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AN OPTIMAL CONTROL PROBLEM FOR HELMHOLTZ EQUATION WITH NON-LOCAL BOUNDARY CONDITIONS AND QUADRATIC FUNCTIONAL

(optimal control/helmholtz equation/elliptic equations)

FRANCISCO CRIADO*, GAMLET MELADZE** and NANA ODISEHLIDZE**

* *Department of Mathematics.* Málaga University. Spain. ** *Department of Applied Mathematics and Computer Science.* Tbilisi State University. Georgia.
E-mail: f_criado@ccuma.sci.uma.es

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ABSTRACT

In the present paper the optimal control problem for Helmholtz equation with non-local boundary conditions and quadratic functional is considered. The necessary and sufficient conditions for optimality in a maximum principle form have been obtained.

RESUMEN

En este trabajo se considera el problema de control óptimo para la ecuación de Helmholtz con condiciones de contorno locales y funcional cuadrático. Se obtienen condiciones necesarias y suficientes de optimalidad en la forma del principio del máximo.

1. INTRODUCTION

The control with distributed systems, described by linear differential elliptic equations with non-local boundary conditions is a serious problem in the optimal control theory.

Bitsadze-Samarski non-local boundary problem [1] arises in connection with mathematical modeling of plasma processes. We can also indicate other areas of important applications, for example, in the investigation of baroclinic sea [8], in the theory of elasticity and shells [2]. An optimal control problem for elliptic equations with classical boundary conditions and quadratic functional has been considered in [3].

2. STATEMENT OF THE PROBLEM AND MAIN RESULTS

Let G be a rectangle $G =]0, l_1[\times]0, l_2[$. Γ the boundary of the rectangular domain, $\gamma = \{(l_1, y) : 0 \leq y \leq l_2\}$ and $\gamma_0 = \{(x_0, y) : 0 \leq y \leq l_2\}$, x_0 the fixed point of interval $]0, l_1[$, V some open subset in $\Re$ and U_{ad} the set of control functions $v : G \to V$. $v \in L_2(G)$.

Let us consider Bitsadze-Samarski problem for Helmholtz equation [4] for each fixed $v \in U_{ad}$ in the domain G:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - q(x, y)\, u = a(x, y)\, v + b(x, y). \quad (x, y) \in G. \tag{1}$$

$$u(x, y) = 0 \qquad (x, y) \in \Gamma/\gamma$$
$$u(l_1, y) = \sigma\, u(x_0, y) \qquad 0 \leq y \leq l_2 \tag{2}$$

where $a \in L_\infty(G)$, $b \in L_2(G)$, $0 \leq q \in L_\infty(G)$, $0 < \sigma < 1$, σ = const

Similarly as [5], it can be shown, that the solution of problem (1)-(2) exists, is unique and belongs to space $H^2(G)$ [10].

Let us consider the functional

$$I(v) = \iint_G \left[c(x, y)u - z_d\right]^2 dx\, dy + \iint_G N(x, y)v^2\, dx\, dy \tag{3}$$

where: $c \in L_\infty(G)$, z_d is given element, $z_d \in L_2(G)$, $0 < N \in L_\infty(G)$.

1. Let us formulate the following control problem: find the function $v_0 \in U_{ad}$, whose corresponding solution of the boundary value problem (1)-(2) together with v_0 results in minimal functional value. The pair (u_0, v_0) thus obtained is called optimal [9].

2. To obtain conditions of optimality following the scheme developed in the works [6] [7].

Let us consider arbitrary permissible control $v_\varepsilon \in U_{ad}$ and u_ε corresponding solution of problem (1)-(2). Let us introduce the notation

$$\tilde{v} = v_\varepsilon - v_0, \quad \tilde{u} = u_\varepsilon - u_0 \tag{4}$$

then we obtain the following problem:

$$\frac{\partial^2 \tilde{u}}{\partial x^2} + \frac{\partial^2 \tilde{u}}{\partial y^2} - q(x, y)\tilde{u} = a(x, y)\tilde{v}, \quad (x, y) \in G \tag{5}$$

$$\begin{aligned} \tilde{u}(x, y) &= 0 & (x, y) &\in \Gamma \setminus \gamma, \\ \tilde{u}(l_1, y) &= \sigma\tilde{u}(x_0, y) & 0 \le y &\le l_2. \end{aligned} \tag{6}$$

Let $\Psi \neq 0$, $\Psi \in H^2(G\setminus\gamma_0) \cap H^1(G)$ [3]. Multiplying equation (5) by Ψ and integrating over domain G, the following equality is obtained:

$$\iint_G \Psi(x, y)\left[\frac{\partial^2 \tilde{u}}{\partial x^2} + \frac{\partial^2 \tilde{u}}{\partial y^2} - q(x, y)\,\tilde{u}\right]dx\,dy = \iint_G a(x, y)\,\Psi(x, y)\tilde{v}\,dx\,dy \tag{7}$$

The increment of functional (3) with fixed v_0, v_ε is:

$$\tilde{I} = I(v_\varepsilon) - I(v_0) \tag{8}$$

$$= \iint_G \left[(c(x, y)\,u_\varepsilon - z_d)^2 + N(x, y)\,v_\varepsilon^2\right]dx\,dy - \iint_G \left[(c(x, y)\,u_0 - z_d)^2 + N(x, y)\,v_0^2\right]dx\,dy =$$

$$= \iint_G \left[2c(x, y)\,\tilde{u}\,(c(x, y)\,u_0 - z_d) + 2N(x, y)\,v_0\tilde{v} + c^2(x, y)\,\tilde{u}^2 + N(x, y)\,\tilde{v}^2\right]dx\,dy$$

From relations (7)-(8), we obtain for the increment the following expression:

$$\tilde{I} = I(v_\varepsilon) - I(v_0) = \tag{9}$$

$$= \iint_G \Psi(x, y)\left[\frac{\partial^2 \tilde{u}}{\partial x^2} + \frac{\partial^2 \tilde{u}}{\partial y^2} - q(x, y)\,\tilde{u}\right]dx\,dy - \iint_G a(x, y)\,\Psi(x, y)\,\tilde{v}\,dx\,dy$$

$$+ \iint_G \left[2c(x, y)\,\tilde{u}\,(c(x, y)\,u_0 - z_d) + N(x, y)\,\tilde{v}^2 + 2N(x, y)\,v_0\tilde{v}\right.$$

$$\left. + c^2(x, y)\,\tilde{u}^2\right]dx\,dy = \iint_G \Psi(x, y)\left[\frac{\partial^2 \tilde{u}}{\partial x^2} + \frac{\partial^2 \tilde{u}}{\partial y^2}\right]dx\,dy +$$

$$+ \iint_G \left[\left(2c(x, y)\,(c(x, y)\,u_0 - z_d) - q(x, y)\,\Psi\right)\tilde{u}\, + \right.$$

$$\left. + \left(2N(x, y)\,v_0 - a(x, y)\,\Psi\right)\tilde{v} + c^2(x, y)\,\tilde{u}^2 + N(x, y)\,\tilde{v}^2\right]dx\,dy$$

To obtain the adjoint equation, let us make the following transformations:

$$\int_0^{l_2}\int_0^{l_1} \Psi(x, y)\,\frac{\partial^2 \tilde{u}}{\partial x^2}\,dx\,dy = \int_0^{l_2}\left(\int_0^{x_0} \Psi(x, y)\,\frac{\partial^2 \tilde{u}}{\partial x^2}\,dx + \int_{x_0}^{l_1} \Psi(x, y)\,\frac{\partial^2 \tilde{u}}{\partial x^2}\,dx\right)dy =$$

$$= \int_0^{l_2}\Big(\Psi(l_1, y)\,\tilde{u}_x(l_1, y) - \Psi(0, y)\,\tilde{u}_x(0, y) + \left[\Psi(x_0^-, y) - \Psi(x_0^+, y)\right]\tilde{u}_x(x_0, y)$$

$$+ \left[\Psi_x(x_0^-, y) - \Psi_x(x_0^+, y) - \sigma\Psi_x(l_1, y)\right]\tilde{u}(x_0, y) + \int_0^{x_0}\frac{\partial^2 \Psi}{\partial x^2}\tilde{u}(x, y)\,dx +$$

$$+ \int_{x_0}^{l_1}\frac{\partial^2 \Psi}{\partial x^2}\tilde{u}(x, y)\,dx\Big)\,dy$$

Since $\Psi \in H^2(G\setminus\gamma_0) \cap H^1(G)$, then $\Psi(x_0^-, y) = \Psi(x_0^+, y)$ [10]

In similar way, we obtain:

$$\int_0^{l_1}\int_0^{l_2} \Psi(x, y)\,\frac{\partial^2 \tilde{u}}{\partial x^2}\,dx\,dy$$

$$= \int_0^{l_1}\left(\int_0^{l_2} \tilde{u}(x, y)\,\frac{\partial^2 \Psi}{\partial y^2}\,dy + \left(\Psi(x, l_2)\,\tilde{u}_y(x, l_2) - \Psi(x, 0)\,\tilde{u}_y(x, 0)\right)\right)dx$$

From these relations we conclude that if Ψ is the solution of the following problem:

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} - q(x, y)\,\Psi = -2c(x, y)\,(c(x, y)\,u_0 - z_d), \quad (x, y) \in G \setminus \gamma_0, \tag{10}$$

$$\begin{aligned} \Psi(x, y) &= 0 & (x, y) &\in \Gamma \\ \Psi_x(x_0^-, y) - \Psi_x(x_0^+, y) &= \sigma\Psi_x(l_1, y), & 0 \le y &\le l_2 \end{aligned} \tag{11}$$

then, the increment of the functional will take the form:

$$\tilde{I} = \iint_G (2N(x, y)\,v_0 - a(x, y)\,\Psi)\,\tilde{v}\,dx\,dy + \iint_G (c^2(x, y)\,\tilde{u}^2 + N(x, y)\,\tilde{v}^2)\,dx\,dy \ge 0 \tag{12}$$

Partially integrating equation (10) and using the properties of Dirac's distribution, we obtain the following problem equivalent to problem (10) - (11).

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} - q(x, y)\,\Psi = -2c(x, y)\,(c(x, y)\,u_0 - z_d) -$$

$$-\delta(x_0 - x)\,\sigma\Psi_x(l_1, y), \quad (x, y) \in G \tag{13}$$

$$\Psi(x, y) = 0 \qquad (x, y) \in \Gamma \tag{14}$$

where $\delta(x_0 - x)$ is the Dirac distribution.

Theorem 1. *Let Ψ_0 be a solution of the adjoint problem (13)-(14) and $N(x, y) > 0$, then for (u_0, v_0) to be optimal a necessary and sufficient condition is that the following relation be true almost everywhere on G:*

$$2N(x, y)v_0 - a(x, y)\,\Psi_0 = 0 \tag{15}$$

Proof. Let (u_0, v_0) be the optimal pair and $N(x, y) > 0$. We are going to show, that condition (15) is satisfied. Let us prove the contrary. Let us assume, that

$$2N(x, y)v_0 - a(x, y)\,\Psi_0 \neq 0$$

on a set of positive Lebesgue measure.

Consequently,

$$0 < \mu[\{(x, y) \in G/2N(x, y)v_0 - a(x, y)\Psi_0 \neq 0\}] =$$
$$= \mu[\{(x, y) \in G/2N(x, y)v_0 - a(x, y)\Psi_0 > 0\} \cup$$
$$\cup \{(x, y) \in G/2N(x, y)v_0 - a(x, y)\Psi_0 < 0\}] =$$
$$= \mu[\{(x, y) \in G/2N(x, y)v_0 - a(x, y)\Psi_0 > 0\} +$$
$$+ \mu[\{(x, y) \in G/2N(x, y)v_0 - a(x, y)\Psi_0 < 0\}$$

where μ is the Lebesgue measure on the G. Let us introduce the notation:

$$G_+ = [\{(x, y) \in G/2N(x, y)v_0 - a(x, y)\Psi_0 > 0\}]$$
$$G_- = [\{(x, y) \in G/2N(x, y)v_0 - a(x, y)\Psi_0 < 0\}]$$

Let us consider two cases: $\mu(G_+) > 0$, $\mu(G_-) > 0$.

Suppose $\mu(G_+) > 0$. Since V is open, there exists $k_0 > 0$ such, that $v_0 + k_{\chi G+} \in V$ with $|k| \leq k_0$, where $\chi G+$ denotes the characteristic function of the set G_+.

Denote by

$$T = 2N(x, y)v_0 - a(x, y)\Psi_0$$

then there exists $\varepsilon > 0$ such that

$$\iint_{G_+} T\,dx\,dy > \varepsilon$$

According to the definition $\|k_{\chi G+}\|_{L_2(G)} \to 0$ for $k \to 0$. Let $k < 0$; and assuming, that $\tilde{v} = k_{\chi G+}$ and $\tilde{u}(x, y, k_{\chi G+})$ is the solution of problem (5) - (6), it then follows that:

$$\tilde{u}(x, y, k_{\chi G+}) = k\tilde{u}(x, y, \chi G_+) \tag{16}$$

Taking into account that $\tilde{v} = k_{\chi G+}$

$$\iint_G (2N(x, y)v_0 - a(x, y)\Psi_0)\,\tilde{v}\,dx\,dy = \iint_G Tk_{\chi G+}\,dx\,dy = k\iint_{G_+} T\,dx\,dy$$

and, further, that

$$\iint_G N(x, y)\tilde{v}^2\,dx\,dy = \iint_G N(x, y)\,(k_{\chi G+})^2\,dx\,dy = k^2\iint_{G_+} N(x, y)\,dx\,dy$$

and taking into account (16), we have

$$\iint_G c^2(x, y)\,\tilde{u}^2\,dx\,dy = \iint_G c^2(x, y)\,\tilde{u}^2\left(x, y, k_{\chi G_+}\right)dx\,dy$$
$$= k^2\iint_G c^2(x, y)\,\tilde{u}^2\left(x, y, \chi G_x\right)dx\,dy.$$

Since v_0 is optimal, then for sufficently small increment $\tilde{v}$

$$I(v_\varepsilon) - I(v_0) \geq 0$$

On the other hand, for $\tilde{v} = k_{\chi G+}$ from equation (12) it follows

$$I(v_0+\tilde{v}) - I(v_0) = \iint_G T\tilde{v}\,dx\,dy + \iint_G N(x, y)\tilde{v}^2\,dx\,dy + \iint_G c^2(x, y)\tilde{u}^2\,dx\,dy =$$
$$= k\left(\iint_{G_+} T\,dx\,dy + k\left(\iint_{G_+} N(x, y)\,dx\,dy + \iint_G c^2(x, y)\,(\tilde{u}(x, y, \chi G_+))^2\,dx\,dy\right)\right).$$

From here, there exists $\delta > 0$ such that for $-\delta \leq k \leq 0$ we have

$$\left|k\left(\iint_{G_+} N(x, y)\,dx\,dy + \iint_G c^2(x, y)\,(\tilde{u}(x, y, \chi G_+))^2\,dx\,dy\right)\right| < \frac{\varepsilon}{2}$$

Consecuently, for $-\delta \leq k \leq 0$

$$0 \leq \frac{\varepsilon}{2} < \iint_{G_+} T\,dx\,dy + k\left(\iint_{G_+} N(x, y)\,dx\,dy + \iint_G c^2(x, y)\,(\tilde{u}(x, y, \chi G_+))^2\,dx\,dy\right),$$

and

$$k\left(\iint_{G_+} T\,dx\,dy + k\left(\iint_{G_+} N(x, y)\,dx\,dy + \iint_G c^2(x, y)\,(\tilde{u}(x, y, \chi G_+))^2\,dx\,dy\right)\right) < 0$$

that is, for $-\delta \leq k \leq 0$

$$I(v_0 + k\chi G_+) - I(V_0) < 0$$

Taking into account, that for $|k| \leq k_0$

$$v_0 + k_{\chi G_+} \in U_{ad} \quad \text{and} \quad \lim_{k \to 0} \left\|k_{\chi G_+}\right\|_{L_2(G)} = 0$$

this contradicts the optimally of v_0. Hence

$$\mu(G_+) = 0$$

Similary, it is proved that $\mu(G-) + 0$.

Thus, the neccessary condition of optimality is proved. To prove sufficiency: Let

$$2N(x, y)v_0 - a(x, y)\Psi_0 = 0$$

almost everywhere on G and $N(x, y) > 0$. It then follows, from our assumptions, that for each v_ε

$$I(v_\varepsilon) - I(v_0) \geq 0$$

i.e., (u_0, v_0) is the optimal pair.

This proves the theorem.

Let us consider the case when V convex set and functional has the following form:

$$I(v) = \iint_G \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \right] dx\,dy + \iint_G v^2\,dx\,dy \tag{17}$$

Let us formulate the following control problem: find the function $v_0 \in U_{ad}$, whose corresponding solution of the boundary value problem (1)-(2) together with v_0 results in minimal functional value (17). The pair (u_0, v_0) thus obtained is called optimal [9].

Let us consider arbitrary permissible control $v_\varepsilon = v_0 + \varepsilon(v - v_0) \in U_{ad}$ and $u_\varepsilon = u_0 + \varepsilon(u - u_0)$ corresponding solution of problem (1)-(2). Let us introduce the notation

$$\delta v = v_\varepsilon - v_0 = \varepsilon(v - v_0), \quad \delta u = u_\varepsilon - u_0 = \varepsilon(u - u_0) \tag{18}$$

Let us $\Psi \neq 0$, $\Psi \in H^2(G\backslash\gamma_0) \cap H^1(G)$.

In a similar way we obtain that, if Ψ is the solution of the following problem:

$$\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} - q(x, y)\,\Psi = 2\big(q(x, y)\,u + a(x, y)\,v +$$

$$+ b(x, y)\big), \quad (x, y) \in G \setminus \gamma_0, \tag{19}$$

$$\Psi(x, y) = 0, \quad (x, y) \in \Gamma. \tag{20}$$

$$\Psi_x(x_0^-, y) - \Psi_x(x_0^+, y) - \sigma\Psi_x(l_1, y) = -2\sigma u_x(l_1, y), \quad 0 \le y \le l_2$$

then, the increment of the functional will take the form:

$$\delta I = \iint_G (2v_0 - a(x, y)\,\Psi)\,\delta v + \iint_G \left[\left(\frac{\partial(\delta(u))}{\partial x}\right)^2 + \left(\frac{\partial(\delta(u))}{\partial y}\right)^2 + (\delta v)^2 \right] dx\,dy$$

Taking into account (18), we obtain:

$$\delta I = \varepsilon \iint_G \big(2v_0 - a(x, y)\,\Psi\big)\,(v - v_0)\,dx\,dy +$$

$$+ \varepsilon^2 \iint_G \left[\left(\frac{\partial(u - u_0)}{\partial x}\right)^2 + \left(\frac{\partial(u - u_0)}{\partial y}\right)^2 + \right.$$

$$\left. + (v - v_0)^2 \right] dx\,dy, \quad \forall v \in U_{ad}, \forall \varepsilon \ge 0.$$

From here we have:

$$\iint_G \big(2v_0 - a(x, y)\,\Psi\big)\,(v - v_0)\,dx\,dy \ge 0, \quad \forall v \in U_{ad} \tag{21}$$

Let us show equivalence of (21) and (22),

$$\inf_{v \in V} \Big[\big(2v_0 - a(x, y)\,\Psi\big)\,v - \big(2v_0 - a(x, y)\,\Psi\big)\,v_0\Big] \ge 0 \tag{22}$$

Let us introduce the notation: $P = 2v_0 - a(x, y)\,\Psi$,

$G_0 = \{(x_0, y_0) \in G$ is a Lebesgue's point for P and Pv_0 functions$\}$

Then $mesG_0 = mesG$.

Let us

$$G_\delta = \{(x, y) \in G \,|\, |x - x_0| < \delta,\ |y - y_0| < \delta\}$$

where (x_0, y_0) is arbitrary fixed point from G_δ. Consider admissible control:

$$v_\delta(x, y) = \begin{cases} v_0(x, y), & (x, y) \in G \setminus G_\delta \\ v, & (x, y) \in G_\delta \end{cases}$$

where v is arbitrary point from V.

From (21) follows:

$$\iint_G P(x, y)\,v_\delta(x, y)\,dx\,dy \ge \iint_G P(x, y)\,v_0(x, y)\,dx\,dy$$

Since $G = G \cup \{G\backslash G_\delta\}$, then

$$\iint_{G_\delta} P(x, y)\,v(x, y)\,dx\,dy \ge \iint_{G\backslash G_\delta} P(x, y)\,v_0(x, y)\,dx\,dy \ge$$

$$\ge \iint_{G_\delta} P(x, y)\,v_0(x, y)\,dx\,dy + \iint_{G\backslash G_\delta} P(x, y)\,v_0(x, y)\,dx\,dy$$

Hence

$$\iint_{G_\delta} P(x, y)\,v(x, y)\,dx\,dy \ge \iint_{G_\delta} P(x, y)\,v_0(x, y)\,dx\,dy,$$

$$mesG_\delta = 4\delta^2$$

From here we get:

$$\lim_{\delta\to 0} \frac{1}{4\delta^2} \iint_{G_d} P(x, y)\,v\,dx\,dy \ge$$

$$\ge \lim_{\delta\to 0} \frac{1}{4\delta^2} \iint_{G_d} P(x, y)\,v_0\,(x, y)\,dx\,dy$$

According to the Lebesgue's theorem

$$P(x_0, y_0)v \ge P(x_0, y_0)v_0\,(x_0, y_0)$$

As (x_0, y_0) is taken arbitrary from G_0 and v from V, then

$$\inf_{v\in V} P(x_0, y_0)v \geq P(x_0, y_0)v_0(x_0, y_0), \quad \text{for any } (x_0, y_0) \in G$$

i.e., (22) is valid.

Since $mesG_0 = mesG$, then

$$\inf_{v\in V} P(x, y)v = P(x, y)v_0(x, y)$$

almost everywhere on G.

Let us prove the converse. From (22) it follows

$$P(x, y)v(x, y) \geq P(x, y)v_0(x, y), \quad \forall v(\cdot) \in U_{ad}$$

almost everywhere on G, since

$$P(x, y)v(x, y) \geq \inf_{w\in V} P(x, y)w = P(x, y)v_0(x, y)$$

almost everywhere on G. Then

$$\iint_G P(x, y)v(x, y)\,dx\,dy \geq \iint_G P(x, y)v_0(x, y)\,dx\,dy$$

for any $v(\cdot) \in U_{ad}$.

Thereby the equivalence of (21) and (22) is proved. Hence the necessity of conditions (22) for optimality of pair (u_0, v_0) directly follows.

To prove sufficiency, let (22) be valid. Let us prove that (u_0, v_0) is an optimal pair. From equivalence of (21) and (22) follows that (21) is valid, and hence we get:

$$I = \iint_G (2v_0 - a(x, y)\Psi)(v - v_0)\,dx\,dy +$$

$$+ \varepsilon^2 \iint_G \left[\left(\frac{\partial(u - u_0)}{\partial x}\right)^2 + \left(\frac{\partial(u - u_0)}{\partial y}\right)^2 + (v - v_0)^2\right] dx\,dy \geq 0$$

i.e., (u_0, v_0) is the optimal pair.

For this problem (1)-(2), (17), when V some open subset from $\Re$, the condition (22) has the following form:

$$2v_0 - a(x, y)\Psi = 0$$

Now consider the optimal control problem (1)-(3) when V convex set. Similarly we conclude, that if Ψ is the solution of (10-11), then taking into account (18) from (12) we have:

$$\iint_G (2N(x, y)v_0 - a(x, y)\Psi)(v - v_0)\,dx\,dy \geq 0, \quad \forall v \in U_{ad}$$

which is equivalent to this:

$$\inf_{v\in V} \left[(2N(x, y)v_0 - a(x, y)\Psi)v - (2N(x, y)v_0 - a(x, y)\Psi)v_0\right] \geq 0$$

The maximum principle has been obtained.

Theorem 2. *Let the cost functional $I(v)$ be given by formula (17) (or (3)). Then a necessary and sufficient condition for the pair (u_0, v_0) to be optimal is that the following relations: (1), (2), (9), (20) (or (1), (2), (10), (11)) and*

$$\iint_G (2v_0 - a(x, y)\Psi)(v - v_0)\,dx\,dy \geq 0, \quad \forall v \in U_{ad}$$

$$(or \quad \iint_G (2N(x, y)v_0 - a(x, y)\Psi)(v - v_0)\,dx\,dy \geq 0,$$

$\forall v \in U_{ad}$) almost everywhere on G, hold.

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