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ON THE LEONTIEF'S PROBLEM IN BANACH LATTICES

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ABSTRACT

The object of this work is to find a solution of the Leontief problem in Banach lattices when a subsolution is known.

INTRODUCTION

The Leontief's model is one of the most useful and known in Mathematical Economics. It consists basically on finding a solution x of the equations x = Ax+e which is a non-negative vector of $\mathbb{R}^n$, bein A a square matrix of order n whose elements are all non-negative and e is a non-negative vector of $\mathbb{R}^n$. In economic terms x represents the production of an economic system, e the external demand and Ax the internal demand, being x = Ax+e the equation of equilibrium.

The Leontief model and problem has been extended in several directions, one of them considering non linear expressions Ax and other stating it for infinite dimensional spaces, like in [1], [7] and [8].

The aim of this work is the study of the Leontief model in Banach lattices, for it we will use the notations and concepts of [12].

From now on x will denote a Banach lattice and $T:X \to X$ will be a positive linear operator. For every $e \in X_+ = \{x \in X: x \geq 0\}$ we call Leontief's problem associated to e, and we will write Pe, to the problem of determine a solution $x \in X_+$, of the Leontief equation x = Tx+e. We say that $x \in X_+$ is a subsolution of Pe if $x \geq Tx + e$, and if x_0 is a subsolution of Pe we will denote by $P(e, x_0)$ the problem of finding a solution $x \in [0, x_0] = \{x' \in X: 0 \leq x' \leq x_0\}$.(X, T) has the subsolution property if for every $e \in X_+$ such that Pe has a subsolution x_0 then $P(e, x_0)$ has a solution. We say that X has the subsolution property if (X, T) has the subsolution property for every positive linear operator $T : X \to X$.

Theorem 1 of [7] states that if X is order complete then X has the subsolution property. As we will see this theorem solves the problem in a large number of cases. We give a constructive proof of this theorem by transfinite aproximations and moreover we find a maximal solution and a minimal one of the problem: P (e, x_0). Also we prove the existence of σ-order complete Banach lattices X non having the subsolutions property and we give sufficient conditions for (X, T) to have the subsolutions property.

Though we use linear operators, many of the results can be extended to the non-linear case. Concretely Theorem 1 can be stated for isotone operators $T : X \to X$ (i.e., $Tx \leq Ty$ if $x \leq y$). Thus Corollary 2 holds also for isotone operators non necessarily continuous. In particular, Theorem 3 of [7] is valid though T is not continuous.

As an application of the obtained results it is proved that if C(K) is σ-order complete and G is the open set of the cozeros of a continuous function $\varphi \in C(K)$, then the closure $\bar{G}$ is equivalent to the Stone-Cech compactification βG.

The problem $P(e, x_0)$ can be non-solvable, but we prove the existence always of $x'' \in X_+^{**}$ such that $x'' = T^{**} x'' + e$ and also the existence of a sequence $(x_n) \subset X_+$ suche that $x_n - Tx_n \to e$.

Theorem 1. *If the Banach lattice X is order complete then it has the subsolution property. Moreover, if x_0 is a subsolution of Pe ($e \in X_+$) then $P(e, x_0)$ has a maximum solution and a minimum solution $x \in [0, x_0]$.*

Proof. Let be $x_0 \geq Tx_0 + e$ ($x_0, e \in X_+$). Then we construct by transfinite induction a family $(x_\alpha)_\alpha$ making $x_1 = Tx_0 + e$ and

$$x_{\alpha_0} = T\left(\bigwedge_{\alpha<\alpha_0} x_\alpha\right) + e \in [0, x_0]$$

for every transfinite ordinal α_0. Clearly $(x_\alpha)_\alpha$ is a non increasing family and $x_\alpha \geq x$ for every solution x = Tx + e

$\in [0, x_0]$. Moreover if $(x_\alpha)_{\alpha<\alpha_0}$ is strictly decreasing then card $\{\alpha : \alpha < \alpha_0\} \leq$ card X and therefore, there exists an ordinal α_0 such that $x_{\alpha_0} = x_{\alpha_0+1} = Tx_{\alpha_0} + e$ and $x_\alpha = x_{\alpha_0}$ for every $\alpha \geq \alpha_0$. Then x_{α_0} is the greatest solution of P(e, x_0).

If we define $(x'_\alpha)_\alpha$ making $x'_0 = e$ and

$$x'_{\alpha_0} = T\left(\bigvee_{\alpha<\alpha_0} x'_\alpha\right) + e \in [0, x_0]$$

for every transfinite ordinal α_0, then it is proved in a similar way that there exists x'_{α_0} which is the least solution of P(e, x_0).

Corollary 2. *If the Banach lattice X is order continuous then X has the subsolution property.*

Proof. If follows immediately from Proposition 1.a.8 of [12], since every order continuous Banach lattice is order complete. However, in this case since $x_n \to x_w = \bigwedge_{n\in N} x_n$, we have that $x_w = Tx_w + e$. Analogously, if

$$x'_w = \bigvee_{n\in N} x'_n = \sum_{n+1\in N} T^n e \text{ then } x'_w = Tx'_w + e.$$

Corollary 3. *X has the subsolution property in the following cases:*

3.1. *X does not contain any latticed copy of c_0.*

3.2. *X is reflexive.*

3.3. *X is σ-order complete and it does not contain any lattice copy of l_∞.*

3.4. *X is order complete and separable.*

Proof. It follows trivially from Theorem 1.a.5 and Proposition 1.a.7 of [12].

Theorem 4. *Every dual space X* has the subsolution property.*

Proof. It is an immediate consequence of Theorem 1 since every dual space X* is order complete (see [12], p. 3).

A measure space (Ω, Σ, μ) (or a measure μ) is said to be localizable if $L^\infty(\Omega, \Sigma, \mu) = L^1(\Omega, \Sigma, \mu)^*$. A measure space (Ω, Σ, μ) is localizable if and only if μ has no atoms of infinite measure and $L^\infty(\Omega, \Sigma, \mu)$ is order complete [11].

Let us see an example of a non localizable measure space. Let Ω a non countable set, Σ the σ-álgebra formed by of the countable subsets $A \subset \Omega$ and their complementaries, $\mu(A) =$ card A if $A \subset \Omega$ is a finite subset and $\mu(A) = \infty$ if A is non finite. It is easily seen now that $L^\infty(\Omega, \Sigma, \mu)$ is not order complete and then it can not be a dual space.

Corollary 5. *If (Ω, Σ, μ) is a localizable measure space, then $L^\infty(\Omega, \Sigma, \mu)$ has the subsolution property.*

Let (Ω, Σ, μ) be a measure space. An space X consisting of equivalence classes, modulo equality almost everywhere, of Σ-measurable (or measurable) real functions on Ω is called a Köthe function space if the following condition hold:

(1) If $|f(w)| \leq |g(w)|$ a.e. on Ω, with f measurable and $g \in X$, then $f \in X$ and $\|f\| \leq \|g\|$.

(2) For every $E \in \Sigma$ with $\mu(E) < \infty$ the characteristic function $\chi_E \in X$.

(3) Every function $f \in X$ is locally integrable, i.e., there exists the integral $\int_E f d\mu$ for every $E \in \Sigma$ with $\mu(E) < \infty$.

This concept generalizes the Definition 1.b.17 of [12], where μ is assumed to be σ-finite and complete.

Theorem 6. *If $L^\infty(\Omega, \Sigma, \mu)$ is order complete, then every Köthe function space X on (Ω, Σ, μ) has the subsolution property.*

Proof. It will be enough to prove that X is order complete. Let $(x_i)_{i\in I}$ be a non void family in $[0, x_0]$, $x_0 \in X_+$. Since $L^\infty(\Omega, \Sigma, \mu)$ is order complete, there exists for every $n \in \mathbb{N}$ the supremum x_n of $(x_i \wedge n)_{i\in I}$ in $L^\infty(\Omega, \Sigma, \mu)$. Clearly $x_i \wedge n \in X$, since $x_i \wedge n \leq x_0$, and x_n is the supremum of $(x_i \wedge n)_{i\in I}$ in X. Then $x = \bigvee_1^\infty x_n \in [0, x_0]$ and it follows that x is the supremum of $(x_i)_{i\in I}$ in X.

Theorem 7. *If X is a Köthe function space on a measure space (Ω, Σ, μ) such that the support of every function $x \in X$ is of σ-finite measure, then X has the subsolution property.*

Proof. It suffices to see that X is order complete. In fact, let be $(x_i)_{i\in I}$ a non void family in $[0, x_0]$, $x_0 \in X_+$, A the support of x_0, $\Sigma_A = \{E \in \Sigma : E \subset A\}$ and $\mu_A = \mu|\Sigma_A$. Then $X_A = \{x|A : x \in X\}$ endowed with the norm $\|x|A\| = \|x\chi_A\|$ $(x \in X)$ is a Köthe function space on the σ-finite measure space (A, Σ_A, μ_A), which is order complete since $L^\infty(A, \Sigma_A, \mu_A) = L^1(A, \Sigma_A, \mu_A)^*$, from where it follows easily the existence of $x = \bigvee_{i\in I} x_i$.

Corollary 8. *Every space $L^p(\Omega, \Sigma, \mu)$, $1 \leq p < \infty$, has the subsolution property.*

A compact Hausdorff space K is said to be an extremally disconnected space if the closure of every open set in K is open.

Theorem 9. *If K is an extremally disconneted compact Hausdorff space, then X = C (K) has the subsolution property.*

Proof. It is enough to realize that X is order complete following the Proposition 1.a.4 of [12].

Lemma 10. *If $X = C(K)$ has the subsolution property then X has also the following property: If $0 \le x_1 \in X$, $0 \le x_2 \le 1$, $x_1, x_2 \in X$ then there exists $x_0 \in X_+$ such that $x_1 x_2 = x_1 x_0$.*

Proof. Clearly we can suppose that $x_1 \neq 0$. Let be

$$Tx = \left(1 - \frac{x_1}{\|x_1\|}\right) x \quad (x \in X),$$

then $T : X \to X$ is a positive linear operator verifying that

$$1 - T1 = \frac{x_1}{\|x_1\|} \ge \frac{x_1 x_2}{\|x_1\|} \ge 0.$$

Therefore, 1 is a subsolution of Pe with $e = x_1 x_2 / \|x_1\|$, and there exists $x_0 \in X_+$ such that

$$\frac{x_1 x_2}{\|x_1\|} = x_0 - Tx_0 = \frac{x_1 x_0}{\|x_1\|}$$

and $x_1 x_2 = x_1 x_0$.

Theorem 11. *If $X = C(K)$ has the subsolution property and the complementary of G is the set of the zeros of a continuous function $x \in X$ (i. e. G is the set of the cozeros of a continuous function), then the closure $\bar{G}$ of G is the Stone-Cech compactification βG of G.*

Proof. Let be $\psi = |x|$ and $0 \le \varphi \le 1$ a continuous function on G, then $0 \le \psi\varphi \in X$ and it follows from Lemma 10 that there exists a function $f \in X_+$ such that $\psi\varphi = \varphi f$, from where it results that $f|\bar{G}$ is the continuous extension to $\bar{G}$ of $f|G$. Therefore, $\bar{G} = \beta G$ [5, Corollary 2, p. 130].

Corollary 12. *If K is an extremally disconnected compact Hausdorff space and G is the set of the cozeros of a continuous function $x \in C(K)$ then $\bar{G} = \beta G$.*

Proof. It follows from Theorems 9 and 11.

Corollary 13. *If K is infinite metrizable compact space then X = C (K) has not the subsolution property.*

Proof. It follows from the hypothesis that there exists a non isolated point $w \in K$. Then $\{w\}$ is the set of zeros of a continuous function and it is easily seen that $\bar{G} \neq \beta G$ with $G = K \setminus \{w\}$.

Theorem 14. *If $T: X \to X$ is a positive linear operator and x_0 is a subsolution of Pe, $e \in X_+$,* then there exists $x'' \in x^{**}$ such that $0 \le x'' \le x_0$ and $x'' = T^{**} x'' + e$.

Proof. Let be $x_{n+1} = Tx_n + e$ $(n = 0, 1, \ldots)$. Then $x_n \in [0, x_0]$ $(=\{x \in X : 0 \le x \le x_0\})$ and $x_n \ge Tx_n + e$ for every $n \in \mathbb{N}$. Let $x'' \in X^{**}$ be an w*-aglomeration point of $(x_n)_{n\in\mathbb{N}}$, whose existence follows from the Alaoglu-Bourbaki theorem, and $x' \in X_+^*$. Then for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that

$$0 \le \langle x_n, x' \rangle - \langle x'', x' \rangle < \varepsilon$$

and therefore

$$\lim_n \langle x_n, x' \rangle = \langle x'', x' \rangle$$

since $(x_n)_{n\in\mathbb{N}}$ is a nonincreasing sequence and $x' \in X_+^*$. Thus $\lim_n x_n = x''$ in the w*-topology and

$$\langle x'', x' \rangle = \lim_n \langle x_{n+1}, x' \rangle =$$

$$= \lim_n \langle x_n, T^* x' \rangle + \langle e, x' \rangle =$$

$$= \langle x'', T^* x' \rangle + \langle e, x' \rangle =$$

$$= \langle T^{**} x'' + e, x' \rangle$$

for every $x' \in X^*$. So $0 \le x'' \le x_0$ and $x'' = T^{**} x'' + e$.

Corollary 15. *If $X = C(K)$ and $T : X \to X$ is a positive linear operator and x_0 is a subsolution of Pe, $e \in X_+$, then there exists a function x'' which is the pointwise limit of a nonincreasing sequence of functions $(x_n)_{n\in\mathbb{N}}$, with $x_n \in [0, x_0]$*, and $x'' = T^{**} x'' + e$, $0 \le x'' \le x_0$.

Corollary 16. *If the linear operator $T : X \to X$ is a weakly compact or a Dunford-Pettis one, then (X, T) has the subsolution property.*

Proof. Let x_0 be a subsolution of Pe, $e \in X_+$ and $x_{n+1} = Tx_n + e$ $(n = 0, 1, \ldots)$. Then, since $(x_n)_{n\in\mathbb{N}}$ is a nonincreasing sequence and T is a weakly compact or a Dunford-Pettis operator, $(Tx_n)_{n\in\mathbb{N}}$ is weakly convergent in X and, therefore, also $(x_{n+1})_{n\in\mathbb{N}} = (Tx_n + e)_{n\in\mathbb{N}}$. So $x'' \in X$ (of the Theorem 14) and $x'' = Tx'' + e$.

Theorem 17. *Let be*

$$x_0 - Tx_0 \ge e \qquad (x_0, e \in X_+)$$

and

$$x_{n+1} = Tx_n + e \ (n = 0, 1, \ldots).$$

Then for every $e > 0$ there exists a convex linear combination x of $(x_n)_{n\in\mathbb{N}}$ such that $\|x - Tx - e\| < \varepsilon$.

Proof. Let C be the convex hull of $(x_n)_{n\in\mathbb{N}}$. Then it suffices to prove that $e \in \overline{(I-T)C}$. Suppose that $e \notin \overline{(I-T)C}$. Then $\overline{(I-T)C}$ is a non void closed convex set and it follows from the separation theorem [10, p. 182] that there exists $x^* \in X^*$ such that

$$x^*\left((I-T)x\right) \geq \alpha > x^*(e)$$

for every $x \in C$ and a real number α. It follows now that if $\tilde{C} \subset X^{**}$ is the w*–closure of C then

$$x^*\left(\left(I-T^{**}\right)x''\right) \geq \alpha > x''(e)$$

for every $x'' \in \tilde{C}$, which is a contradiction with Theorem 14 which states that $e \in \left(I-T^{**}\right)\tilde{C}$.

Corollary 18. *Let be $x_0 - Tx_0 = e$ (x_0, $e \in X_+$). Then the closure of the set of the elements $e' \in [0, x_0]$ for which there exists $x' \in [0, x_0]$ such that $e' = x' - Tx'$, contains $[0, e]$.*

As a generalization of Theorem 1 we pose the following:

Problem. Is there any σ-order complete Banach lattice non having the subsolution property?

The answer is yes as we will see in the following.

Theorem 19. *There is a space $X = L^\infty(\Omega, \Sigma, \mu)$ which has not the subsolution property.*

Proof. Le Ω be the set of countable ordinal numbers i.e. less than ω_1, Σ the σ-algebra of the countable subsets $A \subset \Omega$ and their complementaries, $\mu(A)$ = card A if $A \in \Sigma$ is finite and $\mu(A) = \infty$ if A is not finite. Then let us consider $X = L^\infty(\Omega, \Sigma, \mu)$.

Let Ω_1, Ω_2 be the set of the numbers $\alpha \in \Omega$ of first and second kind respectively, $\Omega' = \{2\alpha + 1 : \alpha \in \Omega\}$ and $\Omega'' = \{2\alpha + 2 : \alpha \in \Omega\}$. For every $\alpha \in \Omega_2$ let be $\mathcal{U}'_\alpha$ and $\mathcal{U}''_\alpha$ two ultrafilters finer than the filter of sections $[\alpha', \alpha)$ ($\alpha' < \alpha$) such that $\Omega' \in \mathcal{U}'_\alpha$ and $\Omega'' \in \mathcal{U}''_\alpha$.

Since every $x \in X$ is of the form $x = c + \sum_{\alpha\in\Omega} c_\alpha e_\alpha$ with $e_\alpha = \chi_{\{\alpha\}}$ and $(c_\alpha)_{\alpha\in\Omega}$ bounded with $c_\alpha = 0$ for α sufficiently large, we can define $T : X \to X$ making

$$T\left(c + \sum_{\alpha\in\Omega} c_\alpha e_\alpha\right) = c\left(1 - e_1 - e_2\right) + \sum_{\alpha\in\Omega_1} c_\alpha e_{\alpha+2}$$

$$+ \sum_{\alpha\in\Omega_2} \lim_{\alpha', \mathcal{U}'_\alpha} c_{\alpha'} \cdot e_{\alpha+1}$$

$$+ \sum_{\alpha\in\Omega_2} \lim_{\alpha'', \mathcal{U}''_\alpha} c_{\alpha''} \cdot e_{\alpha+2}.$$

It is easily seen that T is a positive linear operator and $I - T1 = e_1 + e_2 > e_1 > 0$.

Then

$$x - Tx = c\left(e_1 + e_2\right) + \sum_{\alpha\in\Omega} c_\alpha e_\alpha - \sum_{\alpha\in\Omega_1} c_\alpha e_{\alpha+2}$$

$$- \sum_{\alpha\in\Omega_2} \lim_{\alpha', \mathcal{U}'_\alpha} c_{\alpha'} \cdot e_{\alpha+1} - \sum_{\alpha\in\Omega_2} \lim_{\alpha'', \mathcal{U}''_\alpha} c_{\alpha''} \cdot e_{\alpha+2}$$

and so if $x - Tx = e_1$ then $c + c_1 = 1$, $c + c_2 = 0$, $c_{\alpha+2} = c_\alpha$ for $\alpha \in \Omega_1$, $c_0 = c_1$ for $\alpha \in \Omega'$ and $c_0 = c_2$ for $\alpha \in \Omega''$. Thus it can not be $c_\alpha = 0$ for α sufficiently large since $c_1 \neq c_2$. Therefore $X = L^\infty(\Omega, \Sigma, \mu)$ has not the subsolution property.

In the last proof we have used the fact that every ordinal number γ can be expressed in an only way in the form $\gamma = 2\alpha + \beta$ with $\beta = 0, 1$.

Let Ω be an uncountable set, Σ the σ-algebra of the countable subsets $A \subset \Omega$ and their complementaries, $\mu(A)$ = card A if $A \in \Sigma$ is a finite set and $\mu(A) = \infty$ in other case, then it follows from Theorem 19 that $X = L^\infty(\Omega, \Sigma, \mu)$ has not the subsolution property.

The constructed operator T verifies that I-T in injective. Then the operator I-T** cannot be injective because in this case it follows from Theorem 24 that (X, T) would have the subsolution property.

Corollary 20. *There exists a σ-order complete space $X = C(K)$ not having the subsolution property.*

Proof. It is enough to realize that following [12, Theorem 1.b.6] the space $L^\infty(\Omega, \Sigma, \mu)$ considered in Theorem 19 is order isometric to some space C(K).

Theorem 21. *If X is σ-order complete and I-T is a positive operator then (X, T) has the subsolution property.*

Proof. Let be $x_0 - Tx_0 \geq e$ (x_0, $e \in X_+$), $x_{n+1} = Tx_n + e$ ($n = 0, 1, \ldots$) and $x_w = \bigwedge_1^\infty x_n$. Then for every convex linear combination x of $(x_n)_{n\in\mathbb{N}}$ we have that

$$e \leq x_w - Tx_w \leq x - Tx,$$

from where it results by Theorem 17 that $x_w = Tx_w + e$.

Corollary 22. *If $X = C(K)$ is σ-order complete and G is the set of cozeros of a function $x_0 \in C(K)$, then $\overline{G} = \beta G$.*

Proof. Clearly we can suppose that $0 \neq x_0 \in X_+$. Let be

$$Tx = \left(1 - \frac{x_0}{\|x_0\|}\right)x \quad (x \in X).$$

Then $T : X \to X$ and I–T are positive linear operators. Therefore, it follows from Theorem 21 that (X, T) has the subsolution property and it result now from the proof of Theorem 11 that $\overline{G} = \beta G$.

Last corollary improves Corollary 12 since X = C(K) is σ-order complete if and only if the closure of every open F_σ set is open [12, Proposition 1.a.4]. It results from this that G is the set of cozeros of a continuous real function if and only if G is a countable union of closed-open subsets of K.

Theorem 23. *Let Ω be an uncountable set, Σ the σ-algebra of all countable subsets $A \subset \Omega$ and their complementaries,* $\mu(A) = \text{card } A$ *if* $A \in \Sigma$ *is finite and* $\mu(A) = \infty$ *in other case, and* $X = L^\infty(\Omega, \Sigma, \mu)$. *If* $x_0 - Tx_0 \geq e$ ($x_0, e \in X_+$) *and* Tx_0 has a countable support then the corresponding problem Pe is solvable.

Proof. Let w_1 be the first uncountable ordinal number. Let us make $x_1 = Tx_0 + e$ and

$$x_{\alpha_0} = T\left(\bigwedge_{\alpha<\alpha_0} x_\alpha\right) + e$$

for every ordinal $\alpha_0 < w_1$, since $L^\infty(\Omega, \Sigma, \mu)$ is σ-order complete. Then the families $(x_\alpha)_{\alpha<w_1}$ and $(Tx_\alpha)_{\alpha<w_1}$ are nonincreasing. Let $\{w_n : n\in\mathbb{N}\}$ the support of Tx_0. Then for every $n\in\mathbb{N}$ there exists $\alpha_n < w_1$ such that $(Tx_\alpha)(w_n) = (Tx_{\alpha_n})(w_n)$ for every $\alpha \geq \alpha_n$. Let be $\alpha_0 = \sup_n \alpha_n \ (< w_1)$, then $(Tx_\alpha)(w_n) = (Tx_{\alpha_0})(w_n)$ for every $\alpha \geq \alpha_0$ and every $n\in\mathbb{N}$ and $(Tx_\alpha)(w) = 0$ for every $w \notin \{w_n : n\in\mathbb{N}\}$. Thus $Tx_\alpha = Tx_{\alpha_0}$ for every $\alpha \geq \alpha_0$, from where it follows that $Tx_{\alpha_0+1} + e = Tx_{\alpha_0} + e = x_{\alpha_0+1}$.

Let X be a Banach lattice of sequences $x = (a_n)_{n\in\mathbb{N}}$ of real numbers such that $x \geq 0$ if and only if $a_n \geq 0$ for every $n\in\mathbb{N}$. Then it can be proved like in Theorem 23 that if X is σ-order complete then it has the subsolution property. Instead of this it can be proved that then X is order complete, but the used method let to prove that there exists a solution x_α with $\alpha < w_1$.

Let us denote by GS(X) the union of all the vector spaces $i_0^{**}(X_0^{**})$ being X_0 a separable subspace of X and i_0 the inclusion $X_0 \to X$.

We define

$B_1(X) = \{x'' \in X^{**}$: there exists $(x_n)_{n\in\mathbb{N}} \subset X$ such that $\lim_n x_n = x''$ in the w*–topology$\}$

and $Ba(X) \subset X^{**}$ the minimum set containing X which is closed under the w*–limit of sequences. Since it is easy to prove that GS(X) is also closed under the w*–limit of sequences and $X \subset GS(X)$, we have that $Ba(X) \subset GS(X)$.

Theorem 24. *If X is σ-order complete and* $\text{Ker}(I - T^{**}) \subset GS(X)$, then (X, T) has the subsolution property.

Proof. Let be $x_0 - Tx_0 \geq e$ ($x_0, e \in X$) and $(x_\alpha)_{\alpha<w_1}$ the family constructed like in Theorem 23.

Let x" be the w*-limit of $(x_\alpha)_{\alpha<w_1}$ (which is nonincreasing) and x_1'' the w*-limit of $(x_n)_{n\in\mathbb{N}}$. Then, since

$$x_1'' - T^{**}x_1'' = e = x'' - T^{**}x'',$$

we have that

$$\left(1 - T^{**}\right)\left(x_1'' - x''\right) = 0$$

and therefore $x_1'' - x'' \in GS(X)$ and $x'' \in GS(X)$ because $x_1'' \in GS(X)$.

Let $D = (x_n^*)_{n\in\mathbb{N}}$ be a countable subset of X_+^*. Then, $(x_n^*(x_\alpha))_{\alpha<w_1}$ is a nonincreasing family of nonnegative real numbers and for every $n \in \mathbb{N}$ there exists $\alpha_n < w_1$ such that $x_n^*(x_\alpha) = x_n^*(x_{\alpha_n})$ for every $\alpha \geq \alpha_n$. Let be $\alpha_0 = \sup_n \alpha_n < w_1$, then $x_n^*(x_\alpha) = x_n^*(x_{\alpha_0})$ for every $\alpha \geq \alpha_0$ and every $n \in \mathbb{N}$ and therefore $x_n^*(x'') = x_n^*(x_{\alpha 0})$ for every $n \in \mathbb{N}$. Since every $x^* \in X^*$ is the difference of two positive elements of X^*, the last resul hold also when x_n^* are non necessarily positive.

It is deduced now from the H. Corson theorem [15, 2.3.2] that x" is w*-continuous on every w*-separable subset.

Being $x'' \in GS(X)$ there exists a separable space $X_0 \subset X$ and $x_0'' \in X_0^{**}$ such that $x'' = i_0^{**}(x_0'')$ for the inclusion $i_0 : X_0 \to X$. Since X_0 is separable and $X_0^* \approx X^*/X_0^\perp$, there exists a sequence $(x_n')_{n\in\mathbb{N}}$ of unit ball X_1^*(of X*) such that $(i_0^* x_n')_{n\in\mathbb{N}}$ is w*-dense in the unit ball X_{01}^* of X_0^*. Let Z' be the w*-closure of $(x_n')_{n\in\mathbb{N}}$, then Z' is w*-separable, x" is w*-continuous on Z' and it is deduced that x_0'' is w*-continuous on X_{01}^*. From this it is deduced (see [14, Lemma 1]) that $x_0'' \in B_1(X_0)$ and therefore, that $x'' \in B_1(X)$.

Let be $K = X_+^* \cap X_1^*$ endowed with the w*-topology and $(y_n)_1^\infty \subset X$ a sequence with $\lim_n y_n = x''$ in the w*-topology. Let be $z_n(t) = \sup_{k\geq n} y_k(t) \ (t \in K)$

Then, since

$$\lim_\alpha x_\alpha(t) = x''(t) < z_n(t) + \frac{1}{k} \ (n, k \in \mathbb{N})$$

there exists $\alpha = \alpha(n, k, t) < w_1$ such that

$$x_\alpha(t) < z_n(t) + \frac{1}{k},$$

and being z_n a lower semicontinuous function on K and $x_\alpha \in X$, there exists a neighborhood V of t such that

$$x_\alpha(t') < z_n(t') + \frac{1}{k}$$

for every $t' \in V$. Since K is compact and $(x_\alpha)_{\alpha<w_1}$ is a nonincreasing family it follows now the existence of $\alpha_{nk} < w_1$ such that

$$x_\alpha(t) < z_n(t) + \frac{1}{k}$$

for every $\alpha \geq \alpha_{nk}$ and every $t \in K$. Let be $\alpha_0 = \sup_{n,k} \alpha_{n,k} < w_1$, then

$$x_\alpha(t) \leq \lim_n z_n(t) = x''(t) \leq x_\alpha(t)$$

for every $\alpha \geq \alpha_0$ and every $t \in K$. Then $x''(x^*) = x_{\alpha_0}(x^*)$ for every $x^* \in X_+^* \cap X_1^*$ and therefore for every $x^* \in X^*$. Then $x'' = x_{\alpha_0} \in X$, $x'' \in [0, x_0]$, and $x'' = Tx'' + e$.

In particular, if I-T** is injective the result can be easily proved. In fact, since $x_1'' \geq x_w \geq x''$ and $(I-T^{**})(x_1'' - x'') = 0$, it follows that $x_1'' = x''$ and so $x_1'' = x_w$ and $x_w = Tx_w + e$. In a similar way it results that $x_w = Tx_w + e$ if Ker (I-T**) is an ideal.

*Corollary 25. If X is σ-order complete and Ker(I-T**) ⊂ Ba(X), then (X, T) has the subsolution property.*

Proof. It is enough to realize that Ba(X) ⊂ GS (X).

Remark 1, It results from [9, Corollary II.10 and Theorem II. 14] the equivalence between the conditions Ker(I-T**) ⊂ Ba(X) and Ker(I-T**) ⊂B_1(X).

2. In Theorem 24 the condition Ker(I-T**) ⊂ GS(X) can be replaced by Ker(I-T**) ∩ DS(X) ⊂ GS(X), where DS(X) denotes the set of the differences $x_1'' - x_2''$ of two functions $x_i'' \in X_+^{**}$ which are lower semicontinuous on $(X_+^*, weak^*)$.

The element $x'' \in X_+^{**}$ of Theorem 24 is an upper semicontinuous function on (X_+^*, weak^*) but it is not immediate that it will be universally measurable on the unit ball of (X_1^*, weak^*). Let us treat now this question.

Lemma 26. *The element $x'' \in X_+^{**}$ of Theorem 24 is universally measurable on the unit ball $(X_+^{**}, \text{weak}^*)$ and moreover it belongs to the space $B_r'(X_1^*)$ and of the functions x" whose restrictions $x''|_L$ has a point of continuity, for every non void w*-closed subset $L \subset X_1^*$.*

Proof. Let $Z = \{x_\alpha : \alpha < w_1\}$ be like in Theorem 24. Since Z is a nonincreasing family then every sequence $Z' \subset Z$ has a pointwise (or w*–) convergent subsequence, from [15, 14.1.7] it follows that Z is pointwise relatively compact in $B_r'(X_1^*)$ and therefore $x'' \in B_r'(X_1^*)$. Also, it follows that Z is topologically stable and then stable for every finite Radon measure μ on (X_1^*, weak^*).

Moreover, for every one of that measures it follows from [15, 9.5.2] that the identity mapping $(\overline{Z}, \tau_p) \to (\overline{Z}, \tau_m)$ is continuous, being τ_p and τ_m the topologies of the pointwise convergence and the convergence in measure respectively, and $\overline{Z}$ the closure of Z in τ_p. Therefore, since $x_\alpha \to x''$ in the topology τ_p (or w*) and τ_m is metrizable, there exists $\alpha_0 < w_1$ such that $x_\alpha = x''$ on supp μ for every $\alpha \geq \alpha_0$ and consequently also on the closed linear hull of supp μ in the w*–topology, as it follows proceeding like in H. Corson theorem [15, 2.4.2].

Theorem 27. *If X is σ-order complete and*

$$Ker(I-T^{**}) \cap DS(X) \cap B_r'(X_1^*) \subset GS(X),$$

then (X, T) has the subsolution property.

Proof. It is enough to proceed like in Theorem 24 using Lemma 26.

Lemma 28. *If X and Y are two Banach spaces and T : X → Y is a bounded linear operator such that T** X** ⊃ Y, then TX = Y.*

Proof. Let us prove first that the range T*Y* of T* is a closed subset. Let be $x^* = \lim_n T^* y_n^*, y_n^* \in Y^*$. Then, since for every y ∈ Y there exists x" ∈ X** such that T** x" = y, we have that

$$\langle x'', x^* \rangle = \lim_n \langle x'', T^* y_n^* \rangle$$

$$= \lim_n \langle T^{**} x'', y_n^* \rangle$$

$$= \lim_n \langle y, y_n^* \rangle.$$

Since this limit exists for very y ∈ Y, it follows that $(y_n^*)_{n\in N}$ is a bounded sequence and therefore it has an w*–aglomeration point y* and so x* = T*y* with y* ∈ Y*.

Being the range of T* a closed subset then the range of T is also closed and then it results from T**X** ⊃ Y and the Hahn-Banach theorem that $TX = \overline{TX} = Y$.

Given X = C(K) and a closed-open subset $H \subset K$, we denote by X_H (=C(H)) the subspace (of X) of all functions x ∈ X which are null otuside H.

Theorem 29. *If (I-T)X ⊂X_H and there exists $x_1 \in X_+$ such that $x_1 - Tx_1 \geq \chi_H$, then (I-T) X = X_H. Moreover if Z = Ker (I-T) is an ideal then (X, T) has the subsolution property.*

Proof. The first part follows from Theorem 14 and Lemma 28. Suppose now that Z = Ker(I-T) is an ideal. Let $x \to \dot{x}$ that natural mapping X → X/Z and $x_0 - Tx_0 \geq e$ (x_0, e ∈ X_+). Then it follows from Theorem 17 that there exists a sequence $(x_n)_{n \in N} \subset [0, x_0]$ such that $(1-T)\, x_n \to e$. Therefore, it results from the Banach homomorphism theorem that

$$\|\dot{x}_m - \dot{x}_n\| \leq M \|(1-T)(x_m - x_n)\| \to 0 \ (M > 0).$$

Then $(\dot{x}_n)_{n\in N}$ is convergent to some $\dot{x} \in [\dot{0}, \dot{x}_0]$ since X/Z is a Banach lattice [12, p.3]. If $u = (x \vee 0) \wedge x_0$ then $\dot{u} = (\dot{x} \vee \dot{0}) \wedge \dot{x}_0 = \dot{x}$ and (1 – T) u = (1 – T) x = e.

In particular, since every L^∞ (Ω, Ω, μ) space is order isometric to some C(K) [12, Theore 1. b. 6], Theorem 29 can be applied to $X = L_\infty(\Omega, \Sigma, \mu)$. This can be also proved directly taking as X_H the space of functions null outside H, with H measurable.

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