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REGULARITY OF PRE-RADON MEASURES

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ABSTRACT

A pre-Radon measure μ in a topological space X is inner regular when X is weakly metacompact, when X is paralindelöf and μ has a concassage of Lindelöf sets and when X is metalindelöf and μ has a concassage of separable sets.

RESUMEN

Una medida pre-Radon μ en un espacio topológico X es interiormente regular cuando X es débilmente metacompacto, cuando X es paralindelöf y μ tiene un concassage de conjuntos Lindelöf y cuando X es metalindelöf y μ tiene un concassage de conjuntos separables.

1. INTRODUCTION AND PRELIMINARIES

Let X be a topological space. By $\mathcal{G}$, $\mathcal{F}$, $\mathcal{K}$ and $\mathcal{B}$ we shall denote, respectively, the families of all open, closed, compact closed and Borel subsets of X.

A nonempty family $\mathcal{A}$ of subsets of X is called *directed upwards* if for each A, B in $\mathcal{A}$ there is C in $\mathcal{A}$ such that $A \cup B \subset C$. If $\mathcal{A}$ is directed upwards and $A_o = \cup$ A, we write $\mathcal{A} \uparrow A_o$.

A family $\mathcal{A}$ of subsets of X is called *point-finite* (respectively, *point-countable*) if each point $x \in X$ belongs only to finite (resp. countable) many sets of $\mathcal{A}$. The family $\mathcal{A}$ is called *locally countable* if each point $x \in X$ has an open neigborhood which meets only countably many sets of $\mathcal{A}$.

The space X is called *metacompact* (resp. *metalindelöf*) if each open cover of X has a point-finite (resp. point-countable) open refinement. X is called *weakly metacompact* if each open cover of X has an open refinement which is a countable union of point-finite families. X is called *paralindelöf* if each open cover of X has a locally countable open refinement.

A *Borel measure* in X is a measure on $\mathcal{B}$. The *support* of a Borel measure μ in X is the set of all $x \in X$ such that $\mu(V) > 0$ for each open neighborhood V of x. We shall denote by supp μ the support of μ. Clearly, supp μ is a closed subset of X.

If μ is a Borel measure in X, a set $B \in \mathcal{B}$ is called

a) μ-*outer regular* if $\mu(B) = \inf\{\mu(G)\colon B \subset G \in \mathcal{G}\}$;

b) μ-*inner regular* if $\mu(B) = \sup\{\mu(F)\colon B \supset F \in \mathcal{F}\}$;

c) μ-*compact* if for each open cover $\mathcal{U}$ of B and each $\varepsilon > 0$ there is a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ such that $\mu(B - \cup \mathcal{V}) < \varepsilon$.

The concept of μ-compact set is introduced by B. Rodríguez-Salinas in [5]. For a extensive treatment of μ-compact sets we refer to [3].

A Borel measure μ in X is called

A) *outer regular* if each $B \in \mathcal{B}$ is μ-outer regular;

B) *inner regular* if each $B \in \mathcal{B}$ is μ-inner regular;

C) *locally finite* if each $x \in X$ has a neigborhood V such that $\mu(V) < +\infty$.

D) τ-*additive* if $\sup\{\mu(G)\colon G \in \mathcal{G}_0\} = \mu(G_0)$ for each $\mathcal{G}_0 \subset \mathcal{G}$ with $\mathcal{G}_0 \uparrow G_0$.

Let $\mathcal{H}$ be a subfamily of $\mathcal{F}$. A Borel measure μ in X is called

(α) a *Riesz measure of type* ($\mathcal{H}$) when it is outer regular, each $H \in \mathcal{H}$ is a μ-compact set with $\mu(H) >$ 0 and

$$\mu(G) = \sup \{\mu(H): G \supset H \in \mathcal{H}\}$$

for each $G \in \mathcal{G}$.

(β) a *pre-Radon measure* when it is locally finite, τ-additive and outer regular, and each $G \in \mathcal{G}$ with $\mu(G) < +\infty$ is μ-inner regular.

The pre-Radon measures are introduced by I. Amemiya, S. Okada and Y. Okazaki in [1].

Since each Riesz measure of type ($\mathcal{H}$) is τ-additive, each locally finite Riesz measure of type ($\mathcal{H}$) is a pre-Radon measure.

P. Prinz establishes in [4] that a *Riesz measure* μ in a Hausdorff space X (i. e. a Riesz measure of type ($\mathcal{K}$) in X) is inner regular when X is metacompact (resp. paralindelöf) and when X is metalindelöf and μ has a concassage of separable sets. In [2] we introduce the Riesz measures of type ($\mathcal{H}$) and we generalize the Prinz's results to this class of measures. In this paper we extend these results to pre-Radon measures.

2. THE RESULTS

Definition 2.1. Let μ be a Borel measure in X. A *concassage* of μ is a disjoint family $\{F_i\}_{i\in I}$ of closed subsets of X of finite measure which satisfies the following properties:

a) supp $\mu_{F_i} = F_i$ for each $i \in I$;

b) $X - \bigcup_{i\in I} F_i$ is a locally negligible set.

c) $\mu(B) = \sum_{i\in I} \mu(B \cap F_i)$ for each $B \in \mathcal{B}$ with $\mu(B) < +\infty$.

Theorem 2.2. *Each pre-Radon measure μ in X has a concassage.*

Proof. See [1, Theorem 6.1].

Lemma 2.3. *Let μ be a pre-Radon measure in X. If* $B = \cup_{n=1}^{+\infty} B_n$ *with* $B_n \in \mathcal{B}$ *and* $\mu(B_n) < +\infty$ *for each* $n \in \mathbb{N}$, *then B is μ-inner regular.*

Proof. Let $\varepsilon > 0$. For each $n \in \mathbb{N}$ there is $G_n \in \mathcal{G}$ with $B_n \subset G_n$ and $\mu(G_n - B_n) < \varepsilon/2$, and there is $G_n' \in \mathcal{G}$ with $G_n - B_n \subset G_n'$ and $\mu(G_n') < \varepsilon/2$. Moreover, there is $F_n \in \mathcal{F}$ with $F_n \subset G_n$ and $\mu(F_n) > \mu(G_n) - \varepsilon/2$. Let $E_n = F_n - G_n'$. Then $E_n \in \mathcal{F}$, $E_n \subset B_n$ and

$$\begin{aligned}\mu(E_n) &= \mu(F_n) - \mu(G_n' \cap F_n)\\ &> \mu(G_n) - \mu(G_n') - \varepsilon/2\\ &> \mu(G_n) - \varepsilon\\ &> \mu(B_n) - \varepsilon.\end{aligned}$$

Thus, each B_n is μ-inner regular. We shall prove that B is also μ-inner regular.

Replacing B_n by $\bigcup_{i=1}^{n} B_i$ if necessary, we may assume that $B_n \subset B_{n+1}$ for each $n \in \mathbb{N}$. Then $\mu(B) = \lim \mu(B_n)$ and since

$$\begin{aligned}\mu(B_n) &= \sup\{\mu(F): B_n \supset F \in \mathcal{F}\}\\ &\leq \sup\{\mu(F): B \supset F \in \mathcal{F}\}\end{aligned}$$

for each $n \in \mathbb{N}$, takint limits we obtain

$$\mu(B) \leq \sup\{\mu(F): B \supset F \in \mathcal{F}\} \leq \mu(B).$$

Corollary 2.4. *Let μ be a pre-Radon measure in X. If μ is σ-finite, then μ is inner regular.*

Theorem 2.5. *Let μ be a pre-Radon measure in X and let* $\{F_i\}_{i\in I}$ *a concassage of μ. Then μ is inner regular whenever one of the following conditions is satisfied:*

a) *X is weakly metacompact;*

b) *X is paralindelöf and* F_i *is Lindelöf for each* $i \in I$.

c) *X is metalindelöf and* F_i *is separable for each* $i \in I$.

Proof. Since μ is τ-additive, we may assume that the support of μ is the whole space, i. e.,

(1) $\mu(G) > 0$ for $\varnothing \neq G \in \mathcal{G}$.

Let $B \in \mathcal{B}$ and let $\mathcal{G}_0$ be the family of all open subsets of X with finite measure. By Lemma 2.3 we may assume that

(2) $B - \bigcup_{n=1}^{+\infty} G_n \neq \varnothing$ for each sequence $(G_n) \subset \mathcal{G}_0$.

Since μ is locally finite, there is an open refinement $\mathcal{A}$ of $\mathcal{G}_0$ such that $\mathcal{A} = \bigcup_{i=1}^{+\infty} \mathcal{A}_i$ with $\mathcal{A}_i$ point-finite for each $i \in \mathbb{N}$ in case (a), $\mathcal{A}$ is locally countable in case (b) and $\mathcal{A}$ is point-countable in case (c). By Zorn's lemma, there is a maximal subset F of B such that

(3) card $(F \cap U) \leq 1$ for each $U \in \mathcal{A}$.

This set F is uncountable for otherwise, since $\mathcal{A}$ is point-countable, the family $\mathcal{A}' = \{U \in \mathcal{A}: U \cap F \neq \varnothing\}$ is countable and, by (2), do not a cover of B, hence we can add a point $x \in B - \cup \mathcal{A}'$ to F such that

$$\text{card } ((F \cup \{x\}) \cap U) \leq 1$$

for each $U \in \mathcal{A}$, which contradicts the maximality of F. Moreover, F is closed; indeed, if $a \notin F$ there is $U \in \mathcal{A}$ such that $a \in U$ and $F \cap U = \varnothing$ or $F \cap U = \{b\}$ with $b \neq a$; it follows that

$$a \in U \subset X - F \text{ or } a \in U - \{b\} \subset X - F.$$

We shall prove that $\mu(F) = +\infty$.

Later we shall see that for each $G \in \mathcal{G}_0$ the family

$$\mathcal{A}_G = \{U \in \mathcal{A}: \mu(U \cap G) > 0\}$$

is countable. Hence, in vief of (1), the family $\{U \in \mathcal{A}: U \cap G \neq \emptyset\}$ is also countable for each $G \in \mathcal{G}_0$. Since F is uncountable, from (3) it follows that F is not contained in an open set of finite measure, hence

$$\mu(F) = \inf \{\mu(G): F \subset G \in \mathcal{G}\} = +\infty.$$

(a) Assume that $\mathcal{A} = \bigcup_{i=1}^{+\infty} \mathcal{A}_i$ with $\mathcal{A}_i$ point-finite for each $i \in \mathbb{N}$ and assume that $\mathcal{A}_G$ is uncountable for some $G \in \mathcal{G}_0$. Then

$$A_{G,i} = \{U \in A_i: \mu(U \cap G) > 0\}$$

is uncountable for some $i \in \mathbb{N}$. Since

$$\mathcal{A}_{G,i} = \bigcup_{k=1}^{+\infty} \{U \in \mathcal{A}_i: \mu(U \cap G) \geq 1/k\},$$

there is $k \in \mathbb{N}$ such that the family $\{U \in \mathcal{A}_i: \mu(U \cap G) \geq 1/k\}$ is uncountable, and we can find a sequence of distinct $U_n \in \mathcal{A}_i$, such that $\mu(U_n \cap G) \geq 1/k$ for each $n \in \mathbb{N}$. It follows that

$$\mu(\limsup U_n) \geq \mu(\limsup (U_n \cap G))$$
$$\geq \limsup \mu(U_n \cap G) \geq 1/k,$$

hence $\limsup U_n \neq \emptyset$ which contradicts the fact that $\mathcal{A}_i$ is point-finite. Thus $\mathcal{A}_G$ is countable.

(b) Assume that $\mathcal{A}$ is locally countable and that F_i is a Lindelöf set for each $i \in I$. Each point of X has an open neighborhood which meets only countably many sets of $\mathcal{A}$; furthermore, for each $i \in I$, a sequence of these neighborhoods is a cover of F_i, hence the family

$$\mathcal{A}_i = \{U \in \mathcal{A}: U \cap F_i \neq \emptyset\}$$

is countable. Moreover, each $G \in \mathcal{G}_0$ meets only countably many sets of $\{F_i\}_{i \in I}$. Indeed, since

$$\sum_{i \in I} \mu(G \cap F_i) = \mu(G) < +\infty,$$

there is a countable subfamily J of I such that $\mu(G \cap F_i) = 0$ for each $i \notin J$, and as supp $\mu_{F_i} = F_i$, we have $G \cap F_i = \emptyset$ for each $i \notin J$. On the other hand, if $G \in G_0$, $U \in \mathcal{A}$ and $\mu(U \cap G) > 0$, then there is $i \in I$ such that $U \cap G \cap F_i \neq \emptyset$ for otherwise,

$$\mu(U \cap G) = \sum_{i \in I} \mu(U \cap G \cap F_i) = 0.$$

Thus, for each $U \in \mathcal{A}_G$ there is $i \in I$ such that $U \in \mathcal{A}_i$ and $G \cap F_i \neq \emptyset$, hence $\mathcal{A}_G \subset \bigcup_{i \in J} \mathcal{A}_i$ and $\mathcal{A}_G$ is countable.

(c) Assume that $\mathcal{A}$ is point-countable and that F_i is separable for each $i \in I$. For every $i \in I$ there is a countable set A_i with $\overline{A_i} = F_i$ and if $U \in \mathcal{A}$ and $U \cap F_i \neq \emptyset$, there is $x \in U \cap \overline{A_i}$. Thus $x \in \overline{A_i}$ and U is an open neighborhood of x, hence $U \cap A_i \neq \emptyset$. This proves that $\mathcal{A}_i$ is contained in the family

$$\{U \in \mathcal{A}: U \cap A_i \neq \emptyset\}$$

which is countable because $\mathcal{A}$ is point-countable and A_i is countable. Hence $\mathcal{A}_i$ is countable for each $i \in I$ and the proof is finished as in (b).

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