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PERIODIC SOLUTIONS FOR AN EVAPORATION PROBLEM WITH A SIGNORINI TYPE BOUNDARY CONDITION

(periodic solution/quasilinear parabolic problems/porous media/evaporation)

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ABSTRACT

We consider the quasilinear parabolic equation of degenerate type $u_t = \varphi(u)_{xx} - b(u)_x$, in $(0, 1) \times (0, T)$, $T \geq \omega$, with nonlinear boundary flux conditions. We prove the existence and uniqueness of periodic solutions for this problem, which describes the evaporation of an incompressible fluid from a homogeneous porous medium. The soil is represented by the vertical layer (0, 1). The Signorini type condition in $x = 0$, means that the flow of fluid leaving the porous media, depends on variable meteorological conditions and in a nonlinear manner on u, with a discontinuity in $u = 0$. In $x = 1$, we have an impervious boundary. The existence of periodic solutions, shall be proved utilizing the Schauder fixed point theorem and a convergence result. The uniqueness, shall be obtained in the class of bounded variation functions.

RESUMEN

Consideramos la ecuación parabólica cuasilineal de tipo degenerado $u_t = \varphi(u)_{xx} - b(u)_x$, in $(0, 1) \times (0, T)$, $T \geq \omega$, con condiciones de contorno de tipo flujo no lineal. Se demuestra la existencia y la unicidad de soluciones periódicas para este problema. El modelo está originado en el estudio de la evaporación de un fluido incomprensible en un medio poroso homogéneo representado por el estrato vertical (0, 1). La condición de tipo Signorini en $x = 0$, es formulada de manera que el flujo de fluido dejando el medio poroso, depende de las condiciones meteorológicas variables y en manera no lineal de u con una discontinuidad en $u = 0$. El borde $x = 1$ se supone impermeable. La demostración de la existencia de soluciones periódicas, utiliza el teorema de punto fijo de Schauder junto a un resultado de convergencia, mientras que la unicidad es obtenida en espacios de funciones de variación acotada.

1. INTRODUCTION

In this paper, we consider the evaporation of an incompressible fluid from a homogeneous, isotropic and rigid soil represented by the vertical column (0, 1). The bottom $x = 1$ of the layer is supposed to be impervious, while at the top $x = 0$ of the layer, a unilateral or Signorini type conditions is considered. We investigate the case in which the rate of evaporation has a discontinuity that appears for $u = 0$. More precisely, we will study the existence and the uniqueness of the periodic solution for the following degenerate parabolic problem with $T \geq \omega$

$$(1)\quad u_t = \varphi(u)_{xx} - b(u)_x, \quad in\ Q_T := (0, 1) \times (0, T)$$

$$(2)\quad \varphi(u(0, t))_x - b(u(0, t)) \in q(t)\, H(u(0, t)), \quad for\ t \in (0, T)$$

$$(3)\quad \varphi(u(1, t))_x - b(u(1, t)) = 0, \quad for\ t \in (0, T)$$

$$(4)\quad u(x, t+\omega) = u(x, t),\ u \geq 0, \omega > 0, \quad in\ Q_T.$$

We assume that throughout the remainder of the paper are satisfied the following assumptions

$$(H_\varphi)\quad \begin{cases} \varphi \in C^1((0, \infty)),\ \varphi(0) = \varphi'(0) = 0 \text{ and } \varphi'(s) > 0 \text{ for } s > 0 \\ \text{and } \varphi^{-1} \text{ is Hölder continuous of order } \theta \in (0, 1); \end{cases}$$

$$(H_b)\quad b \in C^{0,1}([0, \infty)),\ b(0) = 0,\ b \text{ has bounded range;}$$

$$(H_q)\quad \begin{cases} q : [0, T] \to \mathbb{R}_+ \text{ is a periodic continuous function} \\ \text{with period } \omega > 0; \end{cases}$$

$$(H_H)\quad \begin{cases} H(.) \text{ is the Heaviside graph, defined by} \\ H(z) = \begin{cases} 0, & \text{if } z < 0 \\ [0, 1], & \text{if } z = 0 \\ 1, & \text{if } z > 0. \end{cases} \end{cases}$$

The problem (1)-(4) describes the evaporation of a homogeneous incompressible liquid from a homogeneous, isotropic and rigid soil. We consider the case in which the evaporation takes place at the top surface $x = 0$. In previous papers of some chinese authors, (see [9] for references) evaporation is modeled by prescribing a constant boundary flux, in particular it is supposed that the medium can be dry on the surface. Also if it seems reasonable with the model to assume that the flux becomes zero when the capillarity piezometric head reaches the critical values ψ_0, we want to investigate the delicate case in which a discontinuity appears for $u = 0$.

We assume that $q(t)$ represents variable metereological conditions and the flux of fluid leaving the soil has a discontinuity in $u = 0$, assuming that the rate of evaporation is given by $q(t)$ if $u(0, t) > 0$, while the flux is between 0 an $q(t)$ when $u(0, t) = 0$. Assumption (3) represents an impervious boundary. Equation (1) is a useful model in many different applications as, for instance, the flow of groundwater in a homogeneous, isotropic, rigid and unsaturated porous medium. If we choose the coordinate x to measure the vertical height from ground level and pointing downward, the soil is represented by the vertical layer (0, 1). If $\theta(x, t)$ denotes the moisture content, defined as the volume of water present per unit volume of soil and $v(x, t)$ is the seepage velocity of the water, the law by which flows through the porous media is given by the experimental Darcy's law

(5) $v = -k(\theta)\, \phi_x$

and the continuity equation

(6) $\theta_t + v_x = 0.$

In (5), $k(\theta)$ is the hydraulic conductivity of soil and ϕ is the total potential. When absorption, chemical osmotic and thermal effect are negligible, the total potential may be expressed as $\phi = \psi(\theta) - x$, where $\psi(\theta)$ is the piezometric head. Combining both equations (5), (6), we obtain (1). In (1)-(4), u denotes the saturation of soil and for this reason we require the condition $u \geq 0$. In the present paper, we prove the existence and the uniqueness of the periodic solution for (1)-(4), considering firstly, a quasilinear parabolic problem of nondegenerate type, approximating the problem (1)-(4).

This nondegenerate problem is obtained by adding a so called «artificial viscosity» term, substituting $\varphi(s)$ with $\varphi(s) + s/k$, $\forall k \in \mathbb{N}$ in (1) and the discontinuous term $H(u)$ by a continuous approximation $H_k(u)$: i.e.

(7) $u_{kt} = \Phi_k(u_k)_{xx} - b(u_k)_x$, in Q_T, $T \geq \omega$

(8) $\Phi_k(u_k(0, t))_x - b(u_k(0, t)) = q(t)\, H_k(u_k(0, t))$, $\forall t \in (0, T)$

(9) $\Phi_k(u_k(1, t))_x - b(u_k(1, t)) = 0$, $\forall t \in (0, T)$

(10) $u_k(x, t + \omega) = u_k(x, t)$, $u_k \geq 0$, $\omega > 0$, in Q_T

where $\Phi_k(s) := \varphi(s) + s/k$, $\forall k \in \mathbb{N}$ and

$$(H_k) \quad H_k \in C^1(\mathbb{R}),\ H_k' \geq 0 \text{ and } H_k(z) = \begin{cases} 1, & \text{if } z \geq 1/k \\ 0, & \text{if } z \leq 0. \end{cases}$$

To prove the existence of periodic solutions for (7)–(10) we use the Schauder fixed point theorem for the Poincaré map of the associated initial-boundary value problem $\forall T > 0$

$$(\mathrm{IBVP}_k) \begin{cases} u_{kt} = \Phi_k(u_k)_{xx} - b(u_k)_x, & \text{in } Q_T \\ \Phi_k(u_k(0, t))_x - b(u_k(0, t)) = q(t)\, H_k(u_k(0, t)), & \forall t \in (0, T) \\ \Phi_k(u_k(1, t))_x - b(u_k(1, t)) = 0, & \forall t \in (0, T) \\ u_k(x, 0) = u_{0k}(x), & \text{in } (0, 1) \end{cases}$$

with u_{0k} an arbitrary function satisfating

$$(H_{0k}) \quad u_{0k} \in H^1(0, 1) : 1/k \leq u_{0k}(x) \leq M_0, \quad \forall x \in [0, 1].$$

As a preliminary step, we study the (IBVP_k) problem by showing the existence and the uniqueness of the solution. To do this, we use a semi-discretized scheme in the time. The existence of a periodic solution for problem (1)–(4) is obtained by passing to the limit over k. The uniqueness of the periodic solution, is proved in the space of the bounded variation functions and is obtained assuming that $b(\varphi^{-1})$ is an Hölder continuous function of order 1/2. This assumption improves our previous result in [2], where the uniqueness of the periodic solution was obtained assuming that $b(\varphi^{-1})$ is Lipschitz continuous and there exists a convex function $\mu \in C^0([0, \infty)) \cap C^2((0, \infty))$ such that $\mu(0) = 0$ and $0 < \mu'(r) \leq \varphi'(r) \leq \forall r > 0$. The evaporation problem of a liquid from a porous media, has been studied in [1] and [8] in which is considered in $x = 0$ a rate of evaporation represented by a continuous monotone function of the water content. The case with boundary condition of Signorini type has been studied in [9]. The study of periodical solutions for the infiltration of a fluid in a porous medium, has been considered by many authors. We recall here the papers [2], [3] and [10]. For the knowledge of the author, the study of periodical solutions for the evaporation problem seems to be new.

2. EXISTENCE OF SOLUTIONS FOR THE APPROXIMATING PROBLEM

We denote with $BV(0, T)$ the space of functions that are locally integrable on $(0, T)$ and whose generalized derivative is an integrable measure of Radon on $(0, T)$. Let $V := H^1(0, 1)$ and V' its dual space, we denote with (.,) both the pairing of duality V', V and the usual inner product in $L^2(0, 1)$. The inner product in V is defined by $(u, v)_1 = (u, v) + (u_x, v_x)$. By Sobolev's embedding theorem $V \subset C([0, 1])$, with continuous injection.

Definition 1. *A function u is called a periodic strong solution for the problem* (1)–(4) *if $u \in C([0, T]; L^2(0, 1))$*

$\cap L^\infty (Q_T)$, $u_t \in L^2 (0, T; V')$, $u \geq 0$, $u(x, t + \omega) = u(x, t)$, $\forall t \in [0, T]$ *and a.e.* $x \in (0, 1)$ *and satisfies*

(11)

$$\int_0^t (u_s, v)\, ds + \int_0^t q(s)\, h(s)\, v(0, s)\, ds + \int_0^t \int_0^1 (\varphi(u)_x - b(u)) v_x dx ds = 0$$

for any $h \in L^\infty (0, T)$ *with* $h(t) \in H(u(0, t))$ *a.e.* $t \in (0, T)$ *and* $\forall v \in L^2(0, T; V)$, $\forall t \in (0, T)$.

To solve the (IBVP$_k$) problem, we consider a semi-discretized scheme in the time. Divide $[0, T]$ in steps of ugual lenght $h = \Delta t = T/N$, $N \in \mathbb{N}$ (discretization time step) so, $[0, T] = \cup_{n=1}^{N} [(n-1)\,h,\, nh]$. Now, consider an approximation of u at time nh defining $u_k^n(x) = u_k(x, nh)$ and set $q^n := (1/h) \int_{(n-1)h}^{nh} q(s) ds$. It is not a priori known that the solutions u_k are nonnegative, therefore we consider the continuations on all $\mathbb{R}$, $\hat{\varphi}$, $\hat{b}$ respectively of φ and b, defined by $\hat{\varphi}(s) = \hat{b}(s) = 0$ for $s \leq 0$. We resolve the following.

Problem (P_n): Let (H_{0k}) be satisfied, to find $u_k^n \in V$, $\forall n \geq 1$ solution of the nonlinear elliptic equation

$$(1/h)\left(u_k^n - u_k^{n-1}, v\right) + q^n v(0)\, H_k\left(u_k^n(0)\right) +$$
$$+ \int_0^1 \hat{\Phi}_k\left(u_k^n\right)_x v_x dx - \int_0^1 \hat{b}\left(u_k^n\right) v_x dx = 0,$$
$$\forall v \in V,\ n = 1, 2, \ldots, N-1$$

$u_k^0 = u_{0k}$.

The proof of the existence of a solution of (P_n) is based on the existence of a solution of the following nonlinear equation

Equation (P_s):

To find $z_k \in V$, such that

$$\mu(z_k, v) + q^k v(0)\, H_k(z_k(0)) + \int_0^1 \hat{\Phi}_k(z_k)_x v_x dx -$$
$$- \int_0^1 \hat{b}(z_k) v_x dx = (g, v)$$
$$\forall v \in V, \quad \mu \geq 0, \quad g \in V.$$

Proposition 1. *If* (H_φ), (H_b), (H_k) *hold, there exists a solution* $z_k \in V$ *of equation* (P_s).

Proof. The existence of solutions is proved by the Schauder fixed point theorem (see [7]).

Proposition 2. *If* (H_φ), (H_b), (H_k) *hold, there exists a solution* $u_k^n \in V$ *for* (P_n).

Proof. Solved (P_s), it is possible to solve (P_n) by recurrence on n (see [7]).

Proposition 3. *If assume* (H_φ), (H_b), (H_k) *and* (H_{0k}), *then* u_k^n *are nonnegative on* $[0, 1]$.

Proof. The proof is the same of Proposition 3 of [1] and we omit it.

To prove that (P_n) has a unique solution, we assume that

(12) $$\left|\hat{b}\left(\hat{\Phi}_k^{-1}(s)\right) - \hat{b}\left(\hat{\Phi}_t^{-1}(t)\right)\right| \leq C|t - s|^{1/2}.$$

Then, we can use a Lipschitz increasing approximation of the function of Heaviside, which is well posed with (12) as, for instance,

(13)

$$s_\varepsilon(w) = \begin{cases} 0, & \text{, if } w \leq \varepsilon < 1 \\ 1 - \log w / \log \varepsilon & \text{, if } 0 < \varepsilon \leq w. \end{cases}$$

In [1] was proved that holds the

Proposition 4. ([1]). *With assumptions* (H_φ), (H_b), (H_k) *and* (12), *problem* (P_n) *has a unique solution.*

From u_k^n, we construct the functions

(14) $$u_{h,k}(x, t) := \sum_{n=0}^{N-1} u_k^n(x)\, \chi^n(t)$$

(step approximation of u_k), where $\chi^n(t)$ is the characteristic function of $[nh, (n+1)h]$ and

(15)

$$\sigma_{h,k}(x, t) =$$
$$\begin{cases} (t - nh)\left(u_k^n(x) - u_k^{n-1}(x)\right)/h + u_k^{n-1}(x), & t \in [nh, (n+1)h],\ n = 1, 2, \ldots, N-1 \\ u_{0k}(x), & t \in [0, h] \end{cases}$$

(piecewise linear approximation of u_k, continuous in t).

Proposition 5. *If* (H_φ), (H_b), (H_k) *and* (12) *hold, then* $u_{n,k}$ *is bounded with respect to h in* $L^2 (0, T; V) \cap L^\infty (0, T; L^2 (0, 1)) - *$ *weak.*

As a consequence of the Proposition 5, there exists a subsequence, denoted again with $u_{h,k}$ such that $u_{h,k} \rightharpoonup u_k$ *as* $h \rightharpoonup 0$ in $L^2 (0, T; V)$ and in $L^\infty (0, T; L^2 (0, 1)) - *$ weak. Therefore, $u_k \in L^2 (0, T; V) \cap L^\infty (0, T; L^2 (0, 1))$. If we consider (15), we obtain

(16) $$\left(\sigma_{h,k}(x, t)\right)_t = \begin{cases} \left(u_k^n(x) - u_k^{n-1}(x)\right)/h, & t \in [nh, (n+1)h],\ n = 1, 2, \ldots, N-1 \\ 0,\ t \in [0, h]. & \end{cases}$$

One has, (cfr. [1])

$$(17)\quad \|\sigma_{h,k}\|^2_{L^2(0,T;V)} \le C_k'; \ \|(\sigma_{h,k})_t\|^2_{L^2(Q_\delta)} \le C_\delta, \ \forall 0 < \delta < T$$

and $Q_\delta := (0, 1) \times (\delta, T)$, with C_δ independent of h, k.

By the problem (P_n), we can obtain

$$(18)\quad \|(\sigma_{h,k})_t\|_{L^2(0,T;V')} \le C_k'.$$

Set

$$(19)\quad W(0,T) := \{v \in L^2(0,T;V) : v_t \in L^2(0,T;V')\},$$

from a classical result it is known that $W(0, T)$ is compactly embedded in $L_2(Q_T)$. Proceeding as in [1] an going to the limit as $n \to \infty$ in (P_n), we obtain

(20)

$$(u_k, v) + q(t)\, H_k(u_k(0,t))\, v(0,t) + (1/k)\,(u_{kx}, v_x) + $$
$$+ \int_0^1 \varphi(u_k)_x v_x dx - \int_0^1 b(u_k)\, v_x dx = 0,$$
$$\forall v(t) \in V, \ a.e.\ t \in (0,T).$$

It is so proved that.

Theorem 6. ([1]). *Assume (H_φ) – (H_{0k}) and (12). There exists a unique strong solution u_k of $(IBVP_k)$ such that $u_k \in L^2(0, T; V)$ and $u_{kt} \in L^2(0, T; V') \cap L^2(Q_T)$, $u_k \in C([0, T]; L^2(0, 1))$, $u_k(x, 0) = u_{0k}(x)$ in $[0, 1]$, $u_k(x, t) \ge 0$ on $\overline{Q}_T$ and $\Phi_k(u_k) \in L^2(\delta, T; H^2(0, 1))$.*

To obtain an uniform estimate for u_{kt}, we assume that

(21)

$$\begin{cases} 1/k \le u_{0k}(x) \le M_0, & \forall x \in [0,1], \\ \varphi(u_{0k}) \in V \text{ and } \varphi(u_{0k})_x - b(u_{0k}) \in BV(0,1) \end{cases}$$

In [1] was proved the following result.

Proposition 7. ([1]). *With assumption (H_φ), (H_b), (H_k), (H_q) and (21), there exists a constant $C > 0$ such that*

(22)

$$\|u_k(t+\tau) - u_k(t)\|_{L^1(0,1)} \le C\tau, \quad \forall \tau \in (0,T), \quad \forall t \in [0, T-\tau]$$

$$(23)\quad \|u_{kt}\|_{L^\infty(0,T;L^1(0,1))} \le C,$$

where C is independent on k.

Because of the monotonicity of $\Phi_k(\cdot)$, $(IBVP_k)$ can be written as follows

$$(IBVP_k)' \begin{cases} c_k(v_k)_t = v_{kxx} - b(c_k(v_k))_x, & \text{in } Q_T \\ v_{kx}(0,t) - b(c_k(v_k(0,t))) = q(t)\, H_k(c_k(v_k(0,t))), & \text{in } (0,T) \\ v_{kx}(1,t) - b(c_k(v_k(1,t))) = 0, & \text{in } (0,T) \\ v_k(x,0) = v_{0k}(x), & \text{in } (0,1) \end{cases}$$

where $c_k := \Phi_k^{-1}$ and $1/k^2 \le v_{0k}(x) \le M_0'$.

Then, we obtain.

Lemma 8. *Let v_k be the solution of $(IBVP_k)'$. Then, there exists a constant M such that*

$$(24)\quad 0 \le v_k(x,t) \le M, \ in\ Q_T.$$

Proof. Define the function $U_k(x)$ as follows

$$x = \int_{M_0'}^{U_k(x)} \frac{ds}{b(c_k(s)) + C}$$

with $C > 0$ a suitable constant. One has $U_k(x) \ge M_0' \ge v_{0k}(x)$. Since b has bounded range, we get that $U_k(x) \le M$. By the maximum principle we have

$$(25)\quad 0 \le v_k(x,t) \le U_k(x) \le M, \ \text{ in } Q_T$$

if C is chosen such that $C > q(t)\, H_k(c_k(v_k(0, t)))$.

Because of (25), it follows that

$$0 \le u_k \le \varphi^{-1}(M).$$

It was showed in [1] that

$$(26)\quad \|v_k\|_{L^2(0,T;V)} = \|\Phi_k(u_k)\|_{L^2(0,T;V)} \le C$$

and that

$$(27)\quad \|v_k\|_{H^{1,1}(0,T;L^1(0,1))} = \|\Phi_k(u_k)\|_{H^{1,1}(0,T;L^1(0,1))} \le C$$

as a consequence of (23). From the above regularity, we also deduce that $v_k(\cdot, t) \in C([0, 1])$, $\forall t \in [0, t]$ and $v_k \in C([0, T]); L^2(0, 1))$.

Define the closed and convex set

$$A_k := \{w \in C([0,1]) : 0 \le w(x) \le U_k(x), \ \forall x \in [0,1]\},$$

we can show the following properties for the Poincaré map F, defined by

$$F(v_{0k}(\cdot)) = v_k(\cdot, \omega), \quad \omega \le T$$

where v_k is the strong solution of $(IBVP_k)'$.

i) F leaves invariant A_k.

This is a consequence of (25);

ii) F restricted to A_k is continuous.

Lemma 9. *If assumptions of* Theorem 6 *and* (21) *hold and let* $v_{0k}^n, v_{0k} \in A_k$ with $v_{0k}^n \to v_{0k}$ *uniformly on* $[0, 1]$ *as* $n \to \infty$. *Then, if* v_k^n, v_k *are the strong solutions of (IBVP)' with initial data* v_{0k}^n, *respectively* v_{0k}, *we get that* $v_k^n(x, t) \to v_{k(x,)}(x, t)$ *uniformly on* $[0, 1]$ *when* $n \to \infty$, $\forall t \in [0, T]$.

Proof. By

$$\left(c_k(v_k^n)_t - c_k(v_k)_t, v\right) + q(t)\, v(0,t)\left(H_k\left(c_k(v_k^n(0,t)\right) - H_k\left(c_k(v_k(0,t))\right)\right) +$$

$$+ \int_0^1 (v_{kx}^n - v_{kx}) v_x dx - \int_0^1 \left(b\left(c_k(v_k^n)\right)\right) - b(c_k(v_k)) v_x \, dx = 0,$$

with $v(t) := \operatorname{sgn}_\eta \left(v_k^n(t) - v_k(t)\right)$ and

$$\operatorname{sgn}_\eta(x) := \begin{cases} 1, & \text{if } x > \eta \\ x/\eta, & \text{if } |x| \le \eta, \quad \eta > 0, \\ -1, & \text{if } x < -\eta \end{cases}$$

one has

(28)

$$\int_0^1 \left| c_k\left(v_k^n(x,t)\right) - c_k\left(v_k(x,t)\right)\right| dx \le \int_0^1 \left|v_{0k}^n(x) - v_{0k}(x)\right| dx.$$

Whence, $c_k\left(v_k^n(\cdot, t)\right)$ converges to $c_k(v_k(\cdot, t))$ strongly in $L^1(0, 1)$ as $n \to \infty$ and $v_k^n(\cdot, t)$ converges to $v_k(\cdot, t)$ a.e. in (0,1). Since $v_k^n(\cdot, t) \le$ M, by Lebesgue's theorem, we conclude: $v_k^n(\cdot, t) \to v_k(\cdot, t)$ in $L^p(0, 1)$, $\forall 1 \le p \le \infty$. Moreover, $v_k^n(\cdot, t), v_k(\cdot, t) \in C([0, 1])$ by which it follows the uniform convergence.

iii) $F(A_k)$ is relatively compact in $C([0, 1])$.

By using regularizing arguments, i.e. convolutions with mollifiers functions, it is possible to approximate v_{0k} and b as follows

(29)

$$\begin{cases} v_{0k}^s \in C^2([0,1]) \\ 0 < 1/k \le v_{0k}^s(x) \le U_k(x), \ \forall x \in [0,1] \\ v_{0k}^s(x) \to v_{0k}(x), \text{ uniformly over } [0,1] \\ \left|v_{0k}^{s'}(x)\right| \le M_1, \ \forall x \in [0,1] \\ b_s \in C^2([0,\infty)) \\ b_s \text{ has the range uniformly bounded} \\ b_s \to b \text{ as } s \to \infty, \text{ uniformly on the bounded set} \\ v_{0k}^{s'}(0) - b_s\left(c_k(v_{0k}^s(0))\right) = q(0)\, H_k\left(c_k(v_{0k}^s(0))\right) \\ v_{0k}^{s'}(1) - b_s\left(c_k(v_{0k}^s(1))\right) = 0. \end{cases}$$

With this assumptions, it is possible to use the result of [5], [12], therefore there exists a unique solution $v_k^s \in C^{2+\alpha, 1+\alpha/2}(\overline{Q}_T)$, $\alpha \in (0, 1)$ for the problem

$$(IBVP_k^s)\begin{cases} c_k(v_k^s)_t = v_{kxx}^s - b_s\left(c_k(v_k^s)\right)_x, & \text{in } Q_T \\ v_{kx}^s(0,t) - b_s\left(c_k(v_k^s(0,t))\right) = q(t)\, H_k\left(c_k(v_k^s(0,t))\right), & \forall t \in (0,T) \\ v_{kx}^s(1,t) - b_s\left(c_k(v_k^s(1,t))\right) = 0, & \forall t \in (0,T) \\ v_k^s(x,0) = v_{0k}^s(x,) & \text{in } [0,1]. \end{cases}$$

We can prove this result.

Lemma 10. *There exists a constant* $M > 0$ *such that*

(30) $\left|v_{kx}^s(x,t)\right| \le M$, *in* Q_T

Proof. Define $V(x, t) := v_{kx}^s - b_s\left(c_k(v_k^s)\right)$, then $V(x, t)$ satisfies $V_x = \left(v_{kx}^s - b_s\left(c_k(v_k^s)\right)\right)_x$, by which because of $\left(IBVP_k^s\right)_1$, one has

(31) $V_t = v_{kxt}^s - b_s'\left(c_k(v_k^s)\right) c_k(v_k^s)_t$.

Deriving $\left(IBVP_k^s\right)_1$ with respect to x, we have

$$c_k''(v_k^s)\, v_{kx}^s v_{kt}^s + c_k'(v_k^s)\, v_{kxt}^s = V_{xx}$$

so, by (31) we obtain

$$c_k'(v_k^s)\, V_t + c_k'(v_k^s)^2\, b_s'\left(c_k(v_k^s)\right) v_{kt}^s + c_k''(v_k^s)\, v_{kx}^s v_{kt}^s = V_{xx}$$

which implies

$$c_k'(v_k^s)\, V_t = V_{xx} - V_x\left(c_k'(v_k^s)\, b_s'\left(c_k(v_k^s)\right)\right) + c_k''(v_k^s)\, v_{kx}^s / c_k'(v_k^s)$$

i.e.

$$V_t = V_{xx}/c_k'(v_k^s) - Vx\left(b_s'\left(c_k(v_k^s)\right) + c_k''(v_k^s)\, v_{kx}^s / \left(c_k'(v_k^s)\right)^2\right).$$

By the assumptions done, $V \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ (see [6]).

Since $V(0, t) \ge 0$, $V(1, t) = 0$ and $V(x, 0) = v_{0k}^{0\,\prime}(x) - b_s\left(c_k(v_{0k}^s(x))\right)$ is bounded over $[0, 1]$, by the maximum principle $V(x, t)$ takes its maximum and its minimum on the parabolic boundary Γ of Q_T. Over Γ, $V(x, t)$ is uniformly bounded with respect to s and k. Whence,

$$\max_{Q_T} |V(x,t)| \le \max_{\Gamma} |v(x,t)| \le L$$

which implies that

$$\left|v_{kx}^s(x,t)\right| \le M.$$

In [11] was proved that $c_k\left(v_k^s(x, t)\right)$ is uniformly (with respect to k and s) Lipschitz continuous in x and uniformly (with respect to k and s) Hölder continuous in t. Therefore, for a subsequence we have that $c_k(v_k^s) \to c_k(\hat{v}_k)$ in $C^\beta(\overline{Q}_T)$, $\beta \in (0, 1)$ as $s \to \infty$. Moreover, $v_k^s \to \hat{v}_k$ pointwise and $L^p(Q_T)$, $\forall 1 \le p \le \infty$ by the theorem of Lebesgue. Now, it is a standard matter to see that $\hat{v}_k$ resolve (IBVP$_k$). By the uniqueness of the solution of (IBVP$_k$), we obtain that $v_k = \hat{v}_k$. Since (30) is stable with respect to the weak convergence in $L_2(Q_T)$, one obtains

(32) $$|v_{kx}(x, t)| \le M.$$

By (32), it follows that $v_k(\cdot, \omega)$ is Lipschitz continuous, whence $F(A_k)$ is relatively compact in $C([0, 1])$. To this point, we can conclude with the following.

Theorem 11. *If* $(H_\varphi) - (H_{0k})$ *and* (21) *hold, there exist* ω *– periodic strong solutions for* (7) – (10).

Proof. It was showed that $F(A_k)$ is continuous and relatively compact, by the fixed point theorem of Schauder, there exists a fixed point v_k for the Poincaré map F. This v_k is an ω – periodic strong solution for (7) – (10).

We must again prove the existence of periodic solutions for the problem (1) – (4). We need of some uniform estimate on u_k.

3. ESTIMATES ON u_k

Set $\gamma(\zeta) := \int_0^\zeta \sqrt{\varphi'(\tau)}\, d\tau$ and $C(\zeta) := \int_0^\zeta b(\tau)\, d\tau$, then, proceeding as in [1] we get.

(33) $${}_{L^2(0, T; V)} \le \sqrt{k}\, D$$

(where D denotes various constants independents of k)

(34) $$\|\gamma(u_k)\|^2_{L^2(0, T; V)} \le D$$

(35) $$\|\varphi(u_k)\|^2_{L^2(0, T; V)} \le D$$

(36) $$\|u_{kt}\|_{L^2(0, T; V')} \le D.$$

Now, if we suppose that

$(H_{\gamma-1})$ $\quad \gamma^{1}$ is Hölder continuous of order $\theta \in (0, 1)$,

one has

(37) $$\|u_k\|^{2/\theta}_{L^{2/\theta}(0, T; W^{\theta s, 2/\theta}(0, 1))} \le D.$$

Then, it was proved in [1] that

(38) $$u_k \rightharpoonup u, \text{ in } L^\infty(0, T; L^2(0, 1)) - * \text{ weak}$$

(39) $$u_k \to u, \text{ in } L^2(Q_T) \text{ and a.e.}$$

(40) $$u_{kt} \rightharpoonup u_t, \text{ in } L^2(0, T; V')$$

and

(41) $$\|u_t\|_{L^2(0, T; V')} \le D.$$

Moreover,

(42) $$\varphi(u_k) \rightharpoonup \varphi(u), \text{ in } L^2(0, T; V)$$

(43) $$\Phi_k(u_k) \rightharpoonup \varphi(u), \text{ in } L^2(0, T; V)$$

(44) $$u_k \to u, \text{ in } C([0, T]; V')$$

(45) $$b(u_k) \to b(u), \text{ in } L^2(Q_T)$$

(46) $$\|\Phi_k(u_k)_t\|_{L^\infty(0, T; L^1(0, 1))} \le D.$$

Since $\Phi_k(u_k)$ is bounded in $L^\infty(Q_T) \cap H^{1,1}(0, T; L^1(0, 1))$, by (26) and (46) it follows that $\Phi_k(u_k) \in W(0, T)$. Thus $\Phi_k(u_k) \to \varphi(u)$ in $L^1(Q_T)$ when $k \to \infty$. Thus, $\varphi(u) \in BV(0, T; L^1(0, 1))$ because it is the limit of a sequence in $L^\infty(Q_T) \cap H^{1,1}(0, T; L^1(0, 1))$.

We know that u_k belongs to $L^\infty(Q_T)$ and by (36) and (39) it follows that u_k is bounded in $L^\infty(Q_T) \cap H^{1,1}(0, T; L^1))$ and $u \in BV(0, T; L^1)(0, 1))$. Moreover (see [1]),

(47) $$\|\varphi(u)_t\|_{L^2(0, T; V')} \le D.$$

Since $\varphi(u) \in C([0, T] : L^2(0, 1)$, by the Hölder continuity of φ^{-1} we obtain that $u \in C([0, T]) : L^2(0, 1)$. In conclusion, we have.

Theorem 12. *Assume* (H_φ), (H_b), (H_q), (H_H). *Then, there exists a periodic strong solution for the problem* (1) – (4).

4. UNIQUENESS OF THE PERIODIC SOLUTION

As in [1], we introduce in $\mathbb{R}^2$ the one dimensional Hausdorff measure $\mathcal{H}_1$ (for a definition see [4]). Since the solution u of (1) – (4) and $\varphi(u)$ are $\mathcal{H}_1$ a.e. $\mathcal{L}^2$–approximately continuous in Q_T (see [1] for references), u_t do not charge the complementary set of $\mathcal{L}^2$–approximate continuity points of u, we can use the integration by parts formula (see [1]).

Let u, $\hat{u}$ be for contradiction two periodic solutions of (1) – (4), then

(48)

$$\int_0^t (u_\tau - \hat{u}_\tau, v)_{V'V}\, \rho(\tau)\, d\tau + \int_0^t q(\tau)\,(h_1(\tau) - h_2(\tau))\, v(0, \tau)\, \rho(\tau) d\tau +$$

$$+ \int_0^t \int_0^1 (\varphi(u)_x - \varphi(\hat{u})_x) v_x\, \rho(\tau)\, dx d\tau - \int_0^t \int_0^1 (b(u) - b(\hat{u}))\, v_x\, \rho(\tau) dx d\tau = 0$$

$\forall v \in L^2(0, T; V)$, $\forall t > 0$, $\forall \rho \in D(0, t)$ with $\rho(t) \geq 0$ and $h_1(t) \in H(u(0, t))$, $h_2(t) \in H(\hat{u}(0, t))$ for a.e. $t \in (0, T)$.

Set $w := \varphi(u) - \varphi(\hat{u})$, $H_\eta(w) := \dfrac{w^{+2}}{w^2 + \eta}$, $\eta > 0$ and assuming that

$$(49) \quad \left|b\left(\varphi^{-1}(t)\right) - b\left(\varphi^{-1}(s)\right)\right| \leq c' \, |t - s|^{1/2}$$

it is easy to prove that, $\forall r \in \mathbb{R}$

$$\lim_{\eta \to 0^+} rH'_\eta(r) = 0, \quad 0 \leq H_\eta(r) \leq 1 \text{ and } 0 \leq rH'_\eta(r) \leq 1/2.$$

Choosing $v := H_\eta(w) \in H^1(Q_T)$ in (48), since

$$\int_0^t q(\tau)\left(h_1(\tau) - h_2(\tau)\right) H_\eta\left(w(0, \tau)\right) \rho(\tau)\, d\tau \geq 0$$

because of the monotonicity of φ, we have

$$\int_0^t \left(u_\tau - \hat{u}_\tau, H_\eta(w)\right)_{V'V} \rho(\tau) d\tau +$$

$$+ \int_0^t \int_0^1 \left|\varphi(u)_x - \varphi(\hat{u})_x\right|^2 H'_\eta(w)\, \rho(\tau) dx d\tau -$$

$$- \int_0^t \int_0^1 \left(b(u) - b(\hat{u})\right) H_\eta(w)_x \rho(\tau) dx d\tau \leq 0.$$

Set $s := \varphi(u)$ and $\hat{s} := \varphi(\hat{u})$, by (49) and the inequality of Young we obtain

$$\int_0^t \int_0^1 \left(b(u) - b(\hat{u})\right) H_\eta(w)_x \rho(\tau) dx d\tau \leq$$

$$\leq (1/2) \int_0^t \int_0^1 \left|(s - \hat{s})_x\right| H'_\eta(s - \hat{s})\, \rho(\tau) dx d\tau +$$

$$+ c_1 \int_0^t \int_0^1 |s - \hat{s}| H'_\eta(s - \hat{s})\, \rho(\tau) dx d\tau$$

by which

(51)

$$\int_0^t \left(u_\tau - \hat{u}_\tau, H_\eta(w)\right)_{V'V} \rho(\tau) d\tau \leq$$

$$\leq c_1 \int_0^t \int_0^1 |s - \hat{s}|\, H'_\eta(s - \hat{s})\, \rho(\tau) dx d\tau.$$

By (49) and Lebesgue's theorem, we have

$$\lim_{\eta \to 0^+} \int_0^t \int_0^1 |s - \hat{s}|\, H'_\eta(s - \hat{s})\, \rho(\tau) dx d\tau = 0.$$

Since $u - \hat{u} \in L^2(Q_T)$, $(u - \hat{u})_t \in L^2(0, T; V')$, $\rho H_\eta(w) \in L^2(0, T; V)$ and $(\rho H_\eta(w))_t \in L^2(Q_T)$, introducing the spaces $Y := [V, L^2]_{1/2}$ and $Y' := [L^2, V']_{1/2}$ one has (cfr. [1])

$$\int_0^t \left(u_\tau - \hat{u}_\tau, H_\eta(w)\right)_{V'V} \rho(\tau) d\tau = - \int_0^t \int_0^1 (u - \hat{u}) \left(\rho H_\eta(w)\right)_\tau dx d\tau.$$

We agree to write

$$- \int_0^t \int_0^1 (u - \hat{u}) \left(\rho H_\eta(w)\right)_\tau dx d\tau = - \int_{Q_t} (u - \hat{u}) \left(\rho H_\eta(w)\right)_\tau$$

because $\rho H_\eta(w) \in BV(0, T; L^1(0, 1))$ and the Borel measure $(\rho H_\eta(w))_t$ is $\mathcal{L}^2$–Lebesgue absolutely continuous with density $(\rho H_\eta(w))_t$. We know that $u - \hat{u} \in BV(0, T; L^1(0, 1)) \cap L^\infty(Q_T)$ and $\rho H_\eta(w) \in L^\infty(Q_T) \cap H^1(Q_T)$, then $(u - \hat{u})\, \rho H_\eta(w) \in BV(0, T; L^1(0, 1))$ (cfr. [1] and the references therein).

Thus, we can use the formula of integration by parts to obtain

$$\int_0^t \left(u_\tau - \hat{u}_\tau, H_\eta(w)\right)_{V'V} \rho(\tau) d\tau = \int_{Q_t} \rho(\tau)\, H_\eta(w)\, (u - \hat{u})_\tau.$$

Now

$\lim_{\eta \to 0^+} H_\eta(w) = \text{sgn}^+(\varphi(u) - \varphi(\hat{u})) = \text{sgn}^+(u - \hat{u})$, $\mathcal{H}_1$ – a.e. and Lebesgue's theorem yields

$$\lim_{\eta \to 0^+} \int_{Q_t} \rho(\tau)\, H_\eta(w)\, (u - \hat{u})_\tau = \int_{Q_t} \rho(\tau) \left((u - \hat{u})^+\right)_\tau.$$

Going to the limit as $\eta \to 0^+$ in (51) we obtain

$$\int_{Q_t} \rho(\tau) \left((u - \hat{u})^+\right)_\tau \leq 0,$$

by which it follows the nonincreasing of $(u(x, \cdot) - \hat{u}(x, \cdot))^+$ as function of t. Since $t \to (u(x, \cdot) - \hat{u}(x, \cdot))^+$ is periodical, this implies that $(u - \hat{u})^+$ is constant i.e. $u \leq \hat{u}$. Changing u with $\hat{u}$ the same argument proves that $u \geq \hat{u}$ by which the uniqueness of the periodic solution follows.

In conclusion, we have proved the following result.

Theorem. 13. *If* $(H_\varphi) - (H_H)$ *and* (49) *hold, the problem* (1) – (4) *has a unique periodic strong solution.*

REFERENCES

1. Badii, M. (1994) Existence and Uniqueness of Solutions for a Degenerate Quasilinear Parabolic Problem. *Publicaciones Matemàtiques* **38,** 327-352.

2. Badii, M. & Diaz, J.I. (1995) Periodic Solutions of a Quasilinear Parabolic Boundary Value Problem Arising in Unsaturated Flow Through a Porous Media. *Applicable Analysis* **56,** 3-4, 279-301.

3. Di Benedetto, E. & Friedman, A. (1986) Periodic Behaviour for the Evolutionary Dam Problem and Related Free Boundary Problems. *Comm. in P.D.E.* **11,** n. 12, 1297-1377.

4. Federer, H. (1969) *Geometric Measure Theory. Die Gründlehren der Math. Wiss.* 153 Springer-Verlag, Berlin.

5. Fokina, T.I. (1975) On a Boundary Value Problem for Parabolic Equations with Strong Nonlinearities. *Vestinik Moskov. Univ. Math. Mech.* **30,** 22-27.

6. Friedman, A. (1964) *Partial Differential Equations of Parabolic Type.* Prentice-Hall.

7. Gagnuex, G. (1986) Une approche analytique nouvelle des modèles de la récupération secondée en ingénierie pétrolière. *J. Mécanique Théorique et Appl.* **5,** 3-20.

8. Huang, Z. (1990) A Filtration Problem with Surface Evaporation. *Bollettino U.M.I.* **(7)** 4-A, 253-261.

9. Huang, Z. & Primicerio, M. (1992) The Surface Evaporation Problem with Signorini Boundary Condition. *SIAM J. Math. Anal.* **23** n. 2, 334-345.

10. Kenmochi, K., Kröner, D. & Kubo, M. (1990) Periodic Behaviour to Porous Media Equations of Parabolic-Elliptic Type. *J.P.D.E.* **3,** 3, 63-77.

11. Van Duyn, C.J. & Peletier, A. (1982) Nonstationary Filtration in Partially Saturated Porous Media. *Arch Rational Mech. Anal.* **78,** 173-198.

12. Ladyzenskaja, O., Solonnikov, A. & Ural'ceva, N.N. (1968) Linear and Quasilinear Equations of Parabolic Type. *American Mathematical Society,* Providence Rhode-Island.