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ANOTHER VERSION OF VIDAV-PALMER'S THEOREM ON C^*-ALGEBRA STRUCTURE

(involution/positive elements/normal cone/C*-algebra structure)

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ABSTRACT

It is shown that a complex Banach algebra which admits a convex normal cone satisfying some additional conditions is necessarily a C^*-algebra under an equivalent norm; these conditions are fullfilled in Vidav-Palmer's theorem.

1. INTRODUCTION

Vidav-Palmer's theorem asserts that a V-algebra A (see below for the definition) is in fact a C^*-algebra with respect to the involution defined by $(h+ik)^*=h-ik$ for h, k in $Her(A)$ ([3, theorem 14, p. 211]). We consider cones which give rise to an algebra involution on a complex Banach algebra and provide a characterization of those involutions which turn it into a C^*-algebra. This appears to be more general than Vidav-Palmer's theorem in which the cone is associated to a specific order. Moreover, the proof given here does not appeal to representation theory as in [3]. We go back to a characterization of C^*-algebras which had been conjectured by Kaplansky. On the way, we give a new proof of that result.

If $(A, \| \|)$ is a C^*-algebra and $Pos(A)$ denotes the set of its positive elements, then $Pos(A)$ is a closed convex cone such that $Pos(A)\cap(-Pos(A))=\{0\}$; moreover if $0 \leq a \leq b$, then $\|a\| \leq \|b\|$ (cf. [4]), whence the normality of the cone $Pos(A)$. Recall that a characterization of a normal cone is the following: If $(x_n)_n$ and $(y_n)_n$ are two sequences of positive elements such that $x_n \leq y_n$ for every n, and if $(y_n)_n$ tends to zero, then $(x_n)_n$ also tends to zero. The aim of this note is to show that a complex Banach algebra endowed with a convex normal cone satisfying some conditions among the numerous ones of $Pos(A)$ is a C^*-algebra.

In the sequel, ρ and υ will designate, respectively, the spectral radius and the numerical radius. An element h of a complex unital Banach algebra A is said to be hermitian if it has real numerical range. The algebra A is said to be a V-algebra if $A = Her(A) + i\, Her(A)$, where $Her(A)$ is the set of all hermitian elements of A ([cf 3, p. 205]). Recall also that a complex Banach algebra endowed with an involution is said to be hermitian if every selfadjoint element has real spectrum. In a Banach algebra with an involution, Ptak's function is defined by $x \mapsto \left[\rho\left(xx^*\right)\right]^{1/2}$.

2. ON TWO CONJECTURES OF KAPLANSKY

In 1949, I. Kaplansky stated the following conjectures ([8]).

Conjecture 1: Let $(A, \| \|)$ be a hermitian Banach algebra such that $\rho(h) \geq \alpha\|h\|$ for some $\alpha > 0$ and every hermitian element h. Then A is a C^*-algebra for an equivalent norm.

Conjecture 2: Let $(A, \| \|)$ be a complex and involutive Banach algebra such that $\|xx^*\| \geq \alpha\|x\|\,\|x^*\|$ for some $\alpha > 0$ and every normal element x. Then A is a C^*-algebra for an equivalent norm.

According to a comment of Aupetit ([2, p. 121]), R. Arens gave an affirmative answer, to both conjectures, in the commutative case ([1]); and B. Yood solved them ([11]) for $\alpha > 0{,}677$ (precisely for α larger than the real root of $4t^3-2t^2 + t-1=0$). Then Aupetit [2] reproduces a proof of Ptak [10, 5) implies 1) of theorem 8.4] which solves conjecture 1.

Here, we show that conjectures 1 and 2 are equivalent. Then, obtaining a stronger inequality than that used by Aupetit, we reduce conjecture 2 to a result of Ptak.

We need the following lemma of Aupetit ([2, p. 3]), which is also an improvement of a lemma of Hirschfeld and Zelazko ([5, lemma 2]). The proof is nearly obvious.

Lemma 2.1.: Let $(A, \| \|)$ be a Banach algebra and $A_1 = A \oplus C$ its unitization. For every x in A and λ in C, we have $\rho(x+\lambda) \leq \rho(x) + |\lambda| \leq 3\rho(x+\lambda)$.

Proof: The first inequality is due to the fact that x and λ commute. The second follows from the fact that $Sp(x+\lambda) = \lambda + Spx$ and from the triangle inequality.

Proposition 2.2: Conjectures 1 and 2 are equivalent.

Proof: Conjecture 2 follows from conjecture 1 by standard arguments. For the converse, it is also standard that the algebra is semi-simple; hence the involution is continuous ([10)]. Let x be a normal element, $x = h + ik$ with h and k hermitian. Then

$$\begin{aligned} \|xx^*\| &\geq \rho(xx^*) \\ &\geq \rho(h^2 + k^2) \\ &\geq \frac{1}{2}\left[(\rho(h))^2 + (\rho(k))^2\right] \\ &\geq \frac{\alpha^2}{2}\left[\|h\|^2 + \|k\|^2\right] \\ &\geq \beta\|x\|^2 \qquad ; \text{ for some } \beta > 0 \\ &\geq \gamma\|x\| \, \|x^*\| \qquad ; \text{ for some } \gamma > 0 \end{aligned}$$

Theorem 2.3: Let $(A, \| \|)$ be a complex Banach algebra with an involution such that $\|xx^*\| \geq c\|x\| \, \|x^*\|$ for every normal element x and a given $c > 0$. Then A is a C^*-algebra for an equivalent norm.

Proof: By lemma 2.1 and proposition 2.2. we may suppose A unitary and with continuous involution. Now one shows by induction that, for every normal element x in A,

$$\left\|(xx^*)^{2^n}\right\| \geq c.c^2 \ldots c^{2^n} \|x\|^{2^n} \, \|x^*\|^{2^n} \quad \text{for } n \in N,$$

whence

$$\begin{aligned} \rho(xx^*) &\geq c^2\|x\| \, \|x^*\| \\ &\geq \alpha\|x\|^2 \qquad \text{for some } \alpha > 0. \end{aligned}$$

Hence $\|u\| \leq \alpha^{\frac{-1}{2}}$ for every unitary element u in A. Now 2.3 follows from a result of Pták ([10, theorem 8.4]); the equivalent norm being exactly Pták's function.

3. VIDAV-PALMER'S THEOREM

The cone of positive elements in a C^*-algebra has many properties. We select some of them which are sufficient to induce a C^*-algebra structure.

Let A be a unitary complex Banach algebra and P a (non void) cone. The real linear subspace of A generated by P is the set $H = P - P = \{u - v: u, v \in P\}$. We assume that the following conditions hold.

(P1) $A = H + iH$.

(P2) H is closed in A.

(P3) H is closed under both real and imaginary Jordan products, i.e.,

$$\frac{1}{2}(xy + yx), \; \frac{1}{2i}(xy - yx).$$

(P4) Every h *in* H can be written $h = p - q$ with p, q in P such that both $p\,q = 0$ and $q\,p = 0$.

(P5) For every u in P, u^2 is also in P.

(P6) For every u in P, $\|u^2\| \geq c\,\|u\|^2$ for some $c > 0$.

Conditions **(P1)**, **(P2)** and **(P3)** are necessary in order to have a continuous involution. Condition **(P4)** establishes a link between the cone and the multiplication. The very strong condition $\|xx^*\| = \|x\|^2$ in C^*-algebras is reduced here to **(P6)** where only squares of elements of P appear. Finally condition **(P4)** allows some calculations.

Lemma 3.1: If the cone P is salient (i.e., $x, -x \in P$ implies $x = 0$), then it endows A with a continuous algebra involution by $(h + ik)^* = h - ik$.

Proof: To have an algebra involution, it is sufficient (by [3, lemma 7, p. 64]) to show that $H \cap iH = \{0\}$. If $x \in H \cap iH$, then, by **(P4)**, $x = p - q = i(p' - q')$. Then, using **(P5)**, $x^2 = (p + q)^2 \in P$ and $x^2 = -(p' + q')^2 \in (-P)$. So $x^2 = 0$ since P is salient. But $x^2 = p^2 + q^2$, whence $p^2 \in P \cap (-P)$ and so $p^2 = 0$ by **(P6)**. Idem for q. The involution is continuous since H is closed.

For the rest, we need the following characterization: a convex cone P is normal if, and only if, there is an $\alpha > 0$ such that $\|u + v\| \geq \alpha(\|u\| + \|v\|)$ for every u and v in P (cf. [9, proposition 2.2.]).

Theorem 3.2.: If the cone P is normal and satisfies properties **(P1)** to **(P6)**, then A is a C^*-algebra for an equivalent norm.

Proof: Since P is normal, it is salient, and so A is endowed with an algebra involution (lemma 3.1). Let $\alpha > 0$ be such that $\|u+v\| \geq \alpha(\|u\|+\|v\|)$, for every u, v in P. For x normal, we have $xx^* = h^2 + k^2$, where $x = h + ik$ with h, k *in* H. Then

$$\begin{aligned}
\|xx^*\| &= \|h^2+k^2\| \\
&\geq \alpha\left(\|h^2\|+\|k^2\|\right) \\
&\geq \alpha\left(\|(p+q)^2\|+\|(p'+q')^2\|\right) \\
&\geq \alpha\, c\left(\|p+q\|^2+\|p'+q'\|^2\right) \\
&\geq \alpha^3\, c\left(\|p\|^2+\|q\|^2+\|p'\|^2+\|q'\|^{2^2}\right) \\
&\geq \alpha^3\, c\left(\|p-q\|^2+\|p'-q'\|^2\right) \\
&\geq \beta\,\|x\|^2 \qquad ;\text{ for some } \beta > 0 \\
&\geq \gamma\|x\|\,\|x^*\| \qquad ;\text{ for some } \gamma > 0.
\end{aligned}$$

We conclude by theorem 2.3.

Remark 3.3: Properties **(P1)**, **(P2)** and **(P6)**, involving the norm $\| \ \|$, are still valid for any equivalent norm. So we cannot expect A to be necessarily a C^*-algebra for the given norm.

Remark 3.4: We clearly have $H = H(A) = \{x \in A: x^* = x\}$. We also have $P = Q$, where Q is the set of positive elements for the C^*-algebra structure. Indeed, by [4, lemma 7, p. 207], every $v \in Q$ can be written $v = u^2$ with $u \in Q$. But $Q \subset H$, hence $u = p - q$, with p, q in P and $pq = 0$, $qp = 0$. Then $v = u^2 = (p+q)^2 \in P$. So $Q \subset P$. Similarly $P \subset Q$.

The previous theorem applies to V-algebras. Indeed, conditions **(P1)** to **(P6)** are more or less explicit in ([3, pp. 205-208]). What remains to be shown is the normality of the cone of positive elements.

Proposition 3.5: The cone P of positive elements in a V-algebra A is normal.

Proof: As for C^*-algebras ([4, proposition 2.1.9]), one shows that a continuous linear form on A such that $\|f\| = f(e)$ is positive on P. Hence $\upsilon(a) \geq \upsilon(b)$ whenever $0 \leq a \leq b$. The normality of the cone follows since $\|x\| \leq \frac{1}{e}\upsilon(x)$ for every x in A.

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