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ON THE IDEAL OF ALL SUBSETS ON ℕ OF DENSITY ZERO

(barrelled space/density zero)

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ABSTRACT

In this note we obtain some strong barrelledness properties concerning the simple function space generated by the hereditary ring $\mathcal{Z}$ of all subsets of density zero of ℕ.

RESUMEN

En esta nota obtenemos algunas propiedades de fuerte tonelación relativas al espacio de las funciones simples generado por el anillo hereditario $\mathcal{Z}$ de todos los subconjuntos de densidad cero de ℕ.

1. PRELIMINARIES

If $\mathcal{R}$ is a ring of subsets of a set Ω, we denote by *ba* $(\mathcal{R})$ the linear space over the field $\mathbb{K}$ of the real or complex numbers consisting of all those bounded finitely additive scalar measures on $\mathcal{R}$. This is a Banach space when it is provided with the supremum-norm

$$\|\mu\| = \sup\left\{|\mu(E)| : E \in \mathcal{R}\right\}$$

for each $\mu \in ba(\mathcal{R})$. A ring $\mathcal{R}$ of subsets of a set Ω is said to have the *Nikodym Property,* or *Property (N)*, if given any subset $\{\mu_i : i \in I\}$ of $ba(\mathcal{R})$ such that $\sup\left\{|\mu_i(E)| : i \in I\right\} < \infty$ for each $E \in \mathcal{R}$, then $\sup_{i\in I}\|\mu_i\| < \infty$. As it is well-known, this is equivalent to the barrelledness of the linear space $l_0^\infty(\mathcal{R})$ of all scalar $\mathcal{R}$-simple functions equipped with the supremum-norm

$$\|f\| = \sup\left\{|f(w)| : w \in \Omega\right\}$$

for each $f \in l_0^\infty(\mathcal{R})$.

It has been recently shown in [4] that the hereditary ring $\mathcal{Z}$ consisting of all those subsets of ℕ with density zero has Property (*N*). Let us recall that a subset A of ℕ is said to be of density zero if

$$\lim_{n\to\infty} \frac{|A \cap \{1, \ldots, n\}|}{n} = 0.$$

Following [4] $\mathcal{Z}(l_p)$ will stand for the subspace of l_p, for $1 \le p \le \infty$, consisting of all those sequences whose support is a set of density zero.

The proof in [4], extended in [5] to the ideal $\mathcal{R}$ of all η-zero sets of any strongly nonatomic submeasure η defined on a σ-algebra of sets, works on *ba* $(\mathcal{Z})$ and it is strongly based upon the following property of the scalar series that it seems to be originally due to Auerbach [1].

Lemma 1.1. *Let $\xi = (\xi_n)$ be a scalar sequence. If $\sum_{n\in A}|\xi_n|$ converges for each $A \in \mathcal{Z}$, then $x \in l_1$.*

The argument given in [5] uses an extension of Auerbach's result given in [11] and works in *ba* $(\mathcal{Z})$, although is easier than the original proof. In this note we obtain some strong barrelledness properties of the normed space $l_0^\infty(\mathcal{Z})$ beyond the barrelledness, working directly on the algebraic and topological structure of $l_0^\infty(\mathcal{Z})$. Naturally this have consequences in the boundedness Nikodym theorem, as one can deduce from 7.3.2. and 7.3.3. of [9].

2. RESULTS

In order to set up the techniques we will use in this paper we start reviewing the proof of the barrelledness of $l_0^\infty(\mathcal{Z})$, to do so we will need three previous results. The proof of the first of them is partially inspired in [3, Lemma 2].

Lemma 2.1. *Let $\mathcal{R}$ be a ring of subsets of Ω which is not an algebra. If $\mathcal{A}$ denotes the algebra generated by the ring $\mathcal{R}$, then $l_0^\infty(\mathcal{Z})$ is a closed hyperplane of $l_0^\infty(\mathcal{A})$.*

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Proof. It is obvious that $l_0^\infty(\mathcal{A}) = l_0^\infty(\mathcal{R}) \oplus \langle\{\chi_\Omega\}\rangle$, since $\mathcal{A} = \mathcal{R} \cup \{\Omega - E : E \in \mathcal{R}\}$. Let us see that $\overline{l_0^\infty(\mathcal{R})}$, where the closure is taken in $l_0^\infty(\mathcal{A})$, coincides with $l_0^\infty(\mathcal{R})$. So let $f = \sum_{i=1}^n a_i\chi_{E_i} \in \overline{l_0^\infty(\mathcal{R})}$, $f \neq 0$, where $\{E_1, ..., E_n\}$ is a partition on Ω by elements of $\mathcal{A}$, $a_i \in \mathbb{K}$ for $1 \leq i \leq n$, with $E_i \cap E_j = \phi$ and $a_i \neq a_j$ if $i \neq j$. Set $\varepsilon := \min\{|a_i - a_j| : 1 \leq i < j \leq n\}$ and choose $g = \sum_{j=1}^m b_j\chi_{F_j} \in l_0^\infty(\mathcal{R})$, where $F_j \in \mathcal{R}$ and $b_j \in \mathbb{K}$, $b_j \neq 0$, for $1 \leq j \leq m$, with $F_i \cap F_j = \phi$ and $b_i \neq b_j$ if $i \neq j$, such that $\|f - g\| < |a_k|$ for each $k \in \{1, ..., n\}$ with $a_k \neq 0$, and $\|f - g\| < \frac{\varepsilon}{2}$. We are going to prove that if $a_i \neq 0$ and $J_i := \{1 \leq j \leq m : F_j \cap E_i \neq \phi\}$, then $E_i = \bigcup_{j\in J_i} F_j$. This shows that $E_i \in \mathcal{R}$ and establishes the theorem.

Note in first place that if $F_j \cap E_i \neq \phi$ for some $j \in \{1, ..., m\}$, then $F_j \subseteq E_i$. Indeed, given any $w \in F_j$ then $|b_j - a_i| < \frac{\varepsilon}{2}$ since $F_j \cap E_i \neq \phi$, so if there would be some $t \in F_j \sim E_i$ then F_j would meet some E_k with $k \neq i$, i.e. $|b_j - a_i| < \frac{\varepsilon}{2}$, and then we would have

$|a_i - a_k| \leq |a_i - b_j| + |b_j - a_k| < \varepsilon$, a contradiction. On the other hand, assume that there is some $w \in E_i \sim \bigcup_{j\in J_i} F_j$. then obviously $g(w) = 0$ and consequently $|a_i| \leq \|f - g\| < |a_i|$, which is again a contradiction. ■

Lemma 2.2. *Each barrel T in $l_0^\infty(\mathcal{Z})$ absorbs the set of the characteristic functions of the finite subsets of $\mathbb{N}$.*

Proof. We proceed by contradiction. Assume there is a barrel T in $l_0^\infty(\mathcal{Z})$ which does not absorb the set $\mathcal{F} = \{\chi_F : F \subseteq \mathbb{N}, F \text{ finite}\}$. Let $F_1 \in \mathcal{F}$ such that $\chi_{F_1} \notin T$. Let $\{P_1, P_2\}$ be a partition of the set $\{n \in \mathbb{N} : n > \max F_1\}$ such that if $m \in P_i$, then $m + 1 \in P_j$, for $1 \leq i, j \leq 2$, $i \neq j$. Obviously, T cannot absorb the characteristic functions of all finite subsets either of P_1 or of P_2. Assume for example that T does not absorb the set $\{\chi_F : F \subseteq P_1, F \text{ finite}\}$. Then there exists a finite set $F_2 \subseteq P_1$ such that $\chi_{F_2} \notin 2T$. Now if $\{P_1', P_2'\}$ is a partition of $\{n \in P_1 : n > \max F_2\}$ such that if $m \in P_i'$ then $m + 2 \in P_j'$, for $1 \leq i, j \leq 2$, $i \neq j$, reasoning as above one shows that T does not absorb for example the set $\{\chi_F : F \subseteq P_1', F \text{ finite}\}$, and so there is finite set $F_3 \subseteq P_1'$ such that $\chi_{F_3} \notin 3T$.

Proceeding by recurrence one finds a sequence $\{F_n : n \in \mathbb{N}\}$ of finite subsets of $\mathbb{N}$ such that $\bigcup_{i=1}^\infty F_i \in \mathcal{Z}$ and $\chi_{F_n} \notin nT$. But setting $E = \bigcup_{i=1}^\infty F_i$, if 2^E stands for the σ-algebra for all subsets of E, obviously $l_0^\infty(2^E)$ is a barrelled space. Hence T absorbs the closed unit ball of $l_0^\infty(2^E)$, and consequently there is some $m \in \mathbb{N}$ such that $\chi_{F_m} \in mT$, a contradiction. ■

Lemma 2.3. ([4, Lemma 1]) *For every sequence $\{A_n : n \in \mathbb{N}\}$ of infinite sets in $\mathcal{Z}$ there exist finite sets $F_n \subseteq A_n$, $n \in \mathbb{N}$, such that $\bigcup_{n=1}^\infty (A_n - F_n) \in \mathcal{Z}$.*

Theorem 2.4. *$l_0^\infty(\mathcal{Z})$ is barrelled.*

Proof. Let $\mathcal{A}$ denote the algebra of subsets of $\mathbb{N}$ generated by the ring $\mathcal{Z}$. Assume that $l_0^\infty(\mathcal{A})$ is not barrelled and let T be a barrel in $l_0^\infty(\mathcal{A})$ which is not a neighborhood of the origien in $l_0^\infty(\mathcal{A})$. As $l_0^\infty(\mathcal{Z})$ is a closed one-codimensional subspace of $l_0^\infty(\mathcal{A})$ by virtue of Lemma 2.1, T cannot absorb the closed unit ball of $l_0^\infty(\mathcal{Z})$ and thus there exists some $A_1 \in \mathcal{Z}$ such that $\chi_{A_1} \notin T$.

Since $l_0^\infty(2^{A_1})$ is barrelled, T absorbs its closed unit ball, and hence T does not absorb the unit ball of the space $l_0^\infty(\mathcal{Z} \cap (\mathbb{N} - A_1))$. So, there is some $A_2 \in \mathcal{Z}$ with $A_2 \cap A_1 = \phi$, such that $\chi_{A_2} \notin 2T$. Again, as $l_0^\infty(2^{A_2})$ is barrelled, T cannot absorb the closed unit ball of $l_0^\infty(\mathcal{Z} \cap (\mathbb{N} - A_1 \cup A_2))$, and so on.

Consequently we obtain by recurrence a sequence $\{A_n : n \in \mathbb{N}\}$ of pairwise disjoint sets of density cero such that $\chi_{A_n} \notin nT$ for each $n \in \mathbb{N}$. But, according to Lemma 2.3, for each $i \in \mathbb{N}$ there is a finite set $F_i \subseteq A_i$ such that $\bigcup_{i=1}^\infty (A_i - F_i) \in \mathcal{Z}$. Hence, by putting $E = \bigcup_{i=1}^\infty (A_i - F_i)$ it follows that $l_0^\infty(2^E)$ is barrelled and, consequently, there is some $q \in \mathbb{N}$ with $\chi_{A_i - F_i} \in qT$ for each $i \in \mathbb{N}$. But, on the other hand, according to Lemma 2.2 there is some $p \in \mathbb{N}$ such that $\chi_{F_i} \in qT$ for each $i \in \mathbb{N}$. Therefore, setting $m \geq p + q$, one has that

$$\chi_{A_m} = \chi_{A_m - F_m} + \chi_{F_m} \in mT,$$

a contradiction. ■

Let us recall the definitions of some well known strong barrelledness properties [see for instance [9]]. A locally convex space is said to be suprabarrelled or barrelled of class 1 if given an increasing sequence of linear subspaces covering E there is one of them which is barrelled and dense in E. Given $n \in \mathbb{N}$, a locally convex space E is called barrelled of class n [barrelled of calls 0 = barrelled] if given an increasing sequence of linear subspaces covering E there is one of them which is barrelled of class $n - 1$ and dense in E. A locally convex space is said to be barrelled of class $\aleph_0$ if E barrelled of class n for each $n \in \mathbb{N}$. More general: if $\alpha + 1$ is a succesor ordinal, a locally convex space E is said to be barrelled of class $\alpha + 1$ [12] if given an increasing sequence of vector subspaces of E covering E, one of them is dense and barrelled of class α, and if $\alpha > 1$ is a limit ordinal a locally convex space E is said to be barrelled of class α if E is barrelled of class β for each $\beta < \alpha$. Following [2] a web in a set Ω is a family $\mathcal{W} = \{C_{n_1,...,n_i} : i, n_1,..., n_i \in \mathbb{N}\}$ of subsets of Ω such that $\Omega = \bigcup_{n_1=1}^\infty C_{n_1}$ and for $n_1,..., n_i \in \mathbb{N}$ then

$C_{n_1,\ldots,n_i} = \bigcup_{n_i+1=1}^{\infty} C_{n_1,\ldots,n_i,n_{i+1}}$. A strand of $\mathcal{W}$ is a sequence of subsets $\{C_{n_1,\ldots,n_i}\}_i$ where $\{n_i\}$ is sequence in $\mathbb{N}$. According to [8] a web of a locally convex space is said to be linear if it is formed by linear subspaces $\{E_{n_1,\ldots,n_i} : i, n_1,\ldots, n_i \in \mathbb{N}\}$ of E such that $E_{n_1} \subseteq E_{n_1+1}$ and $E_{n_1,\ldots,n_i,n_{i+1}} \subseteq E_{n_1,\ldots,n_i,n_{i+1}+1}$ for all $i, n_1,\ldots, n_i \in \mathbb{N}$. Baireled spaces [8] are defined as those locally convex spaces E such that each linear web in E contains a strand formed by barrelled and dense subspaces. It has been shown in [13] that baireled spaces, called superbarrelled therein, are precisely the barrelled spaces of class $\aleph_1$.

For the next lemma let $\{E_n : n \in \mathbb{N}\}$ be an increasing sequence of linear subspaces of $l_0^\infty(\mathcal{Z})$, T_n a barrel in E_n, B_n the closure of T_n in $l_0^\infty(\mathcal{Z})$ and —denoting by $\langle B_n\rangle$ the linear span of B_n— let $H_n := \bigcap_{m\geq n} \langle B_m\rangle$ for each $n \in \mathbb{N}$.

Lemma 2.5. *There exists a positive integer p such that*

$$\{\chi_F : F \subseteq \mathbb{N},\ F \text{ finite}\} \subseteq H_p.$$

Proof. Assume $\{\chi_F : F \subseteq \mathbb{N},\ F \text{ finite}\} \not\subseteq H_n$ for each $n \in \mathbb{N}$. Then there is a finite set F_1 in $\mathbb{N}$ such that $\chi_{F_1} \notin H_1$. Then, as in the proof of Lemma 2.2, let $\{P_1, P_2\}$ be a partition of $\{n \in \mathbb{N} : n > \max F_1\}$ such that if $m \in P_i$, then $m + 1 \in P_j$, for $1 \leq i, j \leq 2$, $i \neq j$.

Clearly, none of the H_n can contain the characteristic functions of the finite subsets of P_1 or of P_2. So we may assume for example that no H_n contains the set $\{\chi_F : F \subseteq P_1,\ F \text{ finite}\}$ for each $n > 1$. So there exists a finite set $F_2 \subseteq P_1$ such that $\chi_{F_2} \notin H_2$. Then, let $\{P_1', P_2'\}$ be a partition of $\{n \in P_1 : n > \max F_2\}$ such that if $m \in P_i'$ them $m + 2 \in P_j'$, for $1 \leq i, j \leq 2$, $i \neq j$. Reasoning as above one shows that no $H_n > 2$ absorbs for example the set $\{\chi_F : F \subseteq P_1',\ F \text{ finite}\}$, and here there is a finite set $F_3 \subseteq P_1'$ such that $\chi_{F_3} \notin H_3$.

Proceeding by recurrence one obtains a sequence $\{F_n : n \in \mathbb{N}\}$ of pairwise disjoint finite subsets of $\mathbb{N}$ such that $\bigcup_{i=1}^{\infty} F_i \in \mathcal{Z}$ and $\chi_{F_n} \notin H_n$. Then, setting $M = \bigcup_{i=1}^{\infty} F_i$, as $l_0^\infty(2^M)$ is suprabarrelled and the sequence $\{H_n : n \in \mathbb{N}\}$ is increasing and covers $l_0^\infty(\mathcal{Z})$, there is a positive integer p such that $H_p \cap l_0^\infty(2^M)$ is a dense and barrelled subspace of $l_0^\infty(2^M)$. Thus $\langle B_m\rangle \cap l_0^\infty(2^M)$ is barrelled and dense in $l_0^\infty(2^M)$ for each $m \geq p$. But, as may be easily shown, the barrelledness of $\langle B_m\rangle \cap l_0^\infty(2^M)$ implies that it must be closed in $l_0^\infty(2^M)$. Hence $\langle B_m\rangle \supseteq l_0^\infty(2^M)$, and consequently B_m absorbs the closed unit ball of $l_0^\infty(2^M)$ for each $m \geq p$. Thus $\chi_{F_p} \in H_p$, a contradiction. ■

Theorem 2.6. *$l_0^\infty(\mathcal{Z})$ is a suprabarrelled space.*

Proof. Let $\mathcal{A}$ denote the algebra of subsets of $\mathbb{N}$ generated by the ring $\mathcal{Z}$. Assuming $l_0^\infty(\mathcal{A})$ is not suprabarrelled there exists an increasing sequence $\{E_n : n \in \mathbb{N}\}$ of dense linear subspaces of $l_0^\infty(\mathcal{A})$ covering $l_0^\infty(\mathcal{A})$ such taht no E_n is barrelled. Thus, for each positive integer n let T_n be a barrel in E_n which is not a neighborhood of the origin in E_n and denote by B_n its closure in $l_0^\infty(\mathcal{A})$. Then define H_n as in the previous lemma.

According to Lemma 2.5 there is not loss of generality by assuming that

$$\{\chi_F : F \subseteq \mathbb{N},\ F \text{ finite}\} \subseteq H_n$$

for each $n \in \mathbb{N}$. Then, as $l_0^\infty(\mathcal{Z})$ is a closed one-condimensional subspace of $l_0^\infty(\mathcal{A})$ by virtue of Lemma 2.1, no H_n contains the closed unit ball of $l_0^\infty(\mathcal{Z})$ and thus there exists some $A_1 \in \mathcal{Z}$ such that $\chi_{A_1} \notin H_1$. So, since $l_0^\infty(2^{A_1})$ is suprabarrelled, reasoning as in the last part of the proof of the previous lemma, there is some positive integer $n_2 > 1$ such that H_m contains the closed unit ball of $l_0^\infty(2^{A_1})$ for each $m \geq n_2$. Consequently no H_m with $m \geq n_2$ contains the closed unit ball of $l_0^\infty(\mathcal{Z} \cap (\mathbb{N} - A_1))$. Hence, there is some $A_2 \in \mathcal{Z}$ with $A_1 \cap A_2 = \emptyset$, such that $\chi_{A_2} \notin H_{n_2}$. Again, as $l_0^\infty(2^{A_2})$ is suprabarrelled, no H_m with $m \geq n_2$ contains the closed unit ball of $l_0^\infty(\mathcal{Z} \cap (\mathbb{N} - A_1 \cup A_2))$, and so on. Then, proceeding by recurrence, we obtain a sequence $\{A_i : i \in \mathbb{N}\}$ of pairwise disjoint sets of density cero such that $\chi_{A_i} \notin H_{n_i}$ for each $i \in \mathbb{N}$, with $n_1 = 1$. But, according to Lemma 2.3, for each $i \in \mathbb{N}$ there is a finite set $F_i \subseteq A_i$ such that $\bigcup_{i=1}^{\infty} (A_i - F_i) \in \mathcal{Z}$. Hence, setting $M = \bigcup_{i=1}^{\infty} (A_i - F_i)$, as $l_0^\infty(2^M)$ is suprabarrelled, there must exists some $p \in \mathbb{N}$ such that $\chi_{A_i - F_i} \in H_{n_p}$ for each $i \in \mathbb{N}$. Therefore one has that

$$\chi_{A_p} = \chi_{A_p - F_p} + \chi_{F_p} \in H_{n_p},$$

a contradiction. ■

For the next lemma consider the following linear subspaces of $l_0^\infty(\mathcal{Z})$. Given a positive integer p let $\{E_{n_1,\ldots,n_s} : n_1, \ldots, n_s \in \mathbb{N},\ 1 \leq s \leq p\}$ be a p-net of linear subspaces of $l_0^\infty(\mathcal{Z})$, i.e., let $\{E_{n_1} : n_1 \in \mathbb{N}\}$ be an increasing sequence of subspaces of $l_0^\infty(\mathcal{Z})$ covering $l_0^\infty(\mathcal{Z})$ and for $1 < s \leq p$ let $\{E_{n_1,\ldots,n_s} : n_s \in \mathbb{N}\}$ be an increasing sequence of linear subspaces of $E_{n_1,\ldots,n_{s-1}}$ covering $E_{n_1,\ldots,n_{s-1}}$. Let $T_{n_1,\ldots,n_p}$ be a barrel in $E_{n_1,\ldots,n_p}$ denote by $B_{n_1,\ldots,n_p}$ its closure in $l_0^\infty(\mathcal{Z})$ and set $Z_{n_1,\ldots,n_p} = Z_{n_1,\ldots,n_p} = \langle B_{n_1,\ldots,n_p}\rangle$ for each $(n_1, \ldots, n_p) \in \mathbb{N}^p$. Then, following either the proof of Theorem 8.4.5. or the preliminaries of Lemma 9.3.3 of [9], define inductively the following subspaces:

$$H_{n_1,\ldots,n_p} = \bigcap \{Z_{n_1,\ldots,n_{p-1},m} : m \geq n_p\},$$

and for $s = p-1, p-2, \ldots, 1$

$$Z_{n_1,\ldots,n_s} = \bigcup \left\{H_{n_1,\ldots,n_s,m} : m \in \mathbb{N}\right\}$$

and

$$H_{n_1,\ldots,n_s} = \bigcap \left\{Z_{n_1,\ldots,n_{s-1},m} : m \geq n_s\right\}$$

It is plain that for $1 \leq s \leq p$ one has $E_{n_1,\ldots,n_s} \subseteq H_{n_1,\ldots,n_s}$, hence the increasing sequence $\left\{H_{n_1,\ldots,n_s} : n_s \in \mathbb{N}\right\}$ covers $E_{n_1,\ldots,n_{s-1}}$ for $1 < s \leq p$.

Lemma 2.7. *There exists some $n \in \mathbb{N}$ such that*

$$\left\{\chi_F : F \subseteq \mathbb{N}, F \text{ finite}\right\} \subseteq H_n.$$

Proof. Reasoning by contradiction let us assume that

$$\left\{\chi_F : F \subseteq \mathbb{N}, F \text{ finite}\right\} \not\subseteq H_n$$

for each $n \in \mathbb{N}$. Since $\{H_n : n \in \mathbb{N}\}$ is an increasing sequence of linear subspaces of $l_0^\infty(\mathcal{Z})$ covering $l_0^\infty(\mathcal{Z})$ there exists a sequence of pairwise disjoint finite sets $\{F_n : n \in \mathbb{N}\} \subseteq \mathbb{N}$ such that $\bigcup_{n=1}^\infty F_n \in \mathcal{Z}$ and $\chi_{F_n} \notin H_n$ for each $n \in \mathbb{N}$.

Setting $M = \bigcup_{n=1}^\infty F_n$, since $l_0^\infty(2^M)$ is barrelled of class p there is some $m_1 \in \mathbb{N}$ such that $H_{n_1} \cap l_0^\infty(2^M)$ is dense in $l_0^\infty(2^M)$ and barrelled of class $p-1$ for each $n_1 \geq m_1$. So, reasoning as in the proof of Lemma 9.3.3 of [9], we may obtain $p-1$ functions $m_2(n_1), \ldots, m_p(n_1, \ldots, n_{p-1})$ such that $H_{n_1,\ldots,n_p} \cap l_0^\infty(2^M)$ is a dense barrelled subspace of $l_0^\infty(2^M)$ for $n_1 \geq m_1$, $n_2 \geq m_2(n_1), \ldots, n_p \geq m_p(n_1, \ldots, n_{p-1})$. Therefore $B_{n_1,\ldots,n_p} \cap l_0^\infty(2^M)$ is a neighborhood of the origin in $H_{n_1,\ldots,n_p} \cap l_0^\infty(2^M)$ for $n_1 \geq m_1$, $n_2 \geq m_2(n_1), \ldots, n_p \geq m_p(n_1, \ldots, n_{p-1})$. Going backwards, this implies that $\left\{\chi_F : F \subseteq \mathbb{N}, F \text{ finite}\right\} \subseteq H_{n_1}$ for each $n_1 \geq m_1$. Hence $\chi_{F_n} \in H_n$ for each $n \geq m_1$, a contradiction. ■

Theorem 2.8. *$l_0^\infty(\mathcal{Z})$ is a barrelled space of class p.*

Proof. Assuming $l_0^\infty(\mathcal{Z})$ is not a barrelled space of class p, there exists a p-net $\left\{E_{n_1,\ldots,n_s} : n_1, \ldots, n_s \in \mathbb{N}, 1 \leq s \leq p\right\}$ of linear subspaces of $l_0^\infty(\mathcal{Z})$ such that no space $E_{n_1,\ldots,n_p}$ is barrelled. If $T_{n_1,\ldots,n_p}$ denotes a barrel in $E_{n_1,\ldots,n_p}$ which is not a neighborhood of the origin in $E_{n_1,\ldots,n_p}$, let $B_{n_1,\ldots,n_p}$ be the closure of $T_{n_1,\ldots,n_p}$ in $E_{n_1,\ldots,n_p}$, and, for $1 \leq s \leq p$, define $Z_{n_1,\ldots,n_s}$ and $H_{n_1,\ldots,n_s}$ as above.

According to the previous lemma there is not loss of generality assuming hat

$$\left\{\chi_F : F \subseteq \mathbb{N}, F \text{ finite}\right\} \subseteq H_n$$

for each $n \in \mathbb{N}$. Then, using the fact that $l_0^\infty(2^E)$ is a barrelled space of class p for each $E \in \mathcal{Z}$, an appropriate modification of the argument used in the proof of Theorem 2.6 allows us to obtain a sequence $\{A_i : i \in \mathbb{N}\}$ of subsets of $\mathbb{N}$ of density zero and a strictly increasing sequence of positive integers (n_i) such that $\chi_{A_i} \notin H_{n_i}$ for each $i \in \mathbb{N}$. So, using Lemma 2.3, for each $i \in \mathbb{N}$ there is a finite set $F_i \subseteq A_i$ such that $\bigcup_{i=1}^\infty (A_i - F_i) \in \mathcal{Z}$. Hence, setting $M = \bigcup_{i=1}^\infty (A_i - F_i) \in \mathcal{Z}$, as $l_0^\infty(2^M)$ is barrelled of class p there exists some $j \in \mathbb{N}$ such that $\chi_{A_i - F_i} \in H_{n_j}$ for each $i \in \mathbb{N}$. So we get the contradiction $\chi_{A_j} \in H_{n_j}$. ■

Corollary 2.9. *$l_0^\infty(\mathcal{Z})$ is a barrelled space of class $\aleph_0$.*

Proof. This is an obvious consequence of the previous theorem. ■

Theorem 2.10. *$\mathcal{Z}(l_p)$ is a barrelled space of class $\aleph_0$* for $1 \leq p \leq \infty$.

Proof. Consider the linear map $T : l_0^\infty(\mathcal{Z}) \otimes_\pi l_p \to \mathcal{Z}(l_p)$ defined by $T(\chi_A \otimes \xi) = \zeta$, where $\zeta_n = \xi_n$ if $n \in A$ and $\zeta_n = 0$ otherwise, for each $A \in \mathcal{Z}$ and each $\xi \in l_p$. Then, if $B_{l_0^\infty(\mathcal{Z}) \otimes_\pi l_p}$ denotes the closed unit ball of $l_0^\infty(\mathcal{Z}) \otimes_\pi l_p$ and $B_{\mathcal{Z}(l_p)}$ represents the unit ball of $\mathcal{Z}(l_p)$, one may easily prove that $T\left(B_{l_0^\infty(\mathcal{Z}) \otimes_\pi l_p}\right) = B_{\mathcal{Z}(l_p)}$. Consequently T is an onto, continuous and open mapping, and hence $\mathcal{Z}(l_p)$ is a quotient of $l_0^\infty(\mathcal{Z}) \otimes_\pi l_p$. As l_p is a Banach space, and according to Theorem 2.7, $l_0^\infty(\mathcal{Z})$ is a barrelled space of class $\aleph_0$, it follows form Proposition 4.3.1 of [9] that $l_0^\infty(\mathcal{Z}) \otimes_\pi l_p$ is barrelled of class $\aleph_0$. Since the class of the barrelled spaces of class $\aleph_0$ is closed under the formation of separated quotients, this shows that $\mathcal{Z}(l_p)$ is also a barrelled space of class $\aleph_0$. ■

Remark 2.1. *Further investigations.* If (Ω, Σ) is a measurable space it its well known [7] that the space $l_0^\infty(\Sigma)$ is barrelled of class $\aleph_0$, a result that has recently been improved by the second author [10] by showing that $l_0^\infty(\Sigma)$ is in fact a baireled [= superbarrelled in [13]] space. So it is natural to wonder whether $l_0^\infty(\mathcal{Z})$ is a baireled space or not. However, it seems that the techniques used in this paper are not strong enough to show that $l_0^\infty(\mathcal{Z})$ is baireled, a property whose study has been suggested by Professor B. Rodríguez-Salinas. So we must postpone the answer to this question to a further research.

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