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REVISITED ISOPERIMETRIC INEQUALITIES FOR THE p-CAPACITY AND APPLICATION TO THE MUSKAT PROBLEM

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ABSTRACT

We give a unified form to various isoperimetric inequalities of p-capacity type and we present an application to a Muskat problem with prescribed flux.

1. INTRODUCTION

Let $1 < p < \infty$ be a real number, N a positive integer and p' the conjugate of p: $\frac{1}{p} + \frac{1}{p'} = 1$. Let $\omega_1 \subset\subset \omega_2$ be given bounded open sets in $\mathbb{R}^N$ having respective boundaries $\partial\omega_1 = \gamma_1$, $\partial\omega_2 = \gamma_2$ and Lebesgue measures m_1, m_2. We define the domain $\Omega = \omega_2 \setminus \overline{\omega}_1$. We denote by $\xi_1 \cdot \xi_2$ the inner product of ξ_1 and $\xi_2 \in \mathbb{R}^N$, by $|\xi|$ the Euclidean norm of $\xi \in \mathbb{R}^N$, by $|\omega|$ the Lebesgue measure of a measurable subset $\omega \subset \mathbb{R}^N$ and by β_N the Lebesgue measure of the unit ball of $\mathbb{R}^N$.

Let α be a function of $L^\infty(\Omega)$, positive almost everywhere with $\frac{1}{\alpha} = \alpha^{-1} \in L^\infty(\Omega)$. Let $u \in W^{1,p}(\Omega)$ be such that

(1.1) $\quad u_{|\gamma 1} = \text{constant} > u_{|\gamma 2} = \text{constant}$

and $\sigma \in L^{p'}(\Omega)^N$ be a vector field which is divergence free (in the sense of distributions):

(1.2) $\quad -\operatorname{div} \sigma = 0$ in Ω.

Furthemore, we assume that the pair (u, σ) satisfies the inequality

(1.3) $\quad \sigma \cdot \nabla u \geq \alpha |\nabla u|^p$ a.e. in Ω.

We will provide some examples of vector fields σ with their underlying functions α.

In this paper, we show that (u, σ) satisfies a general isoperimetric inequality which brings in a function U of $W^{1,p}\left(\tilde{\Omega}\right)$ verifying

$$\begin{cases} -\,div\, \Sigma = -\,div\, \tilde{\mathcal{A}}\, U = 0 \text{ in } \tilde{\Omega}, \\ U_{|\tilde{\gamma}1} = \text{constant} > U_{|\tilde{\gamma}2} = \text{constant}, \end{cases}$$

with

- $\tilde{\Omega} = \tilde{\omega_2} / \overline{\tilde{\omega}_1}$, where $\tilde{\omega_i}$ are the balls of $\mathbb{R}^N$ centered at the origin and having the same measures as ω_i and $\tilde{\gamma}_i = \partial\tilde{\omega}_i$ for $i = 1$ or 2,

- $\sum(x) = \left(\tilde{\mathcal{A}}\, U\right)(x) = \tilde{\alpha}(x) \left|\nabla U(x)\right|^{p-2} \nabla U(x)$ where $\tilde{\alpha}$ is the spherical radially increasing rearrangement of α on $\tilde{\Omega}$. We will give later on a precise definition of this rearrangement introduced by A. Alvino and G. Trombetti [AlTr1, AlTr2].

Remark 1. The constant values $U_{|\tilde{\gamma}_1}$ and $U_{|\tilde{\gamma}_2}$ are not necessarily the same as $u_{|\gamma_1}$ and $u_{|\gamma_2}$.

Remark 2. Indeed, the condition (1.1) can be replaced by

$$u_{|\gamma 1} = \text{constant} < u_{|\gamma 2} = \text{constant}$$

(set $u' = -u$ and $\sigma' = -\sigma$).

Our main tool is the theory of rearrangement of functions of Sobolev type, introduced by G. Talenti [Ta].

We will present some applications of this general result to various problems of Mathematical Physics such as the Muskat problem, a model arising from Oil Engineering.

2. EXAMPLES

We give in this section some examples of vector fields σ with their corresponding functions α.

Let $A : \Omega \to \mathbb{R}^{N\times N}$ be a matrix with measurable coefficients defined almost everywhere in Ω and

$g : \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N\ (x, \zeta, \xi, \eta) \to g(x, \zeta, \xi, \eta)$ a function defined for almost every x in Ω and for any $(\zeta, \zeta, \eta) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N$.

We assume that the function g and the matrix A are related by the following hypothesis:

$$\begin{cases} \text{There exists a functions } \alpha : \Omega \to \mathbb{R},\ \alpha \in L^\infty(\Omega),\ \alpha > 0 \text{ and } \frac{1}{\alpha} = \alpha^{-1} \in L^\infty(\Omega) \\ \text{such that a.e. } x \in \Omega,\ \forall \zeta \in \mathbb{R},\ \forall \xi \in \mathbb{R}^N,\ g(x, \zeta, \xi, A(x)\xi)\cdot\xi \ge \alpha(x)|\xi|^p. \end{cases}$$

For $v \in W^{1,p}(\Omega)$, we denote by $\mathcal{A}v$ the function defined for almost every x in Ω by

$$\mathcal{A}v(x) = g(x, v(x), \nabla v(x), A(x)\nabla v(x)).$$

We assume that u is a function of $W^{1,p}(\Omega)$ verifying

- $\mathcal{A}u \in (L^{p'}(\Omega))^N$,
- $-\operatorname{div} \mathcal{A}u = 0$ in Ω (in the sense of distributions),
- $u_{|\gamma_1} = \text{constant} > u_{|\gamma_2} = \text{constant}$.

The vector field σ is then given by $\mathcal{A}u$. We precise hereafter some operators g:

1. We consider $g(x, \zeta, \xi, \eta) \equiv g(x, \eta)$ satisfying the condition $g(x, \eta)\cdot\eta \ge \rho(x)|\eta|^p$ with $\rho(x) > 0$. We choose the matrix $A(x) \equiv a(x)\ Id$ *with* $a(x) > 0$. We get

$$g(x, \zeta, \xi, A(x)\xi)\cdot\xi = g(x, a(x)\xi)\cdot\xi$$
$$= \frac{1}{a(x)} g(x, a(x)\xi)\cdot a(x)\xi \ge \rho(x)(a(x))^{p-1}|\xi|^p.$$

We suppose that the function α defined by $\alpha(x) = \rho(x)(a(x))^{p-1}$ for almost every $x \in \Omega$, belongs to $L^\infty(\Omega)$ as well as α^{-1}. The equation satisfied by u is

$$-\operatorname{div} g(x, a(x)\nabla u(x)) = 0 \text{ in } \Omega.$$

2. We choose a function $g(x, \zeta, \xi, \eta) \equiv \rho(x)(\eta\cdot\xi)^{\frac{p}{2}-1}\eta$ with $\rho(x) > 0$ and a matrix A such that a.e. $x \in \Omega$, $\forall\xi \in \mathbb{R}^N$, $A(x)\xi\cdot\xi \ge a(x)|\xi|^2$ with $a(x) > 0$. Then, we have

$$g(x, \zeta, \xi, A(x)\xi)\cdot\xi = \rho(x)(A(x)\xi\cdot\xi)^{\frac{p}{2}} \ge \rho(x)(a(x))^{\frac{p}{2}}|\xi|^p.$$

In this case, we assume that the function $\alpha(x) = \rho(x)(a(x))^{\frac{p}{2}}$ is in $L^\infty(\Omega)$ as well as its inverse. The equation satisfied by u is

$$-\operatorname{div}\left[\rho(x)(A(x)\nabla u(x)\cdot\nabla u(x))^{\frac{p}{2}-1}A(x)\nabla u(x)\right] = 0 \text{ in } \Omega.$$

We recover the operator of [Bo1, Bo2, BoMos1, BoMos2, Mos].

3. Let $g(x, \zeta, \xi, \eta) \equiv g(x, \zeta, \xi)\cdot\xi \ge C|\xi|^p$ where C is a real positive constant. Here α is a function defined by $\alpha(x) = C$ for $x \in \Omega$. The function u verifies the equation

$$-\operatorname{div} g(x, u(x), \nabla u(x)) = 0 \text{ in } \Omega.$$

This case has been studied by J.I. Diaz [Di].

3. ISOPERIMETRIC INEQUALITIES

First, we will prove a general isoperimetric inequality. We recall that the pair (u, σ) is a solution of the problem denoted by $(\mathcal{P})$:

$$(\mathcal{P}) \begin{cases} u \in W^{1,p}(\Omega), \\ u_{|\gamma_1} = \text{constant} > u_{|\gamma_2} = \text{constant}, \\ \sigma \in (L^{p'}(\Omega))^N, \\ -\operatorname{div}\sigma = 0 \text{ in } \Omega, \\ \sigma\cdot\nabla u \ge \alpha|\nabla u|^p \text{ a.e. in } \Omega \end{cases}$$

and the pair (U, Σ) is the solution of $(\tilde{\mathcal{P}})$:

$$(\tilde{\mathcal{P}}) \begin{cases} U \in W^{1,p}(\tilde\Omega),\ \Sigma = \tilde{\mathcal{A}}U = \tilde\alpha|\nabla U|^{p-2}\nabla U \in (L^{p'}(\Omega))^N, \\ U_{|\tilde\gamma_1} = \text{constant} > U_{|\tilde\gamma_2} = \text{constant}, \\ -\operatorname{div}\Sigma = 0 \text{ in } \tilde\Omega. \end{cases}$$

The function $\tilde\alpha$ is defined on $\tilde\Omega$ by $\tilde\alpha(x) = \alpha^*(\beta_N|x|^N - |\omega_1|)$ where β_N is the measure of the unit ball of $\mathbb{R}^N$, $|\omega_1|$ and $|x|$ are respectively the Lebesgue measure on $\mathbb{R}^N$ of ω_1 and the Euclidean norm of the vector $\overrightarrow{Ox}$, α^* is the unidimensional increasing rearrangement of α, defined on $\overline{\Omega^*} = [0, |\Omega|]$ by $\alpha^*(0) = \operatorname{ess\,inf}\alpha$, $\alpha^*(|\Omega|) = \operatorname{ess\,sup}\alpha$, $\alpha^*(s) = \inf\{\theta \in \mathbb{R}, |\alpha < \theta| \ge s\}$ for $s \in (0, |\Omega|)$, with $|\alpha < \theta|$ = Lebesgue measure of $\{x \in \Omega, \alpha(x) < \theta\}$. Similarly, we define α_* the unidimensional decreasing rearrangement of α on $\overline{\Omega^*} = [0, |\Omega|]$ by $\alpha_*(s) = \alpha^*(|\Omega| - s)$ for $s \in [0, |\Omega|]$.

Lemma 1. *Under the conditions (1.1), (1.2) and (1.3), we have for almost every* $x \in \Omega$

$$u_{|\gamma_1} \ge u(x) \ge u_{|\gamma_2} \text{ and } \int_\Omega \sigma\cdot\nabla u\,dx > 0.$$

Proof: Let us show that for almost every $x \in \Omega$, we have $u_{|\gamma 1} \geq u(x)$. Since $\sigma \in (L^{p'}(\Omega))^N$ and $-\operatorname{div} \sigma = 0$ in Ω in the sense of distributions, we get

$$(3.1) \quad \int_\Omega \sigma \cdot \nabla w\, dx = 0, \quad \forall w \in W_0^{1,p}(\Omega).$$

We take as test function $w_1 = \left(u - u_{|\gamma 1}\right)_+$, the positive part of $u - u_{|\gamma 1}$. We have by (3.1) and then by (1.3),

$$0 = \int_\Omega \sigma \cdot \nabla \left(u - u_{|\gamma 1}\right)_+ dx = \int_{u > u_{|\gamma 1}} \sigma \cdot \nabla u\, dx$$

$$\geq \int_{u > u_{|\gamma 1}} \alpha(x) |\nabla u|^p dx = \int_\Omega \alpha(x) \left|\nabla \left(u - u_{|\gamma 1}\right)_+\right|^p dx.$$

Since $\alpha > 0$, we deduce that $w_1 = \left(u - u_{|\gamma 1}\right)_+ = \text{constant} = 0$ in Ω. It holds $u(x) \leq u_{|\gamma 1}$ for almost every $x \in \Omega$. Similarly, using the test function $w_2 = \left(u_{|\gamma 2} - u\right)_+ = \left(u - u_{|\gamma 2}\right)_-$, we show that for almost every $x \in \Omega$, $u_{|\gamma 2} \leq u(x)$.

Let us show now that the quantity $\int_\Omega \sigma \cdot \nabla u\, dx$ is positive. Assume that $\int_\Omega \sigma \cdot \nabla u\, dx$ vanishes. Using the inequality (1.3), we deduce easily that u is constant in each connected components of Ω. This is in contradiction with $u_{|\gamma 1} > u_{|\gamma 2}$.

The general isoperimetric inequality is given in the following proposition.

Proposition *Assume that (u, σ) verifies ($\mathcal{P}$). Let*

$$C(u, \sigma) = \int_\Omega \sigma \cdot \nabla u\, dx.$$

Then, for all real numbers θ, θ' satisfying $u_{|\gamma 2} \leq \theta \leq \theta' \leq u_{|\gamma 1}$, the following isoperimetric inequality holds

$$\theta' - \theta \leq N^{-p'} \beta_N^{\frac{-p'}{N}} \left(\frac{C(u, \sigma)}{u_{|\gamma 1} - u_{|\gamma 2}}\right)^{\frac{p'}{p}}$$

$$\int_{\mu(\theta')}^{\mu(\theta)} (s + m_1)^{\frac{p'}{N} - p'} \left(\alpha^*\right)^{\frac{-p'}{p}} \left(s - \mu(\theta')\right) ds$$

with $\mu(\theta) = |u > \theta| =$ measure of $\{x \in \Omega,\ u(x) > \theta\}$, $\mu(\theta') = |u > \theta'|$, $m_1 = |\omega_1|$, $\left(\alpha^*\right)^{\frac{-p'}{p}}(s - \mu(\theta'))$ indicates the value of the function $\left(\alpha^*\right)^{\frac{-p'}{p}}$ at the point $s - \mu(\theta')$.

Proof: We follow L. Boukrim [Bo1, Bo2] for the proof of this proposition. By Lemma 1, any value of u is in $\left[u_{|\gamma 2}, u_{|\gamma 1}\right]$. What follows is valid for almost every $\tau \in \left(u_{|\gamma 2}, u_{|\gamma 1}\right)$. We set

$$z_\tau = \tau - (u - \tau)_- = \begin{cases} u \text{ if } u \leq \tau \\ \tau \text{ if } u > \tau, \end{cases}$$

$$v_\tau = \frac{u - u_{|\gamma 2}}{u_{|\gamma 1} - u_{|\gamma 2}} - \frac{z_\tau - u_{|\gamma 2}}{\tau - u_{|\gamma 2}} = \begin{cases} \left(u - u_{|\gamma 2}\right)\left(\dfrac{1}{u_{|\gamma 1} - u_{|\gamma 2}} - \dfrac{1}{\tau - u_{|\gamma 2}}\right) & \text{if } u \leq \tau \\ \dfrac{u - u_{|\gamma 2}}{u_{|\gamma 1} - u_{|\gamma 2}} - 1 & \text{if } u > \tau. \end{cases}$$

Since $u \in W^{1,p}(\Omega)$, we have $v_\tau \in W_0^{1,p}(\Omega)$. Taking $w = v_\tau$ in (3.1), it follows

$$0 = \int_\Omega \sigma \cdot \nabla v_\tau dx = \left(\frac{1}{u_{|\gamma_1} - u_{|\gamma_2}} - \frac{1}{\tau - u_{|\gamma_2}}\right)$$

$$\int_{u \leq \tau} \sigma \cdot \nabla u\, dx + \frac{1}{u_{|\gamma_1} - u_{|\gamma_2}} \int_{u > \tau} \sigma \cdot \nabla u\, dx,$$

that is

$$\int_{u \leq \tau} \sigma \cdot \nabla u\, dx = \frac{\tau - u_{|\gamma 2}}{u_{|\gamma 1} - u_{|\gamma 2}} C(u, \sigma).$$

In consequence, one has

(3.2)

$$-\frac{d}{d\tau} \int_{u > \tau} \sigma \cdot \nabla u\, dx = \frac{d}{d\tau} \int_{u \leq \tau} \sigma \cdot \nabla u\, dx = \frac{C(u, \sigma)}{u_{|\gamma 1} - u_{|\gamma 2}}.$$

On the other hand, for $h > 0$

$$\frac{1}{h} \int_{\tau < u \leq \tau + h} |\nabla u|\, dx = \frac{1}{h} \int_{\tau < u \leq \tau + h} \alpha^{\frac{-1}{p}} \alpha^{\frac{1}{p}} |\nabla u|\, dx$$

$$\leq \left[\frac{1}{h} \int_{\tau < u \leq \tau + h} \alpha^{\frac{-p'}{p}} dx\right]^{\frac{1}{p'}} \left[\frac{1}{h} \int_{\tau < u \leq \tau + h} \alpha |\nabla u|^p\, dx\right]^{\frac{1}{p}}$$

$$\leq \left[\frac{1}{h} \int_{\tau < u \leq \tau + h} \alpha^{\frac{-p'}{p}} dx\right]^{\frac{1}{p'}} \left[\frac{1}{h} \int_{\tau < u \leq \tau + h} \sigma \cdot \nabla u\, dx\right]^{\frac{1}{p}}.$$

The first inequality above arises from the Hölder inequality and the second one comes from the condition (1.3). Letting h tend to 0, one gets at the limit

$$-\frac{d}{d\tau} \int_{u > \tau} |\nabla u|\, dx \leq \left[-\frac{d}{d\tau} \int_{u > \tau} \alpha^{\frac{-p'}{p}} dx\right]^{\frac{1}{p'}} \left[-\frac{d}{d\tau} \int_{u > \tau} \sigma \cdot \nabla u\, dx\right]^{\frac{1}{p}}.$$

Using the relation (3.2), we are led to

(3.3)

$$-\frac{d}{d\tau} \int_{u > \tau} |\nabla u|\, dx \leq \left[-\frac{d}{d\tau} \int_{u > \tau} \alpha^{\frac{-p'}{p}} dx\right]^{\frac{1}{p'}} \left(\frac{C(u, \sigma)}{u_{|\gamma 1} - u_{|\gamma 2}}\right)^{\frac{1}{p}}.$$

Thanks to the isoperimetric inequality for the generalized perimeter of De Giorgi relative to Ω of the set $\{u > \tau\}$, denoted by $P_\Omega(u > \tau)$ [De] and a result of Fleming and Rishel [FlRi], we have with $\mu(\tau) = |u > \tau|$,

$$-\frac{d}{d\tau}\int_{u>\tau}|\nabla u|dx = P_\Omega(u>\tau) =$$

$$= P_{\mathbb{R}^N}(\{u>\tau\} \cup \overline{\omega}_1) \geq N\beta_{N^{\frac{1}{N}}}(\mu(\tau) + m_1)^{1-\frac{1}{N}}$$

since the set $\{u > \tau\}$ does not meet γ_2 and its boundary includes γ_1. Therefore, by (3.3)

$$N\beta_N^{\frac{1}{N}}(\mu(\tau) + m_1)^{1-\frac{1}{N}} \leq \left[-\frac{d}{d\tau}\int_{u>\tau}\alpha^{\frac{-p'}{p}}dx\right]^{\frac{1}{p'}}\left(\frac{C(u,\sigma)}{u_{|\gamma 1} - u_{|\gamma 2}}\right)^{\frac{1}{p}}.$$

Furthermore, thanks to the derivation formula (*see* Rakotoson and Temam [RaTe])

$$\frac{d}{d\tau}\int_{u>\tau}\alpha^{\frac{-p'}{p}}dx = \mathcal{W}'(\mu(\tau))\,\mu'(\tau)$$

where $\mathcal{W}'(s) = \left(\alpha^{\frac{-p'}{p}}\right)_{*u}(s)$, that is, the relative rearrangement of $\alpha^{\frac{-p'}{p}}$ with respect to u defined by J. Mossino and R. Temam [MosTe], we obtain

(3.4)

$$1 \leq N^{-p'}\beta_N^{\frac{-p'}{N}}\left(\frac{C(u,\sigma)}{u_{|\gamma 1} - u_{|\gamma 2}}\right)^{\frac{p'}{p}}(\mu(\tau) + m_1)^{\frac{p'}{N}-p'}\mathcal{W}'(\mu(\tau))\,(-\mu'(\tau)).$$

Integrating the inequality (3.4) between θ and θ', we get

$$\theta' - \theta \leq N^{-p'}\beta_N^{\frac{-p'}{N}}\left(\frac{C(u,\sigma)}{u_{|\gamma 1} - u_{|\gamma 2}}\right)^{\frac{p'}{p}}$$

$$\int_\theta^{\theta'}(\mu(\tau) + m_1)^{\frac{p'}{N}-p'}\mathcal{W}'(\mu(\tau))\,(-\mu'(\tau))\,d\tau$$

$$\leq N^{-p'}\beta_N^{\frac{-p'}{N}}\left(\frac{C(u,\sigma)}{u_{|\gamma 1} - u_{|\gamma 2}}\right)^{\frac{p'}{p}}\int_0^{|\Omega|}\chi_{[\mu(\theta'),\,\mu(\theta)]}(s)\,(s+m_1)^{\frac{p'}{N}-p'}\left(\alpha^{\frac{-p'}{p}}\right)_{*u}(s)\,ds.$$

According to a result of Rakotoson [Ra], the integral

$$\int_0^{|\Omega|}\chi_{[\mu(\theta'),\,\mu(\theta)]}(s)\,(s+m_1)^{\frac{p'}{N}-p'}\left(\alpha^{\frac{-p'}{p}}\right)_{*u}(s)\,ds$$

is bounded above by

$$\int_0^{|\Omega|}\left(\chi_{[\mu(\theta'),\,\mu(\theta)]}(\cdot)\,(\cdot+m_1)^{\frac{p'}{N}-p'}\right)_*(s)\left(\alpha^{\frac{-p'}{p}}\right)_*(s)\,ds$$

$$= \int_0^{|\Omega|}\chi_{[0,\,\mu(\theta)-\mu(\theta')]}(s)\,(s+m_1+\mu(\theta'))^{\frac{p'}{N}-p'}(\alpha^*)^{\frac{-p'}{p}}(s)\,ds$$

$$= \int_{\mu(\theta')}^{\mu(\theta)}(s+m_1)^{\frac{p'}{N}-p'}(\alpha^*)^{\frac{-p'}{p}}(s-\mu(\theta'))\,ds.$$

Consequently

$$\theta' - \theta \leq N^{-p'}\beta_N^{\frac{-p'}{N}}\left(\frac{C(u,\sigma)}{u_{|\gamma 1} - u_{|\gamma 2}}\right)^{\frac{p'}{p}}\int_{\mu(\theta')}^{\mu(\theta)}(s+m_1)^{\frac{p'}{N}-p'}(\alpha^*)^{\frac{-p'}{p}}(s-\mu(\theta'))\,ds,$$

and this ends the proof of the propostion.

Applying this proposition with $\theta = u_{|\gamma 2}$ and $\theta' = u_{|\gamma 1}$, one obtains the

Theorem 1. *Assume that (u, σ) and (U, Σ) verify respectively $(\mathcal{P})$ and $(\tilde{\mathcal{P}})$. Set*

$$C(u,\sigma) = \int_\Omega \sigma\cdot\nabla u\,dx \text{ and } \tilde{C}(U,\Sigma) =$$

$$= \int_{\tilde{\Omega}}\Sigma\cdot\nabla U\,dx = \int_{\tilde{\Omega}}\tilde{\alpha}\,|\nabla U|^p\,dx = \tilde{C}(U).$$

Then one obtains the isoperimetric inequality

$$\frac{\left(u_{|\gamma 1} - u_{|\gamma 2}\right)^p}{C(u,\sigma)} \leq N^{-p}\beta_N^{\frac{-p}{N}}\left[\int_{m_1}^{m_2}s^{\frac{p'}{N}-p'}(\alpha^*)^{\frac{-p'}{p}}(s-m_1)\,ds\right]^{\frac{p}{p'}} =$$

$$= \frac{\left(U_{|\tilde{\gamma}_1} - U_{|\tilde{\gamma}_2}\right)^p}{\tilde{C}(U,\Sigma)}$$

with $m_i = |\omega_i|$ $(i = 1, 2)$.

Proof: Taking $\theta = u_{|\gamma 2}$ and $\theta' = u_{|\gamma 1}$ in the **Proposition,** one gets

$$u_{|\gamma 1} - u_{|\gamma 2} \leq N^{-p'}\beta_N^{\frac{-p'}{N}}\left(\frac{C(u,\sigma)}{u_{|\gamma 1} - u_{|\gamma 2}}\right)^{\frac{p'}{p}}$$

$$\int_{\mu(u_{|\gamma 1})}^{\mu(u_{|\gamma 2})}(s+m_1)^{\frac{p'}{N}-p'}(\alpha^*)^{\frac{-p'}{p}}\left(s-\mu\left(u_{|\gamma_1}\right)\right)ds.$$

According to **Lemma 1,** the inequality $u(x) \leq u_{|\gamma 1}$ holds for almost every $x \in \Omega$. In consequence, we have $\mu(u_{|\gamma 1}) = 0$ and $\mu(u_{|\gamma 2}) \leq |\Omega|$. One gets the announced inequality by bounding above the previous integral between 0 and $\mu(u_{|\gamma 2})$ by the integral between 0 and $|\Omega|$. The equality in the theorem is classical.

4. SOME APPLICATIONS

We denote by (p, α)-capacity of $\Omega = \omega_2 \setminus \bar{\omega}_1$, the quantity $\int_\Omega \alpha|\nabla v|^p dx$ where v is the solution of $-\operatorname{div}\left(\alpha|\nabla v|^{p-2}\nabla v\right) = 0$ in Ω, $v_{|\gamma_1} = 1$ and $v_{|\gamma 2} = 0$. We give below some applications of **Theorem 1.**

4.1. The (p, α)-capacity problem

We assume that $u_{|\gamma_i} = C_i (i = 1 \text{ or } 2)$ are given constants with $C_1 > C_2$. We consider the problem $(\tilde{\mathcal{P}})$ with the same constants $U_{|\tilde{\gamma}_i} = C_i$. We obtain the isoperimetric inequality

$$\int_\Omega \sigma \cdot \nabla u \, dx = C(u, \sigma) \geq \tilde{C}(U, \Sigma) = \int_{\tilde{\Omega}} \tilde{\alpha}|\nabla U|^p dx$$

$$= N^p \beta_N^{\frac{p}{N}} (C_1 - C_2)^p \left[\int_{m_1}^{m_2} s^{\frac{p'}{N} - p'} \left(\alpha^*\right)^{\frac{-p'}{p}} (s - m_1) \, ds \right]^{\frac{-p}{p'}}.$$

With $\sigma = \alpha|\nabla u|^{p-2}\nabla u$, $C_1 = 1$ and $C_2 = 0$, we recover the isoperimetric inequality for the (p, α)-capacity given in [AlTr2] and [Fe].

4.1.1. Application

Let α_n be a sequence of functions defined on Ω such that their unidimensional increasing rearrangement α_n^* satisfy

$$\int_{m_1}^{m_2} s^{\frac{p'}{N} - p'} \left(\alpha_n^*\right)^{\frac{-p'}{p}} (s - m_1) \, ds \to 0$$

when n tends to infinity. If (σ_n, u_n) verifies the problem $(\mathcal{P})$ of the (p, α)-capacity, then we have

$$\int_\Omega \sigma_n \cdot \nabla u_n dx \to \infty.$$

Let's precise this application by assuming for instance that α_n takes two values: $\alpha_n = A_n^1$ in Ω_n^1 with $|\Omega_n^1| = \frac{1}{n}$ and $\alpha_n = A_n^2 > A_n^1$ in $\Omega_n^2 = \Omega \setminus \Omega_n^1$. Hence

$$\left(\frac{p'}{N} - p' + 1\right) \int_{m_1}^{m_2} s^{\frac{p'}{N} - p'} \left(\alpha_n^*\right)^{\frac{-p'}{p}} (s - m_1) \, ds = \left(A_n^1\right)^{\frac{-p'}{p}} \left[\left(m_1 + \frac{1}{n}\right)^{\frac{p'}{N} - p' + 1} - (m_1)^{\frac{p'}{N} - p' + 1} \right]$$

$$+ \left(A_n^2\right)^{\frac{-p'}{p}} \left[(m_2)^{\frac{p'}{N} - p' + 1} - \left(m_1 + \frac{1}{n}\right)^{\frac{p'}{N} - p' + 1} \right]$$

if $p' \neq \frac{N}{N-1}$ and

$$\int_{m_1}^{m_2} s^{-1} \left(\alpha_n^*\right)^{\frac{-p'}{p}} (s - m_1) \, ds = \left(A_n^1\right)^{\frac{-p'}{p}} \ln\left(1 + \frac{1}{n\, m1}\right) + \left(A_n^2\right)^{\frac{-p'}{p}} \ln\left(\frac{m_2}{m_1 + \frac{1}{n}}\right)$$

if $p' = \frac{N}{N-1}$. In consequence, taking the equivalents, we get

$$\int_{m_1}^{m_2} s^{\frac{p'}{N} - p'} \left(\alpha_n^*\right)^{\frac{-p'}{p}} (s - m_1) \, ds \cong$$

$$\cong m_1^{\frac{p'}{N} - p'} \frac{1}{n} \left(A_n^1\right)^{\frac{-p'}{p}} + \frac{m_2^{\frac{p'}{N} - p' + 1} - m_1^{\frac{p'}{N} - p' + 1}}{\frac{p'}{N} - p' + 1} \left(A_n^2\right)^{\frac{-p'}{p}}$$

if $p' \neq \frac{N}{N-1}$ and

$$\int_{m_1}^{m_2} s^{-1} \left(\alpha_n^*\right)^{\frac{-p'}{p}} (s - m_1) \, ds \cong \frac{1}{m_1} \frac{1}{n} \left(A_n^1\right)^{\frac{-p'}{p}} + \ln\left(\frac{m_2}{m_1}\right) \left(A_n^2\right)^{\frac{-p'}{p}}$$

if $p' = \frac{N}{N-1}$. In this case, in order to let the following integral

$$\int_{m_1}^{m_2} s^{\frac{p'}{N} - p'} \left(\alpha_n^*\right)^{\frac{-p'}{p}} (s - m_1) \, ds$$

tend to *zero*, it is enough to take $A_n^2 \to \infty$ and $\frac{1}{n}\left(A_n^1\right)^{\frac{-p'}{p}} \to 0$. We can choose for instance $A_n^2 \to \infty$ whereas $A_n^1 \to 0$ but with the condition $A_n^1 >> \left(\frac{1}{n}\right)^{p-1}$ (e.g. $A_n^1 = \left(\frac{1}{n}\right)^{p-2}$ if $p \geq 2$).

4.2. The prescribed flux problem

We denote by $Q(u, \sigma)$ the quantity

$$Q(u, \sigma) = \frac{C(u, \sigma)}{u_{|\gamma 1} - u_{|\gamma 2}}.$$

For regular open sets Ω, for regular u and suitable σ, the quantity $Q(u, \sigma)$ is a physical parameter (see the remark below). It is the total flux. For this reason, we also call «flux» the quantity $Q(u, \sigma)$ without any regularity assumption on Ω, u or σ. Assume that u and U satisfy $(\mathcal{P})$ and $(\tilde{\mathcal{P}})$ as well as the condition $\frac{C(u, \sigma)}{u_{|\gamma 1} - u_{|\gamma 2}} = Q = \frac{\tilde{C}(U, \Sigma)}{U_{|\tilde{\gamma}1} - U_{|\tilde{\gamma}2}}$. The value $Q > 0$ is given but the values of $u_{|\gamma 1}$, $u_{|\gamma 2}$, $U_{|\tilde{\gamma}1}$ and $U_{|\tilde{\gamma}2}$ remain undetermined. The **Theorem 1** gives an optimal estimate for the variation of u, that is, a precise comparaison of the quantities $u_{|\gamma 1} - u_{|\gamma 2}$ and $U_{|\tilde{\gamma}1} - U_{|\tilde{\gamma}2}$:

$$u_{|\gamma 1} - u_{|\gamma 2} \leq Q^{\frac{p'}{p}} N^{-p'} \beta_N^{\frac{-p'}{N}} \int_{m_1}^{m_2} s^{\frac{p'}{N} - p'} \left(\alpha^*\right)^{\frac{-p'}{p}} (s - m_1) \, ds = U_{|\tilde{\gamma}1} - U_{|\tilde{\gamma}2}.$$

In particular, if $u_{|\gamma 2} = U_{|\tilde{\gamma}2} = 0$, one obtains an optimal estimate for $u_{|\gamma 1}$.

4.2.1. Application

Let α_n be a sequence of functions defined on Ω such that their unidimensional increasing rearrangements α_n^* satisfy

$$\int_{m_1}^{m_2} s^{\frac{p'}{N}-p'} \left(\alpha_n^*\right)^{\frac{-p'}{p}} (s-m_1)\, ds \to 0$$

when n tends to infinity. Let (σ_n, u_n) be any solution of the prescribed flux problem $(\mathcal{P})$ and such that $(u_n)_{|\gamma 2} = 0$. Then, one has

$$(u_n)_{|\gamma 1} = \operatorname*{ess\,sup}_{x\in\Omega} |u_n(x)| \to 0.$$

If there exists α such that for any $n \in \mathbb{N}$, one has $\alpha_n \geq \alpha$, then

$$\alpha\int_\Omega |\nabla u_n|^p dx \leq \int_\Omega \alpha_n |\nabla u_n|^p \leq \int_\Omega \sigma_n \cdot \nabla u_n dx = Q(u_n)_{|\gamma 1} \to 0$$

and finally $u_n \to 0$ in $W^{1,p}(\Omega)$ and $L^\infty(\Omega)$ (strongly).

Remark 3. Let n be the unitary outer normal to Ω at $\gamma_1 \cup \gamma_2$. We assume that Ω, u, σ are regular enough in order to define

$$Q'(\sigma) = \int_{\gamma 1} \sigma \cdot n\, d\gamma$$

(where $d\gamma$ is the measure on $\gamma_1 \cup \gamma_2$) and in order to apply the Green formula. It appears that the quantity $Q'(\sigma)$ is in fact $Q(u, \sigma)$. This equality is shown in

Lemma 2. *We assume that Ω, u, σ are regular enough. Then we have*

$$C(u, \sigma) = \left(u_{|\gamma 1} - u_{|\gamma 2}\right) Q'(\sigma).$$

Proof: Let's remark that

$$\int_{\gamma 2} \sigma \cdot n\, d\gamma = -\int_{\gamma 1} \sigma \cdot n\, d\gamma$$

which is a straightforward consequence of the Green formula:

$$0 = \int_\Omega div\, \sigma\, dx = \int_{\gamma 1 \cup \gamma 2} \sigma \cdot n\, d\gamma.$$

This yields

$$C(u, \sigma) = -\int_\Omega u\, div\, \sigma\, dx + \int_{\gamma 1\cup\gamma 2} u\sigma \cdot n\, d\gamma = \int_{\gamma 1\cup\gamma 2} u\sigma\cdot n\, d\gamma.$$

Afterwards, using the conditions on u on the boundary of Ω, we obtain

$$\int_{\gamma 1\cup\gamma 2} u\, \sigma\cdot n\, d\gamma = \left(u_{|\gamma 1} - u_{|\gamma 2}\right) Q'(\sigma).$$

This leads to the formula of **Lemma 2.**

4.3. The problem of domains with given (p, α)-capacity

We are given the boundary $\gamma_1 = \partial\omega_1$ of a regular open set ω_1, a real $Q > 0$ and α: $\mathbb{R}^N \setminus \overline{\omega_1} \to \mathbb{R}^+$ a measurable function which is bounded as well as its inverse. Assume furthermore that α is rearrangebale in the sense

$$\forall s > 0,\ \exists t \geq 0,\ |\alpha < t| \geq s$$

(see e.g. B. Simon [Si]). We consider the sphere $\tilde{\gamma}_1 = \partial\tilde{\omega}_1$. Thus we define the unidimensional rearrangement of α on $(0, +\infty)$ by

$$\alpha^*(s) = \inf\left\{t,\ \text{mes}\left\{x \in \mathbb{R}^N \setminus \overline{\omega}_1,\ \alpha(x) < t\right\} \geq s\right\},\quad \forall s \in]0, +\infty[$$

and $\tilde{\alpha} : \mathbb{R}^N \setminus \overline{\tilde{\omega}_1} \to \mathbb{R}^+$ by $x \to \alpha^*\left(\beta_N|x|^N - |\omega_1|\right)$. There exists a unique sphere Γ_2 such that, denoting by $\mho$ the annulus with boundaries $\tilde{\gamma}_1$ and Γ_2, the $(p, \tilde{\alpha})$-capacity of $\mho$ is equal to Q: the measure M_2 of the ball bounded by Γ_2 is the unique solution of the equation

$$N^p \beta_N^{\frac{p}{N}} \left[\int_{m_1}^{M_2} s^{\frac{p'}{N}-p'} \left(\alpha^*\right)^{\frac{-p'}{p}} (s-m_1)\, ds\right]^{1-p} = Q.$$

If ω_2 is any domain containing strongly ω_1 $(\omega_2 \supset\supset \omega_1)$ and such that the (p, α)-capacity of $\Omega = \omega_2 \setminus \overline{\omega_1}$ is equal to Q, the **Theorem 1** says that we have necessarily $m_2 \geq M_2$. Indeed, by **Theorem 1,** we have

$$\frac{1}{Q} \leq N^{-p} \beta_N^{\frac{-p}{N}} \left[\int_{m_1}^{m_2} s^{\frac{p'}{N}-p'} \left[\left(\alpha_{|\Omega}\right)^*\right]^{\frac{-p'}{p}} (s-m_1)\, ds\right]^{\frac{p}{p'}}.$$

Since

$$\left(\alpha_{|\Omega}\right)^*(s) = \inf\left\{t,\ \text{mes}\left\{x \in \Omega,\ \alpha(x) < t\right\} \geq s\right\},\quad \forall s \in]0, |\Omega|[$$

and

$$\text{mes}\left\{x \in \Omega,\ \alpha(x) < t\right\} \leq \text{mes}\left\{x \in \mathbb{R}^N \setminus \overline{\omega}_1,\ \alpha(x) < t\right\},$$

we have, for $s \in [0, |\Omega|]$,

$$\left\{t,\ \text{mes}\left\{x \in \Omega,\ \alpha(x) < t\right\} \geq s\right\} \subset \left\{t,\ \text{mes}\left\{x \in \mathbb{R}^N \setminus \overline{\omega}_1,\ \alpha(x) < t\right\} \geq s\right\}.$$

Hence

$$\forall s \in [0, |\Omega|],\quad \alpha^*(s) \leq \left(\alpha_{|\Omega}\right)^*(s).$$

In consequence, we have

$$\frac{1}{Q} = N^{-p}\beta_N^{\frac{-p}{N}}\left[\int_{m_1}^{M_2} s^{\frac{p'}{N}-p'}\left(\alpha^*\right)^{\frac{-p'}{p}}(s-m_1)\,ds\right]^{\frac{p}{p'}}$$

and

$$\frac{1}{Q} \le N^{-p}\beta_N^{\frac{-p}{N}}\left[\int_{m_1}^{m_2} s^{\frac{p'}{N}-p'}\left[\left(\alpha_{|\Omega}\right)^*\right]^{\frac{-p'}{p}}(s-m_1)\,ds\right]^{\frac{p}{p'}}$$

$$\le N^{-p}\beta_N^{\frac{-p}{N}}\left[\int_{m_1}^{m_2} s^{\frac{-p'}{N}-p'}\left(\alpha^*\right)^{\frac{-p'}{p}}(s-m_1)\,ds\right]^{\frac{p}{p'}}.$$

Finally, we get

$$\int_{m_1}^{M_2} s^{\frac{p'}{N}-p'}\left(\alpha^*\right)^{\frac{-p'}{p}}(s-m_1)\,ds \le \int_{m_1}^{m_2} s^{\frac{p'}{N}-p'}\left(\alpha^*\right)^{\frac{-p'}{p}}(s-m_1)\,ds$$

and therefore $M_2 \le m_2$ necessarily.

5. APPLICATION TO A MUSKAT PROBLEM

5.1. Recall of the Muskat problem and of previous results

Let $1 < p < \infty$, a_1, a_2 and k be positive real constants. Let ω_1, ω_2 with $\omega_1 \subset\subset \omega_2$ be bounded open sets of $\mathbb{R}^N$ with regular boundaries $\partial\omega_1 = \gamma_1$, $\partial\omega_2 = \gamma_2$. At time $t = 0$ (initial time), we are given $\omega = \omega(0)$, an open set with regular boundary $\partial\omega(0) = \gamma(0)$ such that $\omega_1 \subset\subset \omega(0) \subset\subset \omega_2$. We set $\Omega = \omega_2 \setminus \overline{\omega_1}$, $\Omega_1(0) = \omega(0) \setminus \overline{\omega_1}$ and $\Omega_2(0) = \omega_2 \setminus \overline{\omega(0)}$. From this initial position $\omega(0)$, ω evolves with the time t (we write $\omega = \omega(t)$ for this dependance on time t): its boundary $\partial\omega(t) = \gamma(t)$ moves according to the normal velocity

$(Q)_1$

$$v_\nu(x, t) = -ka_1|\nabla u_1(x, t)|^{p-2}\nabla u_1(x, t)\cdot\nu(x, t) = -ka_1|\nabla u_1(x, t)|^{p-2}\frac{\partial u_1}{\partial\nu}(x, t)$$

$$= -ka_2|\nabla u_2(x, t)|^{p-2}\nabla u_2(x, t)\cdot\nu(x, t) = -ka_2|\nabla u_2(x, t)|^{p-2}\frac{\partial u_2}{\partial\nu}(x, t)$$

where $u_1(., t) = u_1(t)$ and $u_2(., t)$ are defined respectively in $\Omega_1(t) = \omega(t) \setminus \overline{\omega_1}$ and in $\Omega_2(t) = \omega_2 \setminus \overline{\omega(t)}$ and are the solutions of the following equations:

$$(Q)_2 \quad \begin{cases} -\mathrm{div}\left(a_1|\nabla u_1(t)|^{p-2}\nabla u_1(t)\right) = 0 & \text{in } \Omega_1(t) \\ -\mathrm{div}\left(a_2|\nabla u_2(t)|^{p-2}\nabla u_2(t)\right) = 0 & \text{in } \Omega_2(t) \\ u_1(t) = 1 & \text{on } \gamma_1 \\ u_2(t) = 0 & \text{on } \gamma_2 \\ u_1(t) = u_2(t) & \text{on } \gamma(t) \\ a_1|\nabla u_1(t)|^{p-2}\frac{\partial u_1}{\partial\nu}(t) = a_2|\nabla u_2(t)|^{p-2}\frac{\partial u_2}{\partial\nu}(t) & \text{on } \gamma(t). \end{cases}$$

The notation $\frac{\partial}{\partial\nu}$ stands for $\nabla_x\cdot\nu(x, t)$ where $\nu(x, t)$ is the unitary outer normal to $\Omega_1(t)$ at $\gamma(t)$. We denote by (Q) the problem $(Q_1) + (Q_2)$. For $p = 2$, this problem is called «Muskat problem» [EO, Mu]. It models the mining of oil (fluid 2 occupying $\Omega_2(t)$) by injection of viscous water (fluid 1 occupying $\Omega_1(t)$). The model suggested above (with $1 < p < \infty$) is a natural generalization of the Muskat one. When $p = 2$ and $N > 1$, F. Abergel and J. Mossino [AbMos] have proved the existence of regular solutions locally in time of the Muskat problem by means of the method of «normal variations». For $p = 2$ and $N = 2$, F. Yi [Y] has also given a proof of the existence and uniqueness of classical solution locally in time by the Newton iteration method. It is well known (*see* [Be]) that the stability or unstability of the interface $\gamma(t)$ corresponds respectively to the condition on the mobility ratio $M = \frac{a_1}{a_2} < 1$ or > 1. A mathematical interpretation of this stability condition can be found in [Ab, AbMos]. When $a_1 < a_2$ (and $1 < p < \infty$), L. Boukrim and J. Mossino [BoMos1, BoMos2] have given isoperimetric inequalities by comparison with an evolution problem with spherical symmetry defined from (Q). They have given an optimal estimate of the «critical time», that is, the time after which no regular solution may exist, as well as an optimal estimate of the respective volumes of the domains $\Omega_1(t)$ and $\Omega_2(t)$.

5.2. A MUSKAT PROBLEM WITH PRESCRIBED FLUX

5.2.1. The most general statement

Let p, ω_1, ω_2 and $\omega(0)$ as in section 5.1. Let β and ξ be functions defined on $\mathbb{R}^+$ such that ξ is positive and β does not vanish and is of constant sign. We assume that $\beta\xi \in L^1(\mathbb{R}^+)$.

Let δ and η be positive functions defined on (m_1, m_2) with m_i = measure of ω_i, $(i = 1, 2)$, such that $\frac{1}{\delta}$ and $\frac{1}{\delta\eta} \in L^1(m_1, m_2)$. Let $a_1, a_2 : \Omega \times [m_1, m_2] \to \mathbb{R}^+$ be positive functions such that for any $m \in [m_1, m_2]$, $a_i(., m)$ and $\frac{1}{a_i(., m)}$ are in $L^\infty(\Omega)$ and

(5.1)

$$\operatorname*{ess\,sup}_{x\in\Omega} a_1(x, m) \le \operatorname*{ess\,inf}_{x\in\Omega} a_2(x, m).$$

From the initial position $\omega(0)$, ω evolces with the time t: we write $\omega = \omega(t)$. We denote $\Omega_1(t) = \omega(t) \setminus \overline{\omega_1}$, $\Omega_2(t) = \omega_2 \setminus \overline{\omega(t)}$ and $m(t) = |\omega(t)|$ = measure of $\omega(t)$ and we define

(5.2)

$$\alpha(x, t) = \alpha_i(x, t) = a_i(x, m(t)) \text{ if } x \in \Omega_i(t),\ i = 1, 2.$$

We consider the very general problem such that at any time t, there exists a domain $\omega(t)$ with $\omega_1 \subset\subset \omega(t) \subset\subset \omega_2$ and a pair $(u(t), \sigma(t)) \in W^{1,p}(\Omega) \times (L^{p'}(\Omega))^N$ satisfying

(5.3) $$-\operatorname{div} \sigma(t) = 0,\ \sigma(t) \cdot \nabla u(t) \geq \alpha(t)|\nabla u(t)|^p \text{ in } \Omega,$$

(5.4) $$u(t) = 0 \text{ on } \gamma_2,$$

(5.5) $$u(t) = \text{undetermined constant on } \gamma_1,$$

(5.6)
$$\int_\Omega \sigma(t) \cdot \nabla u(t)\, dx = \xi(t)\, \eta(m(t))\, u(t)_{|\gamma_1},\ \frac{d}{dt} m(t) = (\beta\xi)(t)(\delta\eta)(m(t)).$$

This general problem is actually a statement coming from the problem that we set below and that we call «Muskat problem with prescribed flux»: $\partial\omega(t) = \gamma(t)$ moves with the normal speed

$$(\mathcal{R}_1)\ v_\nu(x, t) = -\beta(t)\, \delta(m(t))\, \sigma_1(x, t) \cdot \nu(x, t) =$$
$$= -\beta(t)\, \delta(m(t))\, \sigma_2(x, t) \cdot \nu(x, t)$$

where $\nu(t) = \nu(x, t)$ is the unitary outer normal to $\Omega_1(t)$ at $\gamma(t)$ and $\sigma_i(., t) = \sigma_i(t)$ satisfy (accordingly with $u_i(t)$)

$$(\mathcal{R}_2) \begin{cases} -\operatorname{div} \sigma_i(t) = 0,\ \sigma_i(t) \cdot \nabla u_i(t) \geq \alpha_i(t)\, |\nabla u_i(t)|^p \text{ in } \Omega_i(t),\ i = 1, 2, \\ u_1(t) = \text{unknown constant on } \gamma_1, \\ u_2(t) = 0 \text{ on } \gamma_2, \\ u_1(t) = u_2(t) \text{ and } \sigma_1(t) \cdot \nu(t) = \sigma_2(t) \cdot \nu(t) \text{ on } \gamma(t), \\ \int_{\gamma_1} \sigma_1(t) \cdot n\, d\gamma = \xi(t)\, \eta(m(t)). \end{cases}$$

Here and in the following, we denote by n the normal to Ω at $\gamma_1 \cup \gamma_2$. The last condition of $(\mathcal{R}_2)$ involves the flux $\int_{\gamma_1} \sigma_1(t) \cdot n\, d\gamma$: imposing this flux is related classically to the fact that $u_1(t)$ is an undetermined constant on γ_1.

Let us prove that for any regular solution of $(\mathcal{R}_1) + (\mathcal{R}_2)$, we have (5.6) and $\sigma(t)$ is divergence free in the sense of distributions. (By regular solution, we mean that the Green formula is valid.)

1. Proof of $-\operatorname{div} \sigma(t) = 0$ in $\mathcal{D}'(\Omega)$ = space of distributions on Ω. For any test function $\varphi \in \mathcal{D}(\Omega) = C_0^\infty(\Omega)$,

$$\int_\Omega \sigma(t) \cdot \nabla\varphi\, dx = \int_{\Omega_1} \sigma_1(t) \cdot \nabla\varphi\, dx + \int_{\Omega_2} \sigma_2(t) \cdot \nabla\varphi\, dx =$$

$$= \sum_{i=1}^{2} \int_{\Omega_i} (-\operatorname{div} \sigma_i(t))\, \varphi\, dx + \int_{\gamma(t)} \varphi(\sigma_1(t) \cdot \nu(t) - \sigma_2(t) \cdot \nu(t))\, d\gamma = 0.$$

2. Proof of (5.6)

2.a.

$$\int_\Omega \sigma(t) \cdot \nabla u(t) dx = \sum_{i=1}^{2} \int_{\Omega_i} (-\operatorname{div} \sigma_i(t))\, u_i(t)\, dx$$

$$+ \int_{\gamma(t)} u_1(t)\, (\sigma_1(t) \cdot \nu(t) - \sigma_2(t) \cdot \nu(t))\, d\gamma + u_1(t)_{|\gamma_1} \int_{\gamma_1} \sigma_1(t) \cdot n\, d\gamma$$

$$= u_1(t)_{|\gamma_1} \int_{\gamma_1} \sigma_1(t) \cdot n\, d\gamma = \xi(t)\, \eta(m(t))\, u_1(t)_{|\gamma_1}.$$

2.b.

$$\frac{dm}{dt}(t) = \int_{\gamma(t)} v_\nu(x, t)\, d\gamma = -\beta(t)\, \delta(m(t)) \int_{\gamma(t)} \sigma_1(t) \cdot \nu(t)\, d\gamma$$

$$= \beta(t)\, \delta(m(t)) \int_{\gamma_1} \sigma_1(t) \cdot n\, d\gamma = (\beta\xi)\, (t)\, (\delta\eta)\, (m(t)).$$

(The penultimate equality comes from the integration on $\Omega_1(t)$ of $-\operatorname{div} \sigma_1(t) = 0$.)

5.2.2. Isoperimetric inequalities

We denote by $(\mathcal{F})$ the statements (5.2.) to (5.6.). Our isoperimetric inequalities arise from the following theorem as a corollary of **Theorem 1**

Theorem 2 *Let p' verifying $\frac{1}{p} + \frac{1}{p'} = 1$ and β_N The measure of the unit ball of $\mathbb{R}^N$. Denote by t_c the critical time of problem $(\mathcal{F})$ that is the maximal time of existence of solution to $(\mathcal{F})$. Set*

$$\Phi(m) =$$
$$= \left[\int_{m_1}^{m} s^{\frac{p'}{N} - p'} (a_1 *)^{\frac{-p'}{p}} (s - m_1, m)\, ds + \int_{m}^{m_2} s^{\frac{p'}{N} - p'} (a_2 *)^{\frac{-p'}{p}} (s - m, m)\, ds \right]^{\frac{p}{p'}}.$$

Then, for any time t for which there exists a solution to $(\mathcal{F})$ on $[0, t]$ (hence for any $t \leq t_c$), one has

$$N^p \beta_N^{\frac{p}{N}} \int_0^t (\Gamma_1(\tau))^{p-1} |\beta(\tau)|\, d\tau \leq sgn(\beta) \left[\int_{m(0)}^{m(t)} \frac{\Phi}{\delta} (s)\, ds \right].$$

where $\Gamma_1(\tau) = u_1(\tau)_{|\gamma_1}$ and $sgn(\beta)$ is the sign of β.

Proof: Let t be a time for which there exists a solution to $(\mathcal{F})$ on $[0, t]$ and let $\tau \in [0, t]$. By (5.2) we have $\alpha(x, \tau) = a_i(x, m(\tau))$ for $x \in \Omega_i(\tau)$ and we set

$$C(u(\tau), \sigma(\tau)) = \int_\Omega \sigma(x, \tau) \cdot \nabla u(x, \tau)\, dx, \quad \Gamma_1(\tau) = u_1(\tau)_{|\gamma_1}.$$

Then **Theorem 1** ensures that

(5.7)

$$\frac{(\Gamma_1(\tau))^p}{C(u(\tau), \sigma(\tau))} \leq N^{-p} \beta_N^{\frac{-p}{N}} \left[\int_{m_1}^{m_2} s^{\frac{p'}{N} - p'} (\alpha^*(\tau))^{\frac{-p'}{p}} (s - m_1)\, ds \right]^{\frac{p}{p'}}$$

where $\alpha^*(\tau)$ is the increasing rearrangement of $\alpha(., \tau)$. It yields from (5.7) and (5.6) that

$$N^p \beta_N^{\frac{p}{N}} (\Gamma_1(\tau))^{p-1} \le \xi(\tau)\, \eta(m(\tau)) \left[\int_{m_1}^{m_2} s^{\frac{p'}{N}-p'} (\alpha^*(\tau))^{\frac{-p'}{p}} (s-m_1) ds\right]^{\frac{p}{p'}}.$$

By (5.6),

$$\frac{dm(\tau)}{d\tau} = \beta(\tau)\, \delta(m(\tau))\, \xi(\tau)\, \eta(m(\tau))$$

and the previous inequality becomes

$$N^p \beta_N^{\frac{p}{N}} (\Gamma_1(\tau))^{p-1} \le \frac{1}{\beta(\tau)\, \delta(m(\tau))} \frac{dm(\tau)}{d\tau} \left[\int_{m_1}^{m_2} s^{\frac{p'}{N}-p'} (\alpha^*(\tau))^{\frac{-p'}{p}} (s-m_1)\, ds\right]^{\frac{p}{p'}}$$

$$= \frac{\mathrm{sgn}(\beta)}{|\beta(\tau)|\, \delta(m(\tau))} \frac{dm(\tau)}{d\tau} \left[\int_{m_1}^{m_2} s^{\frac{p'}{N}-p'} (\alpha^*(\tau))^{\frac{-p'}{p}} (s-m_1)\, ds\right]^{\frac{p}{p'}}.$$

It follows from (5.1) that

$$\left[\int_{m_1}^{m_2} s^{\frac{p'}{N}-p'} (\alpha^*(\tau))^{\frac{-p'}{p}} (s-m_1)\, ds\right]^{\frac{p}{p'}} \le \Phi(m(\tau)).$$

We get finally

$$N^p \beta_N^{\frac{p}{N}} (\Gamma_1(\tau))^{p-1} |\beta(\tau)| \le \mathrm{sgn}(\beta)\, \frac{\Phi}{\delta}(m(\tau))\, \frac{dm(\tau)}{d\tau}$$

which, after integration between 0 and t, gives the inequality of the theorem above.

We will estimate the unknown value of $u(t)$ on γ_1, denoted $\Gamma_1(t)$, and show that this estimate is optimal by comparison of the problem $(\mathcal{F})$ with an evolution problem $(\tilde{\mathcal{F}})$ with spherical symmetry defined from $(\mathcal{F})$. Clearly, we obtain $(\tilde{\mathcal{F}})$ by replacing ω_1, $\omega = \omega(0)$ and ω_2 respectively by $\tilde{\omega}_1$, $\tilde{\omega} = \tilde{\omega}(0)$ and $\tilde{\omega}_2$, the balls of $\mathbb{R}^N$ centered at the origin and having the same measures as ω_1, $\omega(0)$ and ω_2. We also replace $a_i(., m)$ by its radially increasing rearrangement $\tilde{a}_i(., m)$ on $\tilde{\Omega} = \tilde{\omega}_2 \setminus \overline{\tilde{\omega}_1}$. We denote by $\tilde{\omega}(t)$, the ball related to $(\tilde{\mathcal{F}})$ at time $t \ge 0$. A priori the domain $\tilde{\omega}(t)$ is not the symmetrization of the domain $\omega(t)$ related to $(\mathcal{F})$. Its measure is denoted by $\tilde{m}(t)$. We set $\tilde{\Omega}_1(t) = \tilde{\omega}(t) \setminus \overline{\tilde{\omega}_1}$, $\tilde{\Omega}_2(t) = \tilde{\omega}_2 \setminus \overline{\tilde{\omega}(t)}$ and $\tilde{\alpha}(x, t) = \tilde{a}_i(x, |\tilde{\omega}(t)|)$ if $x \in \tilde{\Omega}_i(t)$, $i = 1, 2$. Finally, in the statement of $(\mathcal{F})$, $(u(t), \sigma(t))$ is replaced by the pair $(U(t), \Sigma(t))$ where $\Sigma(t) = \tilde{\alpha}(t)|\nabla U(t)|^{p-2}\nabla U(t)$; More precisely the problem $(\tilde{\mathcal{F}})$ is

(5.8)

$$-\operatorname{div} \Sigma(t) = 0 \text{ in } \tilde{\Omega} \text{ with } \Sigma(t) = \tilde{\alpha}(t)|\nabla U(t)|^{p-2}\nabla U(t),$$

(5.9) $\quad U(t) = 0 \text{ on } \tilde{\gamma}_2 = \partial\tilde{\omega}_2,$

(5.10) $\quad U(t) = \text{undetermined constant on } \tilde{\gamma}_1 = \partial\tilde{\omega}_1,$

(5.11)

$$\int_{\tilde{\Omega}} \Sigma(t) \cdot \nabla U(t)\, dx =$$

$$= \xi(t)\, \eta(\tilde{m}(t))\, U(t)_{|\tilde{\gamma}_1}, \frac{d}{dt}\tilde{m}(t) = (\beta\xi)(t)(\delta\eta)(\tilde{m}(t)).$$

The problem $(\tilde{\mathcal{F}})$ admits a unique solution and we have the

Theorem 3. *Let $\tilde{m}(t)$ be the measure of $\tilde{\omega}(t)$, $\tilde{\Gamma}_1(t)$ be the undetermined value of $U(t)$ on $\tilde{\gamma}_1$ and $\tilde{t}_c$ the critical time for the symmetrized problem, that is, the time such that $\tilde{\gamma}(t)$ touches $\tilde{\gamma}_1$ (if $\beta < 0$) or $\tilde{\gamma}_2$ (if $\beta > 0$).* The values *$\tilde{t}_c$, $\tilde{m}(t)$ and $\tilde{\Gamma}_1(t)$ are explicitly given by*

(1)

$$\int_0^{\tilde{t}_c} (|\beta|\xi)(\tau)\, d\tau = \begin{cases} \displaystyle\int_{m_1}^{m(0)} \frac{ds}{(\delta\eta)(s)} & \text{if } \beta < 0, \\ \displaystyle\int_{m(0)}^{m_2} \frac{ds}{(\delta\eta)(s)} & \text{if } \beta > 0, \end{cases}$$

(2)

$$\int_{m(0)}^{\tilde{m}(t)} \frac{ds}{(\delta\eta)(s)} = \int_0^t (\beta\xi)(\tau)\, d\tau \text{ for } 0 \le t \le \tilde{t}_c,$$

(3)

$$N^p \beta_N^{\frac{p}{N}} \left(\tilde{\Gamma}_1(t)\right)^{p-1} = (\Phi\eta)(\tilde{m}(t))\, \xi(t) \text{ for } 0 \le t \le \tilde{t}_c.$$

(4) *If $(\mathcal{F})$ admits a solution on* $[0, t]$, *then one has*

(i) $t \le \tilde{t}_c$ *(hence $t_c \le \tilde{t}_c$),*

(ii) $m(t) = \tilde{m}(t)$,

(iii) $\Gamma_1(t) \le \tilde{\Gamma}_1(t)$.

Proof:

1. Let $\tilde{t}_c$ be the critical time for the symmetrized problem and $\tau \in [0, \tilde{t}_c]$. We have

$$\frac{d\tilde{m}(\tau)}{d\tau} = (\beta\xi)(\tau)\, (\delta\eta)\, (\tilde{m}(\tau))$$

hence

(5.12)

$$(\beta\xi)(\tau) = sgn(\beta)\, (|\beta|\xi)(\tau) = \frac{1}{(\delta\eta)(\tilde{m}(\tau))} \frac{d\tilde{m}(\tau)}{d\tau}.$$

This leads to

$$(|\beta|\xi)(\tau) = \frac{sgn(\beta)}{(\delta\eta)(\tilde{m}(\tau))}\frac{d\tilde{m}(\tau)}{d\tau}$$

that we integrate between 0 an $\tilde{t}_c$. We get the equality

(5.13)

$$\int_0^{\tilde{t}_c}(|\beta|\xi)(\tau)\,d\tau = sgn(\beta)\int_0^{\tilde{t}_c}\frac{1}{(\delta\eta)(\tilde{m}(\tau))}\frac{d\tilde{m}(\tau)}{d\tau}\,d\tau =$$

$$= sgn(\beta)\int_{\tilde{m}(0)}^{\tilde{m}(\tilde{t}_c)}\frac{ds}{(\delta\eta)(s)}.$$

We obtain the announced equality by using the definition of the critical time for the symmetrized problem and $m(0) = \tilde{m}(0)$.

2. Let $t \in [0, \tilde{t}_c]$. From (5.12), we obtain, since $m(0) = \tilde{m}(0)$

$$\int_{m(0)}^{\tilde{m}(t)}\frac{ds}{(\delta\eta)(s)} = sgn(\beta)\int_0^t(|\beta|\xi)(\tau)\,d\tau.$$

3. From **Theorem 1** and (5.11), we obtain as in the proof of **Theorem 2** for any $t \le \tilde{t}_c$

$$N^p\beta_N^{\frac{p}{N}}\left(\tilde{\Gamma}_1(t)\right)^{p-1} = \xi(t)\,\eta(\tilde{m}(t))\left[\int_{m_1}^{m_2}s^{\frac{p'}{N}-p'}(\alpha^*)^{\frac{-p'}{p}}(s-m_1)\,ds\right]^{\frac{p}{p'}}$$

$$= \xi(t)\,\eta(\tilde{m}(t))\,\Phi((\tilde{m}(t)))$$

4. We assume that the problem $(\mathcal{P})$ admits a solution on $[0, t]$.

(i) Let $\tau \in [0, t]$. By the second relation of (5.6)

$$(\beta\xi)(\tau) = sgn(\beta)(|\beta|\xi)(\tau) = \frac{1}{(\delta\eta)(m(\tau))}\frac{dm(\tau)}{d\tau},$$

we have for any t such that $0 \le t \le t_c$,

$$\int_0^t(|\beta|\xi)(\tau)\,d\tau = sgn(\beta)\int_{m(0)}^{m(t)}\frac{ds}{(\delta\eta)(s)}.$$

Since $\beta(\tau)$ and $\frac{dm(\tau)}{d\tau}$ have the same signs, we are led to

$$\int_{m(0)}^{m(t)}\frac{ds}{(\delta\eta)(s)} \le \int_{m(0)}^{m_2}\frac{ds}{(\delta\eta)(s)} = \int_{m(0)}^{\tilde{m}(\tilde{t}_c)}\frac{ds}{(\delta\eta)(s)}$$

if $\beta > 0$ (the function $t \to \tilde{m}(t)$ is increasing). By the same way, we have

$$\int_{m(0)}^{m(t)}\frac{ds}{(\delta\eta)(s)} \ge \int_{m(0)}^{m_1}\frac{ds}{(\delta\eta)(s)} = \int_{m(0)}^{\tilde{m}(\tilde{t}_c)}\frac{ds}{(\delta\eta)(s)}$$

if $\beta < 0$. hence, using (5.13)

$$\int_0^t(|\beta|\xi)(\tau)\,d\tau \le \text{sgn}(\beta)\int_{\tilde{m}(0)}^{\tilde{m}(\tilde{t}_c)}\frac{ds}{(\delta\eta)(s)} = \int_0^{\tilde{t}_c}(|\beta|\xi)(\tau)\,d\tau$$

for any $t \le t_c$. Particularly, we have for $t = t_c$,

$$\int_0^{t_c}(|\beta|\xi)(\tau)\,d\tau \le \int_0^{\tilde{t}_c}(|\beta|\xi)(\tau)\,d\tau.$$

That is to say that $t_c \le \tilde{t}_c$ and consequently, there exists no regular solution after $\tilde{t}_c$.

(ii) We recall that for any t such that $0 \le t \le \min(t_c, \tilde{t}_c) = t_c$,

$$\int_{\tilde{m}(0)}^{\tilde{m}(t)}\frac{ds}{(\delta\eta)(s)} = \text{sgn}(\beta)\int_o^t(|\beta|\xi)(\tau)\,d\tau,$$

$$\int_{m(0)}^{m(t)}\frac{ds}{(\delta\eta)(s)} = \text{sgn}(\beta)\int_o^t(|\beta|\xi)(\tau)\,d\tau.$$

These two equalities and $m(0) = \tilde{m}(0)$ give

$$\int_{\tilde{m}(0)}^{\tilde{m}(t)}\frac{ds}{(\delta\eta)(s)} = \int_{m(0)}^{m(t)}\frac{ds}{(\delta\eta)(s)} = \int_{\tilde{m}(0)}^{m(t)}\frac{ds}{(\delta\eta)(s)}.$$

In conclusion, one has $\tilde{m}(t) = m(t)$ for any $t \le t_c$. This proves that $\tilde{\omega}(t)$ is the symmetrized domain of $\omega(t)$ for any $t \le t_c$.

(iii) For any $t \le t_c$, we have

$$\begin{aligned}(\Gamma_1(t))^{p-1} &\le N^{-p}\beta_N^{\frac{-p}{N}}(\Phi\delta)(m(t))\,\xi(t)\\ \left(\tilde{\Gamma}_1(t)\right)^{p-1} &= N^{-p}\beta_N^{\frac{-p}{N}}(\Phi\delta)(\tilde{m}(t))\,\xi(t)\\ &= N^{-p}\beta_N^{\frac{-p}{N}}(\Phi\delta)(m(t))\,\xi(t)\end{aligned}$$

because $\tilde{m}(t) = m(t)$ as it has just been proved. We deduce immediately that $\Gamma_1(t) \le \tilde{\Gamma}_1(t)$ for any $t \le t_c$.

Remark that from this inequality, one also has

$$C(u(t), \sigma(t)) = \int_\Omega \sigma(t)\cdot\nabla u(t)\,dx = \Gamma_1(t)\eta(m(t))\xi(t)$$

$$\le \tilde{\Gamma}_1(t)\eta(m(t))\xi(t) = \tilde{C}(U(t), \Sigma(t)) = \sum_{i=1}^{i=2}\int_{\tilde{\Omega}_i(t)}\tilde{a}_i|\nabla U_i(t)|^p\,dx$$

for any $t \le t_c$.

5.3. AN EXAMPLE OF MUSKAT PROBLEM WITH PRESCRIBED FLUX

5.3.1. The problem

Let p, ω_i, $\omega(0)$, $a_i(x, m)$, β, ξ, δ and η as in the section 5.2. For $i = 1, 2$ we consider functions G_i veryfing

i) $G_i : (x, m\ x) \in \Omega \times [m_1, m_2] \times \mathbb{R}^N \to G_i(x, m, \xi) \in \mathbb{R}$ are Caratheodory functions (that is, measurable with respect to x, continuous with respect to (m, ξ)),

ii) For almost every $x \in \Omega$, for any $m \in [m_1, m_2]$, $G_i(x, m,.)$ is strictly convex and admits a gradient $g_i(x, m,.)$,

iii) There exists c^1, c^2, $c^4 > 0$ and $c_i^3 \in L^1(\Omega)$ such that a.e. $x \in \Omega$, $\forall \xi \in \mathbb{R}^N$ and $\forall m \in [m_1, m_2]$

$$c^1|\xi|^p \le G_i(x, m, \xi) \le c^2|\xi|^p + c_i^3(x),$$
$$|g_i(x, m, \xi)| \le c^4\left(1+|\xi|^{p-1}\right),$$
$$g_i(x, m, \xi)\cdot\xi \ge a_i(x, m)|\xi|^p.$$

We set for $i = 1, 2$

$$\sigma_i(x, t) = g_i(x, m(t), \nabla u_i(t)).$$

Then, we have for $u_i(t) \in W^{1,p}(\Omega_i(t))$

$$\sigma_i(x, t)\cdot\nabla u_i(t) = g_i(x, m(t), \nabla u_i(t))\cdot\nabla u_i(t) \ge a_i(x, m(t))\,|\nabla u_i(t)|^p = \alpha_i(t)|\nabla u_i(t)|^p.$$

With such σ_i, we consider the statements $(\mathcal{R}_1)$, $(\mathcal{R}_2)$ of the Muskat problem with prescribed flux.

Example: We set

$$G_i(x, m, \xi) = \frac{1}{p}\left(A_i(x, m)\xi\cdot\xi\right)^{\frac{p}{2}}$$

where $A_i = \Omega \times [m_1, m_2] \to \mathbb{R}^{N\times N}$ are symmetric matrices such that for any $m \in [m_1, m_2]$, $A_i(., m) \in L^\infty(\Omega)^{N\times N}$ and

$$a.e.\ x \in \Omega,\ \forall \xi \in \mathbb{R}^N\ A_i(x, m)\xi\cdot\xi \ge a_i(x, m)|\xi|^2.$$

W recover the operator of [BoMos2].

5.3.2. Existence of solution for a given time t

For a given time t, we obtain the problem $(\mathcal{R}_2)$, with $Q(t) = \xi(t)\ \eta(m(t))$

$$(\mathcal{R}_2)\begin{cases} -\text{div}\, g_i(x, m(t), \nabla u_i(t)) = 0 \text{ in } \Omega_i(t),\ i = 1, 2 \\ u_1(t) = \text{unknown constant on } \gamma_1, \\ u_2(t) = 0 \text{ on } \gamma_2, \\ u_1(t) = u_2(t) \text{ on } \gamma(t), \\ g_1(x, m(t), \nabla u_1(t))\cdot\nu(t) = g_2(x, m(t), \nabla u_2(t))\cdot\nu(t) \text{ on } \gamma(t), \\ \int_{\gamma 1} g_1(x, m(t), \nabla u_1(t))\cdot n\, d\gamma = Q(t). \end{cases}$$

We relate to $(\mathcal{R}_2)$ the minimization problem $(\mathcal{M}_t)$: $Inf\{J(v),\ v \in V\}$ where

$$V = \left\{v \in W^{1,p}(\Omega),\ v = 0 \text{ on } \gamma_2,\ v_{|\gamma 1} = \text{undetermined constant}\right\}$$

and

$$J(v) = \int_{\Omega_1(t)} G_1(x, m(t), \nabla v)\, dx + \int_{\Omega_2(t)} G_2(x, m(t), \nabla v)\, dx - Q(t)\ v_{|\gamma 1}.$$

Then V is a closed subspace of the reflexive Banach space $W^{1,p}(\Omega)$ $(1 < p < \infty)$ with its usual norm. We equip V with the «gradient» norm $\|v\|_V = \|\nabla v\|_{L^p(\Omega)}$, $v \in V$. By the Poincaré inequality, this norm is equivalent to the one induced by $W^{1,p}(\Omega)$. One cheks that J is strictly convex, continuous and coercive on V. There exists, in consequence, a unique solution to the minimization problem $(\mathcal{M}_t)$. This solution, denoted by u, is characterized by the variational equation

$$\begin{cases} u \in V, \\ \int_{\Omega_1(t)} g_1(x, m(t), \nabla u)\cdot\nabla v\, dx + \int_{\Omega_2(t)} g_2(x, m(t), \nabla u)\cdot\nabla v\, dx = \\ = \xi(t)\ \eta(m(t))\ v_{|\gamma 1},\ \forall v \in V. \end{cases}$$

Finally, using the Green fromula, we get formally the problem $(\mathcal{R}_2)$.

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