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ON SPACES OF CONTINUOUS FUNCTIONS WITH VALUES IN COECHELON SPACES

(Coechelon sequence space/space of vector-valued continuous functions/LB-space/bornological space/Mackey completion/Montel space)

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ABSTRACT

It is proved that for any coechelon space $k_p(V)$ of order p, $1 \leq p \leq \infty$, and any compact set K, the space of conti nuous functions C(K, $k_p(V)$) is bornological. This is a partial solution of the problem of Schmets and Bierstedt on bornologicity of LB-spaces of continuous functions. Moreover, if $k_p(V)$ is Montel, then C(K, $k_p(V)$) is even the local completion of $C(K) \otimes_\varepsilon k_p(V)$.

Grothendieck asked the still open question if every regular LB-space is necessarily complete [Bi, p. 78], [PCB, Problem 13.8.6]. In fact the problem is the key question from the whole complex of related problems. Its positive solution would imply answers to some other natural questions in the theory of LB-spaces.

Let us recall that the smallest locally complete space Y, $X \subseteq Y \subseteq \tilde{X}$, is called the *local completion* (or the *Mackey completion)* of a locally convex space X, where $\tilde{X}$ denotes the completion of X (see [PCB, 5.1.5 and 5.1.21]). It is known that the local completion of each LB-space is a regular LB-space [PCB, 6.2.8 and 7.3.3]. Thus the positive solution of the Grothendieck problem would imply that:

(i) the local completion of each LB-space is equal to its completion;

(ii) the completion of any LB-space must be an LB-space [PCB, Problem 13.8.1].

Now, let $E = \text{ind}_{n\in\mathbb{N}} E_n$ be any complete LB-space. Clearly, the space $F := \text{ind}_{n\in\mathbb{N}} C(K, E_n)$ is an LB-space for any compact K. It is known (see [Sch2, I.7.2] or [Mu]) that $C(K, E)$ is the completion of F and F contains $C(K) \otimes_\varepsilon E$ as a locally dense subspace. Thus the positive solution to the Grothendieck ploblem would imply:

(iii) $C(K, E)$ is the local completion of F or, equivalently, of $C(K) \otimes_\varepsilon E$;

(iv) $C(K, E)$ is bornological ([Schl, p. 103], comp. [PCB, Problem 13.6.2]).

Up to now, it is not known if any of the statements (i)—(iv) is generally true.

The statement (iii) is trivially true if $C(K,E) = F$ (i.e., E is a compactly regular LB-space, see [PCB, Def. 8.5.32]). It was proved in [DiDo2] that (iii) also holds if E is a coechelon space $k_\infty(V)$ of order ∞ and K is the one-point compactification of the natural numbers. As far as the author knows these are the only cases where (iii) has been established.

Our main result (Th. 3.1) shows that (iii) holds if E is an arbitrary Montel coechelon space $k_p(V)$ for any p, $1 \leq p \leq \infty$, and for an arbitrary compact set K. The proof is quite involved and somehow similar to that of [DiDo3]. It leads to a criterion on the range of a continuous function $h: K \to k_p(V)$ which implies that h belongs to the α-th Mackey derivative of F.

In the forthcoming paper [DD] J. C. Díaz and the author prove that for an arbitrary compact set K the corresponding space of *weakly* continuous E-valued functions is the Mackey completion of the inductive limit of spaces of *weakly* continuous E_n-valued functions for $E = \text{ind}_{n\in\mathbb{N}} E_n = k_\infty(V)$ if and only if $\lambda_1(A)$ is distinguished for $k_\infty(V) = \lambda_1(A)_i'$. Analogon of the sufficiency part is proved there also for the spaces of continuous functions.

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The question if (iv) is always true for any LB-space E was posed by Bierstedt and Schmets [Schl, p.103]. Only some partial solutions are known. If E is a Montel LB-space, then $C(K, E)$ is bornological for K either the one-point or the Stone-Čech compactification of the natural numbers (see [DiDo2, Th. 1], [BiBo2, Th. 1.5 (b) (2)]). Moreover, by [BoD, Obs. 9 (a)] (comp. [BiBo2, Prop. 2.9(b)] and [BDM, Cor. 11]), $C(K, k_p(V)) \simeq L_b(\lambda_q(A), C(K))$ is bornological for any Montel space $k_p(V)$, $1 \le p \le \infty$, where $k_p(V) \simeq \lambda_q(A)'_b$. If K is the one-point compactification of the natural numbers and $p = \infty$, then the same holds even for non-Montel $k_\infty(V)$ [DiDo2, Cor. 4]. Of course, the same holds if E is a compactly regular LB-space [Sch2, Th. IV.4.4] (see also [Me], [Bol], [BoS1] and [BoS2]) .

We also show (Theorem 1.3) that $C(K, k_p(V))$ is bornological for any compact set K and any coechelon space $k_p(V)$, $1 \le p \le \infty$. The proof is much more elementary than the proof of Th. 3.1.

Let us remark that for coechelon spaces $k_0(V)$ of order 0 both our results hold if only $k_0(V)$ is complete. Indeed, by [Bi, Th. 4.7 and p. 103], completeness of $k_0(V)$ implies compact regularity.

Our notation and terminology is standard and follows in general [J], [PCB] and [Bi]. We denote by $E^{(1)}$ the *Mackey derivative* of E, i.e., the set of all local limit points of E in the completion $\tilde{E}$. Inductively, we define Mackey derivatives of higher order:

$$E^{(\alpha+1)} := \left(E^{(\alpha)}\right)^{(1)} \quad \text{and} \quad E^{(\beta)} := \bigcup_{\alpha<\beta} E^{(\alpha)} \text{ for limit ordinals } \beta.$$

It is known that the local completion of E is equal to $\bigcup_\alpha E^{(\alpha)}$ where the union is taken over all countable ordinals α.

By V we always denote a matrix, (v_{ik}) with $v_{ik} \ge v_{ik+1} > 0$ for any $i, k \in \mathbb{N}$. Then for $a_{ik} := \frac{1}{v_{ik}}$, $A = (a_{ik})$ is a *Köthe matrix*. By *coechelon space of order p* we mean:

$$k_p(V) := \operatorname*{ind}_{k\in\mathbb{N}} l_p\left((v_{ik})_{i\in\mathbb{N}}\right), \text{ where}$$

$$l_p\left((v_{ik})_{i\in\mathbb{N}}\right) := \left\{x = (x_i) : \|x\|_k := \left(\sum_{i\in\mathbb{N}} |x_i|^p v_{ik}\right)^{1/p} < \infty\right\},$$

where the latter space is equipped with the norm $\|\cdot\|_k$ (for $p = \infty$ we take $\|x\|_k := sup_{i\in\mathbb{N}}|x_i|v_{ik}$). It is known that for $1 \le p \le \infty$ the space $k_p(V)$ is always a complete LB-space (even the inductive dual of some Frechet space), see [Bi, 2.9 and 2.10].

1. Bornological spaces of vector-valued continuous functions.

Let $(a_i)_{i\in\mathbb{N}}$ be a sequence of positive numbers and let $1 \le p \le \infty$. A subset D of the space of all sequences is called an *l_p-ball* with axes $(a_i)_{i\in\mathbb{N}}$ iff

$$D = \left\{x = (x_i) : \sum_{i\in\mathbb{N}} \left|\frac{x_i}{a_i}\right|^p \le 1\right\} \qquad \text{for } p < \infty,$$

$$D = \left\{x = (x_i) : \sup_{i\in\mathbb{N}} \frac{|x_i|}{a_i} \le 1\right\} \qquad \text{for } p = \infty.$$

Observe that the unit balls B_k in steps of $k_p(V)$ are the l_p-balls with axes $\left(a_{ik}^{1/p}\right)_{i\in\mathbb{N}}$. Moreover, each l_p-ball is pointwise closed in any coechelon space $k_p(V)$, $1 \le p \le \infty$.

Let us recall that a topology on a sequence space is called *locally solid* if it has a 0-neighbourhood basis consisting of sets U such that if $x \in U$, then every y smaller than x with respect to the pointwise order also belongs to U. In particular, each coechelon space $k_p(V)$, $1 \le p \le \infty$, has a locally solid topology.

Lemma 1.1. *Let $1 \le p \le \infty$ and let b be a positive number.*

(a) If D is the l_p-ball with axes (a_i), then bD is the l_p-ball with axes (ba_i).

(b) If $D_1,\ldots,D_n$ are the l_p-balls with axes $(a_{ik})_{i\in\mathbb{N}}$, $k = 1,\ldots, n$, resp., then $\sum_{k=1}^n D_k \subseteq D$, where D is the l_p-ball with axes $\left(\sum_{k=1}^n 2^k a_{ik}\right)_{i\in\mathbb{N}}$.

(c) If D is the l_p-ball with axes $\left(\sum_{k=1}^n w_{ik}\right)$ and C_k are the l_p-balls with axes $(w_{ik})_{i\in\mathbb{N}}$ respectively, then there are functions f_k: $D \to C_k$, $k = 1,\ldots, n$, continuous with respect to any linear locally solid topology on lin $\left(\sum_{k=1}^n C_k\right)$ *and satisfying $\sum_{k=1}^n f_k(x) = x$ for any $x \in D$. In particular, $D \subseteq \sum_{k=1}^n C_k$.*

Proof. (a): Obvious.

(b): Let $x = x_1 + \ldots + x_n$, $x_k = (x_{ik})_{i\in\mathbb{N}} \in D_k$. Then

$$\left(\sum_{i\in\mathbb{N}} \left|\frac{\sum_{k=1}^n x_{ik}}{\sum_{j=1}^n 2^j a_{ij}}\right|^p\right)^{1/p} \le \sum_{k=1}^n \left(\sum_{i\in\mathbb{N}} \frac{|x_{ik}|^p}{\left|\sum_{j=1}^n 2^j a_{ij}\right|^p}\right)^{1/p}$$

$$\le \sum_{k=1}^n \left(\sum_{i\in\mathbb{N}} \frac{|x_{ik}|^p}{2^{kp} a_{ik}^p}\right)^{1/p} \le \sum_{k=1}^n 2^{-k} \le 1.$$

(c): Let $x = (x_i) \in D$, then we define

$$f_k(x) := \left(\frac{w_{ik} x_i}{\sum_{j=1}^n w_{ij}}\right)_{i\in\mathbb{N}}.$$

Moreover, we get

$$\sum_{i\in N}\left|\frac{w_{ik}x_i}{\sum_{j=1}^{n} w_{ij}}\right|^p \frac{1}{w_{ik}^p} \le \sum_{i\in N}\frac{|x_i|^p}{\left|\sum_{k=1}^{n} w_{ik}\right|^p} \le 1$$

and $f_k(x) \in C_k$. It is clear that $|f_k(x) - f_k(y)| \le |x-y|$, which implies the required continuity.

Lemma 1.2. *Let K be a compact set in an LB-space $E = ind_{n\in N} E_n$ having a weaker topology which makes the unit balls B_n of E_n compact. Then for any sequence of positive numbers $(b_k)_{k\in N}$ there is $m \in \mathbb{N}$ such that for any $x \in K$ the set $\left(x + \sum_{k=1}^{m} b_k B_k\right) \cap K$ is a neighbourhood of x in K.*

Proof. Since $2K$ is metrizable (see [Pf], [CaOr1] and [CaOr2]), we may apply density type arguments of Bierstedt and Bonet [DiDo1, Th. 1.3] (comp. [BiBo1], [BiBo2]). Thus $\overline{\sum_{k=1}^{m} b_k B_k} \cap 2K$ is a 0-neighbourhood in 2K for some $m \in \mathbb{N}$. Since there is a weaker topology making B_k compact, $\sum_{k=1}^{m} b_k B_k = \overline{\sum_{k=1}^{m} b_k B_k}$. Finally, there is a 0-neighbourhood U in E such that

$$U \cap 2K \subseteq \sum_{k=1}^{m} b_k B_k.$$

For $x \in K$:

$$(x + U) \cap K \subseteq (x + (U \cap 2K)) \subseteq \left(x + \sum_{k=1}^{m} b_k B_k\right).$$

Theorem 1.3. For *any $1 \le p \le \infty$, any matrix V as above and any compact set K, the space $C(K, k_p(V))$ is bornological.*

Proof. Let us take B_k to be the unit ball in $l_p((v_{ik})_{i\in N})$, as noted above B_k is the l_p-ball with axes $\left(a_{ik}^{1/p}\right)_{i\in N}$. We will show that for any $(b_k)_{k\in N}$ the bornivorous set

$$W := \bigcup_{m\in N}\sum_{k=1}^{m} b_k C(K, B_k)$$

contains the 0-neighbourhood

$$U := C\left(K, \bigcup_{m\in N}\sum_{k=1}^{m} 2^{-k-1} b_k B_k\right).$$

Let $h \in U$, then $h(K) \subseteq \bigcup_{m\in N}\sum_{k=1}^{m} 2^{-k-1} b_k B_k$. By Lemma 1.2, there are finitely many elements $\{t_i: i = 1,..., l\} \subseteq K$, $n \in \mathbb{N}$, and neighbourhoods W_i of t_i in K such that

$$\bigcup_{i=1}^{l} W_i = K \text{ and } h(W_i) \subseteq h(t_i) + \sum_{k=1}^{n} 2^{-k-1} b_k B_k.$$

On the other hand

$$h(t_i) \in \sum_{k=1}^{m_i} 2^{-k-1} b_k B_k.$$

Thus, for $m = \max(m_1,..., m_l, n)$ we have:

$$h(K) \subseteq \sum_{k=1}^{m} 2^{-k} b_k B_k.$$

By Lemma 1.1.(b),

$$\sum_{k=1}^{m} 2^{-k} b_k B_k \subseteq D,$$

where D is the l_p-ball with axes $\left(\sum_{k=1}^{m} b_k a_{ik}^{1/p}\right)_{i\in N}$. By Lemma 1.1 (c), there are continuous functions f_k: $D \to b_k B_k$, $k = 1,..., m$, such that $\sum_{k=1}^{m} f_k(x) = x$ for any $x \in D$. Finally, $h = \sum_{k=1}^{m} f_k \circ h$ and $f_k \circ h \in C(K, b_k B_k)$; this completes the proof.

2. Combinatorial preliminaries. In the proof of our second main result we will use some combinatorial arguments contained in [DiDo3]. For the sake of completeness we give them also here.

We call a family T of finite sequences of natural numbers a *tree* if

(i) $\emptyset \in T$;

(ii) $(n_0,...,n_k) \in T \Rightarrow (n_0,...,n_{k-1}) \in T$.

We call T a *blooming tree* if additionally

(iii) $(n_0,...,n_k) \in T \Rightarrow (n_0,..., n_{k-1}, l) \in T$ for each $l \in \mathbb{N}$;

(iv) for any infinite sequence $(n_i)_{i\in N} \subseteq \mathbb{N}$ there is m such that $(n_0,...,n_m) \notin$ T.

A sequence of elements of the tree of the form:

$$(n_0,...,n_k),\ (n_0,...,n_k, n_{k+1}),...,\ (n_0,...,n_k,...,n_l)$$

is called a *branch* of the tree. The *rank* of a blooming tree T will be the crucial notion used in the paper. If $T = \{\emptyset\}$, then rank(T) := 0, otherwise we define the rank as follows. First we construct a new blooming tree $T^{(1)}$ equal to

$$\{(n_0,...,n_k) \in T: \forall m \exists (l_k,...,l_{k+m}): (n_0,...,n_{k-1}, l_k,...,l_{k+m}) \in T\} \cup \{\varnothing\}.$$

It is easily seen that if $T^{(1)} = \{\varnothing\}$, then there is m such that no sequence $(l_0,...,l_m)$ belongs to T. Thus the tree T is bounded, i.e., all branches of T have length bounded by some fixed m.

We have the whole family of blooming (!) trees defined inductively:

$$T^{(\alpha+1)} := \left(T^{(\alpha)}\right)^{(1)} \text{ and } T^{(\beta)} := \bigcap_{\alpha<\beta} T^{(\alpha)} \text{ for limit ordinals } \beta.$$

It is known that the family strictly decreases to $\{\varnothing\}$ and thus for every blooming tree T there exists a countable ordinal number α such that $T^{(\alpha)} = \{\varnothing\}$ [DiDo3, Prop. 5 and Cor. 6]. We define rank(T) to be the minimal ordinal number α such that $T^{(\alpha)} = \{\varnothing\}$. It is easily seen, that if $T_1 \subseteq T_2$, then rank(T_1) $\leq$ rank(T_2).

Proposition 2.1. *The ordinal number* rank *(T) is never a limit ordinal.*

Proof. If rank$(T) = \sup_n \alpha_n$, then there is $n \in \mathbb{N}$ such that (1) $\notin T^{(\alpha_n)}$ and, by (iii) and (ii), $T^{(\alpha_n)} = \{\varnothing\}$.

Lemma 2.2. *Let T be a blooming tree and let $(\lambda_1,...,\lambda_m) \notin T^{(\alpha)}$, then* rank *$(S) \leq \alpha$, whenever*

$$S := \{(\eta_k)_{k=1,...,l} : (\lambda_1,...,\lambda_m, \eta_{m+1},...,\eta_l) \in T\}.$$

Proof. By the same argument as in the proof of Prop. 2.1, there is $\gamma < \alpha$ such that

$$(\lambda_1,...,\lambda_m) \notin T^{(\gamma+1)}.$$

As easily seen

$$S^{(\gamma)} = \{(\eta_k)_{k=1,...,l} : (\lambda_1,...,\lambda_m, \eta_{m+1},...,\eta_l) \in T^{(\gamma)}\}.$$

Since $(\lambda_1,...,\lambda_m) \notin T^{(\gamma+1)}$, there is $n \in \mathbb{N}$ such that for each

$$(\lambda_1,...,\lambda_m, \eta_{m+1},...,\eta_l) \in T^{(\gamma)}$$

we have $l \leq n$. Thus $S^{(\gamma+1)} = \{\varnothing\}$ and rank$(S) \leq \gamma + 1 \leq \alpha$.

3. The completion and the local completion of $C(K) \otimes_\varepsilon k_p(V)$ coincide in the Montel case. We will prove the following main result:

Theorem 3.1. *Let $k_p(V)$, $1 \leq p \leq \infty$, be a Montel coechelon space and let K be an arbitrary compact set. The space $C(K, k_p(V))$ is the local completion of the space $ind_{k\in\mathbb{N}}\, C(K, l_p((v_{ik})_{i\in\mathbb{N}}))$ or, equivalently, of $C(K) \otimes_\varepsilon k_p(V)$.*

Remark. Since the local completion of any LB-space is an LB-space [PCB, 7.3.3 and 6.2.8], the above result gives another proof of bornologicity of $C(K, k_p(V))$ in case of Montel $k_p(V)$.

The proof will be based on a sequence of lemmas.

Lemma 3.2. *Let $1 \leq p \leq \infty$ and let C, D be the l_p-balls with axes (u_i) and (w_i) respectively. If there is a finite set $\{y_1,...,y_r\} \subseteq D$ such that $D \subseteq \bigcup_{l=1}^r y_l + C$, then*

$$\exists j\, \forall i \geq j \quad u_i \geq w_i.$$

Proof. Let us assume that for each j there is $i > j$ such that $w_i > u_i$. Thus there are $i_1,...,i_{r+1}$ such that $w_{i_l} > (1 + \delta)u_{i_l}$, for $l = 1,...,r+1$ and some $\delta > 0$. Let P be the projection on the linear span X of the unit vectors with indices $i_1,...,i_{r+1}$ and let q_D, q_C be the gauge functionals of $P(D)$ and $P(C)$ respectively. Clearly, $(1 + \delta)q_D \leq q_C$. The linear space $Y = \mathrm{lin}\{P(y_l): l = 1,...,r\}$ is a proper closed subspace of X. By the Riesz Lemma [Ds, p.2], there is a vector $x \in P(D)$ such that its q_D-distance from Y is bigger than $\left(1/\left(1+\frac{\delta}{2}\right)\right)$. Of course, the q_C-distance of x from Y is bigger than $(1+\delta)/\left(1+\frac{\delta}{2}\right) > 1$, which contradicts the assumption $P(D) \subseteq \bigcup_{l=1}^r P(y_l) + P(C)$.

Lemma 3.3. *Let $1 \leq p \leq \infty$ and let the l_p-ball D with axes $(w_i)_{i\in\mathbb{N}}$ be a compact subset of the coechelon space $k_p(V)$. Then for each sequence (b_k) of positive numbers the following holds:*

$$\exists n, j\, \forall i \geq j \quad \left(\sum_{k=1}^n b_k a_{ik}^{1/p}\right) \geq w_i.$$

Proof. Let us denote by B_n the unit balls in the steps of $k_p(V)$. Then B_k is the l_p-ball with axes $\left(a_{ik}^{1/p}\right)_{i\in\mathbb{N}}$. By Lemma 1.2, there is n and a finite set $\{y_1,...,y_r\} \subseteq D$ such that

$$D \subseteq \bigcup_{j=1}^r y_j + \sum_{k=1}^n 2^{-k} b_k B_k.$$

By Lemma 1.1 (b),

$$D \subseteq \bigcup_{j=1}^r y_j + D_0,$$

where D_0 is the l_p-ball with axes $\left(\sum_{k=1}^n b_k a_{ik}^{1/p}\right)_{i\in\mathbb{N}}$. Applying Lemma 3.2 we get the conclusion.

Now, we define a hierarchy of families of sequences of positive numbers. Let us fix a matrix $V = (v_{ik})$ as usual with $a_{ik} = \frac{1}{v_{ik}}$. A sequence of positive numbers $(w_i) \in S_1$ iff there is $k \in \mathbb{N}$ such that $w_i = o\left(a_{ik}^{1/p}\right)$ as $i \to \infty$. Let S_α be defined for all ordinal numbers $\alpha < \beta$. If β is a limit ordinal, then $S_\beta := \bigcup_{\alpha<\beta} S_\alpha$. Otherwise, $\beta = \alpha + 1$ and $(w_i) \in S_\beta$ if and only if the following condition holds:

$$\exists k \, \forall \varepsilon > 0 \qquad w_i = \varepsilon a_{ik}^{1/p} + z_i, \text{ where } (z_i) \in S_\alpha \text{ depends on } \varepsilon.$$

It is easily seen that if $(w_i) \in S_\alpha$ and $u_i \le w_i$ for every i, then $(u_i) \in S_\alpha$ as well.

Lemma 3.4. *Let $1 \le p \le \infty$ and let $w = (w_i)_{i\in\mathbb{N}}$ be a sequence of positive numbers. If the l_p-ball D with axes $(w_i)_{i\in\mathbb{N}}$ is a compact subset of the coechelon space $k_p(V)$, then there is a countable ordinal α such that $(w_i) \in S_\alpha$.*

Proof. For any sequence $w = (w_i)$ of axes of a compact l_p-ball we define a tree T_w as follows. For every sequence of natural numbers $(\lambda_k)_{k\in\mathbb{N}}$ we say that $(\lambda_1,\dots,\lambda_n) \in T_w$ iff

$$\forall j \, \exists i \ge j \sum_{k=1}^{n-1} \lambda_k^{-1} a_{ik}^{1/p} < w_i.$$

It is easily seen that the relation $(\lambda_1,\dots,\lambda_n) \in T_w$ does not depend on λ_n. Thus, by Lemma 3.3, T_w is a blooming tree. If the tree is bounded (i.e., rank$(T_w) = 1$), then

$$\exists n \, \forall (\lambda_k)_{k\in N} \, \exists j \, \forall i \ge j \quad \sum_{k=1}^{n} \lambda_k^{-1} a_{ik}^{1/p} \ge w_i.$$

In particular, if $\sum_{k=1}^{\infty} \lambda_k^{-1} < \varepsilon$, then

$$\exists j \, \forall i \ge j \quad w_i \le \varepsilon a_{in}^{1/p}.$$

We have proved that $(w_i) \in S_1$.

Now, assume that for $\alpha < \beta$, if rank$(T_w) \le \alpha$, then $w = (w_i) \in S_\alpha$. We will show that if rank$(T_w) \le \beta$ then $w = (w_i) \in S_\beta$.

If β is a limit ordinal, then (Prop. 2.1) rank$(T_w) < \beta$, i.e., there exists $\alpha < \beta$ such that rank$(T_w) \le \alpha$ and $(w_i) \in S_\alpha \subseteq S_\beta$.

If $\beta = \alpha + 1$ and rank$(T_w) = \beta$, then $T_w^{(\alpha)}$ is a bounded tree. Let $(\lambda_k)_{k\in\mathbb{N}}$ satisfy $\sum_{k\in\mathbb{N}} \lambda_k^{-1} < \varepsilon$. There is m not depending on ε such that

$$(\lambda_1,\dots,\lambda_m) \notin T_w^{(\alpha)}.$$

If $(\lambda_1,\dots,\lambda_m, \eta_{m+1},\dots,\eta_l) \notin T_w$, then

$$\exists j \, \forall i \ge j \sum_{k=1}^{m} \lambda_k^{-1} a_{ik}^{1/p} + \sum_{k=m+1}^{l-1} \eta_k^{-1} a_{ik}^{1/p} \ge w_i.$$

We take $q_i := \min\left(w_i, \sum_{k=1}^{m} \lambda_k^{-1} a_{ik}^{1/p}\right)$, $s_i := w_i - q_i$. Since $s_i \le w_i$ for any $i \in \mathbb{N}$, the l_p-ball with axes (s_i) is also compact in $k_p(V)$. Clearly, if for $(\eta_k)_{k\in\mathbb{N}}$, the sequence $(\lambda_1,\dots,\lambda_m, \eta_{m+1},\dots,\eta_l)$ does not belong to T_w, then

$$\exists j \, \forall i \ge j \sum_{k=m+1}^{l-1} \eta_k^{-1} a_{ik}^{1/p} \ge s_i.$$

We have proved that the tree T_s of the sequence $s = (s_i)$ is contained in the tree

$$S := \left\{ (\eta_k)_{k=1,\dots,l} : (\lambda_1,\dots,\lambda_m, \eta_{m+1},\dots,\eta_l) \in T_w \right\}.$$

By Lemma 2.2, rank$(T_s) \le$ rank$(S) \le \alpha$ and $(s_i) \in S_\alpha$ by the inductive hypothesis. We obtain $w_i = q_i + s_i$, where $q_i \le \varepsilon a_{im}^{1/p}$, $(s_i) \in S_\alpha$, hence $(w_i) \in S_\beta$.

Lemma 3.5. *Let $1 \le p \le \infty$ and let K be a compact set. If $h\colon K \to k_p(V)$ is a continuous function, with $h(K)$ contained in the l_p-ball D with axes (w_i) and $(w_i) \in S_\alpha$, then $h \in F^{(\alpha)}$ (the α-th Mackey derivative), where*

$$F := \operatorname*{ind}_{k\in N} C\left(K, l_p\left((v_{ik})_{i\in N}\right)\right).$$

Proof. If $(w_i) \in S_1$, then there is k such that $w_i = o\left(a_{ik}^{1/p}\right)$ and D is compact in $l_p((v_{ik})_{i\in\mathbb{N}})$ This means that the topology of the latter space and the one of $k_p(V)$ coincide on D and $h \in C\left(K, l_p\left((v_{ik})_{i\in N}\right)\right) \subseteq F \subseteq F^{(1)}$.

Now, let us assume that the result holds for $\alpha < \beta$. Take $(w_i) \in S_\beta$. If β is a limit ordinal, then $(w_i) \in S_\alpha$ for some $\alpha < \beta$ and we are done.

Let $\beta = \alpha + 1$, for some $k \in \mathbb{N}$ and each $\varepsilon > 0$

$$w_i = \varepsilon a_{ik}^{1/p} + z_i,$$

where $(z_i) \in S_\alpha$ depends on ε. Let C be the l_p-ball with axes (z_i). By Lemma 1.1 (c), there are continuous maps $f\colon D \to \varepsilon B_k$, $g\colon D \to C$ such that $f(x) + g(x) = x$ for each $x \in D$. Clearly, $g \circ h\colon K \to C$ and, by the inductive hypothesis, $g \circ h \in F^{(\alpha)}$. On the other hand, $f \circ h\,(K) \subseteq \varepsilon B_k$ and $h = g \circ h + f \circ h$. We have shown that $h \in F^{(\alpha+1)}$.

Proof of Theorem 3.1. If $k_p(V)$ is a Montel space, then the unit balls B_k are compact. Since they are l_p-balls, then, by Lemma 3.4, for each k there is a countable ordinal α_k such that $\left(a_{ik}^{1/p}\right)_{i\in N} \in S_{\alpha_k}$. Now, if $h : K \to k_p(V)$ is continuous, then without loss of generality we may assume that $h(K)$ is contained in some B_k. By Lemma 3.5, $h \in F^{(\alpha_k)}$; this completes the proof.

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