalization of the above result.

Theorem 4.3. *Let $D \subset X$ be closed convex, bounded, A be the infinitesimal generator of a compact semigroup $S(t)$, $f : \mathbb{R}^+ \times D \to X$ be continuous bounded, and ω-periodic, suppose*

$$\liminf_{h \to 0^+} h^{-1} d\big(S(h)x + hf(t,x),\, D\big) = 0 \text{ for all } (t,x) \in \mathbb{R}^+ \times D$$

Then, (5) has a ω-periodic mild solution in D.

Remark 4.1. *1) It should be noted that this statement complements the one given in* [7] *where the method used seems to be restricted to* $\overset{\circ}{D} \neq \emptyset$

2. Under enough strong conditions of the main result of Becker [2], *for example (X is a Hilbert, B and f satisfy certain boundedness and continuity conditions), he established the existence of a mild periodic solution by applying Schauder theorem.*

5. EXAMPLES

Example 5.1. *Suppose that X is the space $C_0([0, 1]; \mathbb{R})$ of all $\varphi \in C([0, 1]; \mathbb{R})$ such that $\varphi(0) = 0$. Define the operator A on X by*

$$D(A) = \{\varphi \in X : \varphi \text{ is abs. cont., } \varphi' \in X \text{ and } \varphi(0) = 0\}$$
$$A\varphi = -\varphi'$$

Then A is the generator of a semigroup $\{S(t)\}$ on X with $|S(t)|_{\mathcal{L}(X)} = 1$.

Suppose that

$f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfy

(H1) there is $L > 0$ such that

$$|f(\tau,\zeta) - f(\tau,\eta)| \le L|\zeta - \eta|$$

for all $(\tau,\zeta),\ (\tau,\eta) \in [0,1] \times \mathbb{R}$.

Suppose also that $g : [0, 1]^2 \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfy.

(H2) for each r, $\varepsilon > 0$, there is a $\delta(r, \varepsilon)$ such that $|g(\tau,s,\zeta) - g(\tau,s,\eta)| \le \varepsilon$ whenever $(\tau,s) \in [0,1]^2$ and ζ, $\eta \in \mathbb{R}$ with $|\zeta| \le r$, $|\eta| \le r$ and $|\zeta - \eta| \le \delta$,

(H3) $\lim\limits_{\zeta \to +\infty} \dfrac{g(\tau,s,\zeta)}{|\zeta|} = 0$, uniformly in $(\tau, s) \in [0, 1]^2$.

Consider the following partial differential equation

(9)

$$\begin{cases} \dfrac{\partial u(t,\tau)}{\partial t} = \dfrac{\partial u(t,\tau)}{\partial \tau} - u(t,\tau) + \lambda(t)\left[f(\tau,u(t,\tau)) + \int_0^1 g(\tau,s,u(t,s))\, ds\right] \\ u(t,0) = 0 \end{cases}$$

for all $(t, \tau) \in \mathbb{R} \times [0,1]$.

Theorem 2.1 of section 2 yield the following existence result.

Corollary 5.1. *Assume (H1) through (H3) hold. Then for each λ, continuous and almost periodic, and for L sufficiently small enough, there exists at least one almost periodic solution of (9).*

Example 5.2. *Suppose that X is the space $C_{0,0}$ ([0, 1]; $\mathbb{R}$) of all $\varphi \in C([0, 1]; \mathbb{R})$ such that $\varphi(0) = \varphi(1) = 0$.*

Define the operator A on X by

$$D(A) = \{\varphi \in X : \varphi,\ \varphi' \text{ are abs. cont., } \varphi'' \in X \text{ and } \varphi(0) = \varphi(1) = 0\}$$
$$A\varphi = \varphi'' \text{ for all } \varphi \in D(A).$$

Then A is the generator of a compact semigroup $\{S(t)\}$, and there are numbers $M \ge 1$, $\delta > 0$ such that $|S(t)|_{\mathcal{L}(X)} \le Me^{-\delta t}$ for all $t > 0$ (see ([4] prop. 6.6.).

Let f: $[0, 1] \times \mathbb{R} \to \mathbb{R}$ be continuous with $f(0, 0) = f(1, 0) = 0$ and $a \le 0 \le b$.

Consider the equation

$$(10) \quad \left.\begin{aligned} \frac{\partial u(t,\tau)}{\partial t} &= \frac{\partial^2 u(t,\tau)}{\partial \tau^2} + \lambda(t) f(\tau,u(t,\tau)) \\ u(t,0) &= u(t,1) = 0 \\ a &\le u(t,\tau) \le b \end{aligned}\right\} (t,\tau) \in \mathbb{R} \times [0,1].$$

and $D = \{\varphi \in X : a \le \varphi(\tau) \le b, \text{ for all } \tau \in [0, 1]\}$.

Corollary 5.2. *For each λ, continuous and almost periodic. Equation* (10) *has at least one almost periodic solution.*

This result is an immediate of theorem 3.1 and remark 2.1.

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