Rev.R.Acad.Cien.Exact.Fis.Nat. (Esp)
Vol. 92, N.º 4, pp 371-376, 1998
Monográfico: Problemas complejos de decisión

ON USING MULTISTAGE LINKING CONSTRAINTS FOR STOCHASTIC OPTIMIZATION AS A DECISION-MAKING AID[1]

(multistage scenario analysis/scenario tree/deterministic equivalent model/parallel computing)

LAUREANO F. ESCUDERO

IBERDROLA Ingeniería y Consultoría and DEIO, *Universidad Complutense de Madrid*, Spain

ABSTRACT

We present a modelling framework for multistage planning problems under uncertainty in the objective function coefficients and right-hand-side. A multistage scenario analysis scheme with partial recourse is used. So, the decision policy can be implemented for a given set of initial time periods (so-called implementable time stage), such that the solution for the other periods does not need to be anticipated and, then, it depends upon the scenario group to occur at each stage. In any case the solution offered for each stage takes into account all scenarios but without subordinating to any of them. A novel scheme is presented for modelling multistage linking constraints through the scenario tree. This type of constraints is modelled by using a splitting variable scheme that allows to produce a sibble of the coupling variables for each scenario group that belongs to the latest stage with nonzero coefficients in the given constraints block. The proposed scheme is very amenable for using decomposition approaches to solve the deterministic equivalent model and, then, for experimenting with parallel computing implementations.

1. INTRODUCTION

Decision making is inherent to all aspects of industrial, business and social activities. In all of them, difficult tasks must be accomplished. One of the most reliable decision support tools available today is Optimization, a field at the confluence of Mathematics and Computer Science. The purpose of the field is to build and solve effectively realistic mathematical models of the situation under study, allowing the decision makers to explore a huge variety of possible alternatives. As reality is complex, many of these models are *large* (in terms of the number of decision variables) and *stochastic* (there are parameters whose value cannot be controlled by the decision maker and are uncertain). The last fact makes the problem difficult to tackle, yet its solution is critical for many leading organizations in fields such as public policy making, supply chain planning, production and distribution planning and assets allocation among many other areas.

Problems with the characteristics given above are transformed into mathematical optimization models. Often there are tens of thousands of constraints and variables for a deterministic situation. The problems can be modelled as large-scale linear programs. Given today's Operations Research state-of-the-art tools, deterministic logistics scheduling optimization should not present major difficulties. However, it has long been recognized (Beale, 1955 and Dantzig, 1955) that traditional deterministic optimization is not suitable for capturing the truly dynamic behavior of most real-world applications. The main reason is that such applications involve data uncertainties which arise because information that will be needed in subsequent decision stages is not available to the decision maker when the decision must be made. See Kall and Wallace (1994), Higle and Sen (1996) and Birge and Louveaux (1997) for good surveys on Stochastic Programming and additional references.

The aim of this work is to present a modeling approach for dealing with multistage linking constraints in an environment under uncertainty in some parameters. The paper is organized as follows. Section 2 presents the stochastic environment to deal with. Section 3 presents the multistage linking constraints problem to address in the stochastic environment. Finally, section 4 introduces the splitting variables approach to model this type of constraints and two alternative mathematical approaches for problem solving.

2. GENERAL APPROACH

Let the following (deterministic) model

[1] This work has been partially supported by the European Commission within the Environment and Climate program, project WARSYP PL970441.

$$\min_{v} c^T v$$
$$s.t.\ Av = p$$
$$v \geq 0 \qquad (2.1)$$

where c is the vector of the objective function coefficients, A is the $m \times n$ constraint matrix, p is the right-hand-side (*rhs*) m-vector and v is the n-vector of the decision variables to optimise. It must be extended in order to deal properly with uncertainty in the values of some parameters, say, c and p. A similar approach can be used for dealing with the uncertainty in the constraints matrix A.

In this case one needs to consider two additional features. In the first place, one must model the availability of information over time, and state what sort of decisions can be made at each of the various stages. Secondly, to compute an optimal solution in the stochastic area any proposed solution should also be compared with other candidate solutions as it is done in the deterministic field. But, in the stochastic setting, the criteria by which this comparison can be performed are much less clear. Thus, one needs an approach to model the uncertainty in the problem data. The traditional approach is to make probabilistic distribution assumptions, estimate the parameters from historical data and, then, develop an stochastic model to take the uncertainty into account. Such an approach may not be appropriate if only limited information is available. On the other hand, in many applications it is often necessary and possible to take into account information that is not reflected in the historical data. In many such cases we may employ a technique so-called *scenario analysis*, where the uncertainty is modelled via a set of scenarios.

Let S denote the set of scenarios to consider, and w^S the likelihood that the decision maker assigns to scenario s. One way to deal with the uncertainty is to obtain the solution v that best tracks each of the scenarios, while satisfying the constraints for each scenario. This can be achieved by obtaining a solution that minimizes a norm of the weighted upper difference between the proposed solution and the optimal solution value for each scenario. The resulting model does not increase the number of variables of the original representation, but now there are $m|G|$ constraints. Unfortunately, this representation does not preserve the structure of the deterministic model (2.1) and the objective function is no longer linear; see in Escudero (1994) some procedures to overcome this difficulty. Models of this form are known as scenario immunization models, or *SI* models for short, see Dembo (1991) and also Mulvey et al. (1995).

As an alternative goal, we could minimize the expected value of the objective function; in this case model (2.1) becomes

$$\min_{v} \sum_{s \in S} w^s c^{s^T} v$$
$$s.t.\ Av = p^s, \ \forall s \in S \qquad (2.2)$$
$$v \geq 0$$

Note that (2.2) gives an implementable policy based on the so-called *simple recourse* scheme. (See that the whole vector of decision variables is anticipated at time period 1).

3. NONANTICIPATIVE POLICIES

The *SI* models do anticipate decisions in v that for multistage environments may not be needed at stage $r=1$. Very frequently the decisions for stage $r=1$ are the decisions to be made since at stage $r=2$ one may realize that some of the data has been changed, some scenarios vanish, etc. In this case, the models will be usually reoptimized in a rolling horizon mode. When only spot decisions (i.e., decisions for the first stage) are to be made, the information about future uncertainty is only taken into account for a better spot decision making. This type of scheme is termed *full recourse*.

Let R denote the set of stages and v_r^s denote the vector of the variables related to stage r under scenario s for $r \in R$ and $s \in S$, and v^s is the set of vectors $\{v_r^s, \forall r \in R\}$.

Rockafellar and Wets (1991), see also Wets (1989), state the so-called *non-anticipative principle*: If two different scenarios, say, s and s' are identical up to stage r on the basis of the information available about them at that stage, then the values of the v-variables must be identical up to stage r. This principle guarantees that the solution obtained from the model is not dependent at stage r on the information that is not yet available. To illustrate this concept, consider a so-called *scenario tree* where each node represents a point in time where a decision can be made. Once a decision is being made several contingencies can happen, and information related to these contingencies is available at the beginning of the next stage. This information structure is visualized as a tree, where each root-to-leave path represents one specific scenario and corresponds to one realization of the uncertain parameters.

In order to introduce the implications of this principle in our approach, we define a set of scenario groups, say, G_r for each stage r, such that all scenarios having the same realizations of the uncertainty up to stage r belong to the same scenario group, say, g for $g \in G_r$. Let $S_{g,r}$ denote the set of scenarios that belong to group g at stage r for $S_{g,r} \subseteq S$. Let a node in the scenario tree be represented by the pair, say, (k,r) for $k \in G_r$, $r \in R$, such that the scenario tree is defined by the set of nodes $\bigcup_{r \in R} (k, r) / k \in G_r$ and the set of directed arcs E, where $(k.\ l) \in E$ iff $S_{l,r+1} \subseteq S_{k,r}$ for $k \in G_r$ and $l \in G_{r+1}$. Let $G_r^k \equiv \{l \in G_{r+1} | (k,l) \in E\}$. Finally, let N denote the set of solutions that satisfy the so-called *nonanticipativity constraints*. That is,

$$v \in N \equiv \{v^s | v_r^s = v_r^{s'} \ \forall s,\ s' \in S_{g,r},\ g \in G_r,\ r \in R\} \qquad (3.1)$$

So, the Deterministic Equivalent Model (DEM) of the so-called *full recourse* version of model (2.1) can be expressed

$$\begin{aligned} \min_{v} \; & \sum_{s \in S} w^s c^{s^T} v^s \\ s.t. \; & A v^s = p^s, \quad \forall s \in S \\ & v \in N \\ & v^s \geq 0 \qquad \forall s \in S. \end{aligned} \tag{3.2}$$

Model (3.2) has a nice structure that we may exploit. Two approaches can be used to represent the nonanticipativity constraints (3.1). One approach is based on a *compact representation,* where (3.1) is used to eliminate variables in (3.2) as well as for reducing model size, so that there is a single variable for each element at each scenario group of each stage, but any special structure of the constraints in (2.1) is destroyed. In this case let the variables vector $v=(x,y,z)$ have the following structure: $x_{g,\,r}$, vector of variables with nonzero coefficients in the constraints related to stage r alone for $g \in G_r$, $r \in R$; $y_{g,\,r}$, vector of variables with nonzero elements in the constraints related to the stages r and $r+1$; and $z_{g,\,r}$, vector of variables with nonzero elements in the constraints related to stage r as well as in the constraints related to sets of stages to be defined below.

Let the following additional notation. U is the set of z-related constraint blocks through stages, so-called *multistage linking constriants*, R_u is the set of time stages related to constraint block u for $u \in U$, $\underline{r}_u$ and $\bar{r}_u$ are the smallest and largest elements from R_u, respectively, and $N_{g,\,u}$ is the set of nodes in the directed path through the set of stages (i.e., set R_u) whose ending node is node $(g, \bar{r}_u)$ and the unique origin node is, say, $(i, \underline{r}_u)$. So, the pair (k, r) index for variable $z_{k,\,r}$ is such that $(k, r) \in N_{g,\,u}$ for ending node $(g, \bar{r}_u)$ and constraint block u. Note: The variable $z_{k,\,r}$ can belong to more than one multistage linking constraint block. This type of constraint block can be represented as follows.

$$Z_{g,u}: \sum_{(k,\tau) \in N_{g,u}} D_{u,\tau} z_{k,\tau} = d_{g,u} \quad \forall g \in G_{\bar{r}_u}, u \in U, \tag{3.3}$$

where $D_{u,\,\tau}$ is the matrix for constraint block u related to the z-variables from stage τ, and $d_{g,\,u}$ is the *rhs* of constraint block u for scenario group g from stage $\bar{r}_u$, $u \in U$. (See that constraint block u has $|G_{\bar{r}_u}|$ versions).

The compact representation of model (3.2) can be expressed as follows.

$$\begin{aligned} \min_{x,\,y} \; & \sum_{r \in R} \; \sum_{g \in G_r} w_{g,\,r}\left(a_{g,\,r}^T x_{g,\,r} + b_{g,\,r}^T y_{g,\,r} + c_{g,\,r}^T z_{g,\,r}\right) \\ s.t. \; 0 \leq \; & x_{g,\,r}, y_{g,\,r}, \quad z_{g,\,r} \in X_{g,\,r} \quad \forall g \in G_r, r \in R \\ & z_{g,\,r} \in Z_{g,\,u} \quad \forall g \in G_{\bar{r}_u}, u \in U \end{aligned} \tag{3.4}$$

such that

$$\begin{aligned} & X_{g,\,r}: A_r x_{g,\,r} + B_r' y_{l,r-1} + B_r y_{g,\,r} + C_{g,\,r} z_{g,\,r} = p_{g,\,r} \\ & \forall g \in G_r, \; r \in R, \end{aligned} \tag{3.5}$$

where $w_{g,\,r}$ gives the weight associated to scenario group g at stage r, such that $w_{g,\,r} = \sum_{s \in S_{g,\,r}} w^s$, $a_{g,\,r}$, $b_{g,\,r}$ and $c_{g,\,r}$ are the x-, y- and z-variables related objective function coefficients for the pair (g,r), A_r, B_r and C_r are the appropriate constraint matrices, $p_{g,\,r}$ is the *rhs*, all with the conformable dimensions, and $l: g \in G_{r-1}^l$ for $g \in G_r$, $r \in R$.

4. SPLITTING VARIABLE REPRESENTATION

One of the main inconveniences of the compact representation (3.3)-(3.5) is the inherent difficulty for its decomposition in smaller models. Given the large-scale instances of the model, easy decomposition is a key for success. It can be obtained from the so-called *splitting variable representation*. It requires to produce sibbles of the y- and z-variables.

For this purpose let $N^{g,\,r}$ denote the set of pairs (k, τ) such that $k \in G_\tau$, $\tau \in R/\tau \geq r$ and $\exists u \in U / \tau = \bar{r}_u$ for $(g, r) \in N_{k,\,u}$. That is, (g, r) and (k, τ) for $g \in G_r$ and $k \in G_\tau$ are any two nodes in the scenario tree for $r, \tau \in R$, such that there is a constraint block u for $u \in U$ where $\tau \equiv \bar{r}_u$ and there is a path from some node, say, $(i, \underline{r}_u)$ to node (k, τ) through node (g, r). Note: There is only one path from node $(i, \underline{r}_u)$ to node (k, τ). See that (k, τ) is the ending node of any of its subpaths through node (g, r). (It is the case where $u' \in U/\underline{r}_{u'} > \underline{r}_u$, $\bar{r}_{u'} = \bar{r}_u = \tau$ such that $(g, r) \in N_{k,\,u'}$ and $k \in G_\tau$).

In order to introduce the new representation, let us rename the y- and z-variables such that $y_{g,\,r}$ and $z_{g,\,r}$ will be replaced by $y_{g,\,r}^0$ and $z_{g,\,r}^{g,\,r}$, respectively, and add the new variables $y_{l,\,r-1}^g$, where $l: g \in G_{r-1}^l$ and $z_{g,\,r}^{k,\,\tau}$ $\forall (k, \tau) \in N^{g,\,r}$, $g \in G_r$, $r \in R$. So, the splitting variable representation is as follows.

$$\begin{aligned} \min_{x,\,y,\,z} \; & \sum_{r \in R} \; \sum_{g \in G_r} w_{g,\,r}\left(a_{g,\,r}^T x_{g,\,r} + b_{g,\,r}^T y_{g,\,r}^0 + c_{g,\,r}^T z_{g,\,r}^{g,\,r}\right) \\ s.t. \; X_{g,\,r}: \; & A_r x_{g,\,r} + B_r' y_{l,\,r-1}^g + B_r y_{g,\,r}^0 + C_r z_{g,\,r}^{g,\,r} = p_{g,\,r} \\ & \qquad \forall g \in G_r, \; r \in R \ni l: g \in G_{r-1}^l \\ Z_{g,u}: \; & \sum_{(k,\tau) \in N_{g,u}} D_{u,\tau} z_{k,\tau}^{g,\bar{r}_u} = d_{g,u} \quad \forall g \in G_{\bar{r}_u}, u \in U, \\ Y_{g,\,r}^l: \; & y_{g,\,r}^l - y_{g,\,r}^{l+1} = 0 \;\; \forall l \in \{0\} \cup G_r^g, \; g \in G_r, \; r \in R \\ Z_{g,\,r}^{k,\,\tau}: \; & z_{g,\,r}^{g,\,r} - z_{g,\,r}^{k,\,\tau} = 0 \;\; \forall (k, \tau) \in N^{g,\,r}, \; g \in G_r, \; r \in R \\ & x, y, z \geq 0 \end{aligned} \tag{4.1}$$

Note: The last two constraint blocks (i.e., the constraint blocks that define the solution spaces $Y^l_{g,r}$ and $Z^{k,\tau}_{g,r}$) are the expressions for the non-anticipativity constraints (3.1). Escudero (1998) and Escudero et al. (1998) present case studies that make use of the linking constraints mechanism for two consecutive stages. They can use the multistage linking constraint mechanism as well.

Different types of decomposition approaches can be used for solving model (4.1); we favour Augmented Lagrangian and Benders Decomposition schemes. These types of schemes are very amenable for using parallel computing approaches, see Ruszczynski (1993), Valdimirou and Zenios (1997), Dempster and Thompson (1998), Escudero et al. (1998a), Escudero and Salmerón (1998) and Vladimirou (1998) among others.

Augmented Lagrangian decomposition methods proceed by moving the nonanticipativity constraints of model (4.1) into the objective function to create a problem with independent sets of constraints, in fact, one set per node in the scenario tree, so that each set keeps any special structure that might be present in the original problem (2.1). The resulting problem becomes

$$\max_{\mu,\sigma} v\left(D_\rho(\mu, \sigma)\right) \tag{4.2}$$

where the function $v(D_\rho(\mu, \sigma))$ is defined as

$$\begin{aligned} v\left(D_\rho(\mu,\sigma)\right) = \min_{x,y,z} \sum_{r\in R} \sum_{g\in G_r} & w_{g,r}\left(a^T_{g,r} x_{g,r} + b^T_{g,r} y^0_{g,r} + c^T_{g,r} z^{g,r}_{g,r}\right) + \\ + \sum_{r\in R} \sum_{g\in G_r} \sum_{l\in\{0\}\cup G^g_r} & \left[\mu^{lT}_{g,r}\left(y^l_{g,r} - y^{l+1}_{g,r}\right) + \frac{\rho}{2}\left\|y^l_{g,r} - y^{l+1}_{g,r}\right\|^2\right] + \\ + \sum_{r\in R} \sum_{g\in G_r} \sum_{(k,\tau)\in N^{g,r}} & \left[\sigma^{k,\tau^T}_{g,r}\left(z^{g,r}_{g,r} - y^{k,\tau}_{g,r}\right) + \frac{\rho}{2}\left\|z^{g,r}_{g,r} - z^{k,\tau}_{g,r}\right\|^2\right] \end{aligned}$$

$$\begin{aligned} s.\ t.\quad 0 \le x, y, z \in X_{g,r} &\quad \forall g\in G_r,\ r\in R \\ z \in Z_{g,u} &\quad \forall g\in G_{\bar r_u},\ u\in U \end{aligned} \tag{4.3}$$

where $\rho>0$ is an appropriate penalty parameter and μ and σ are the vectors of the Lagrange multipliers for the nonanticipativity constraints.

Model (4.3) is a quasi-separable quadratic problem with independent constraint subsystems that can be solved in reasonable computing time, being very amenable for parallel computing implementations. Mulvey and Ruszczynski (1992), Ruszczynski (1993), Escudero (1994, 1998), Escudero et al. (1998) and Escudero and Salmerón (1998) among others present detailled algorithms for solving model (4.3).

On the other hand, Benders (1962) based methods exploit the structure of a model by creating the so-called master program and the auxiliary programs, the last ones included by the structured constraints. Its first application to 2-stage stochastic programming is due to Van Slyke and Wets (1969). Well known extensions to multistage stochastic programming have been presented by Birge (1985), Gassmann (1990) and Dempster and Thompson (1998) among many others.

Our Benders decomposition approach for solving model (4.1) requires the dual vectors $\lambda_{g,r}$ for the $X_{g,r}$-constraints, $\pi_{g,u}$ for the $Z_{g,u}$-constraints where $u\in U/\bar r_u \equiv r$, $\sigma^l_{g,r}$ for the $Y^l_{g,r}$-constraints where $l\in\{0\}\cup G^g_r$ and $\mu^{k,\tau}_{g,r}$ for the $Z^{k,\tau}_{g,r}$-constraints where $(k,\tau)\in N^{g,r}$, such that $g\in G_r$, $r\in R$. Before applying the decomposition scheme, the dual of model (4.1) must be obtained.

We can observe in model (4.4) that, by fixing the (coupling) σ-and μ-variables, it results into a system included by independent sets of constraints (in fact, one per node in the scenario tree). This observation gives the motivation for using a Benders Decomposition scheme since, after the variables fixing, the new model from (4.4) results in a set of independent systems. From here, the

$$\begin{array}{lllll} \max & \sum_{r\in R}\sum_{g\in G_r} p_{g,r}\lambda_{g,r} + & \sum_{u\in U}\sum_{g\in G_{\bar r_u}} d_{g,u}\pi_{g,u} & & \\ s.t. & A^T_r\lambda_{g,r} & & & \le a_{g,r}\ \ \forall g\in G_r,\ r\in R \\ & B^T_r\lambda_{g,r} + \sigma^0_{g,r} - \sigma^{|G_r|}_{g,r} & & & \le b_{g,r}\ \ \forall g\in G_r,\ r\in R \\ & B'^T_{r+1}\lambda_{l,r+1} + \sigma^l_{r,g} - \sigma^{l-1}_{r,g} & & & \le 0\ \ \forall l\in G^g_r,\ g\in G_r, r\in R \\ & C^T_r\lambda_{g,r} + & \sum_{u\in U/\bar r_u = r} D^T_{u,r}\pi_{g,u} + & \sum_{(k,\tau)\in N^{g,r}} \mu^{k,\tau}_{g,r} & \le c_{g,r}\ \ \forall g\in G_r,\ r\in R \\ & & \sum_{u\in U/\bar r_u=\tau} D^T_{u,r}\pi_{k,u} - & \mu^{k,\tau}_{g,r} & \le 0\ \ \forall (k,\tau)\in N^{g,r} - \{(g,r)\},\ g\in G_r,\ r\in R \end{array} \tag{4.4.}$$

procedure for obtaining the optimal solution of model (4.4) is very standard; see in Escudero and Salmerón (1988) the details.

5. CONCLUSIONS

In this paper we have presented a modelling approach to generalize the framework to deal with tage indexed variables that have nonzero elements in constraints related to the proper tage and the next one in stochastic optimization for decision aid via scenario analysis. The generalization consists of a very useful modelling scheme for multistage linking constraints in a full recourse decision policy environment. We present novel schemes based on a splitting variables approach to create the appropriate sibbles of the multistage variables. The new modelling framework allows to decompose the Deterministic Equivalent Model of the stochastic problem by considering the special structure of the siblings. Decomposition schemes such as Augmented Lagrangian based and Benders based Decomposition schemes are very appropriate for the treatment of the sibling modelling.

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