Rev.R.Acad.Cienc.Exact.Fís.Nat. (Esp)
Vol. 91, N.º 1, pp 47-52, 1997
Matemáticas

MAXIMUM PRINCIPLE AND EXISTENCE OF SOLUTIONS FOR ELLIPTIC SYSTEMS INVOLVING SCHRÖDINGER OPERATORS

(cooperative elliptic systems/maximum principle/schrödinger operators/existence of solution/semilinear systems)

BÉNÉDICTE ALZIARY, LAURE CARDOULIS and JACQUELINE FLECKINGER-PELLÉ*

Ceremath, Université Toulouse 1, place A. France, 31042 TOULOUSE CEDEX.
alziary@ceremath.univ-tlse1.fr, cardouli@ceremath.univ-tlse1.fr, jfleck@cict.fr

Presentado por J. I. Díaz, 10 de febrero de 1995

ABSTRACT

We study here a cooperative elliptic system defined on $\mathbb{R}^n$; we obtain necessary or sufficient conditions for having the Maximum Principle and the existence of a positive solution for linear systems involving Schrödinger operators. Then we deduce existence of positive solutions for semilinear systems.

RESUMEN

En este trabajo se estudia un sistema cooperativo definido sobre todo $\mathbb{R}^n$ dado por operatores de tipo Schrödinger. Se dan condiciones necesarias o suficientes para la validez del principio del máximo. Finalmente, se analiza la existencia de una solución positiva de tal sistema.

1. INTRODUCTION

Given an integer $N > 1$, we consider the following system (in its variational form), for any $1 \le i \le N$:

$$\left\{ (1.i) \quad L_{q_i} u_i := (-\Delta + q_i)\, u_i = \sum_{j=1}^{j=N} a_{ij}\, u_j + f_i \text{ in } \mathbb{R}^n \right. \tag{1}$$

Here a_{ij} are given numbers such that

$$a_{ij} > 0 \text{ for } i \neq j; \tag{2}$$

$$\begin{cases} q_i \text{ is a positive continuous function, larger than 1;} \\ q_i \text{ tending to } +\infty \text{ at infinity;} \end{cases} \tag{3}$$

Hypothesis on f_i, which are given functions, will be specified later.

Such systems where (2) is satisfied are called cooperative. They appear in some physical and biological problems. They have been studiend on Ω, bounded open set of $\mathbb{R}^n$ and when $q_1 = q_i \equiv 0$ in [1] to [14]... Only few papers deal with systems defined on unbounded domains [15] to [17]. Here, System (1), which involves Shrödinger operators and which is defined on $\mathbb{R}^n$, appears for example in laser theory. It has been studied, when $N = 2$ and $f_2 \equiv 0$, in [15] or [16]; we extend here some of these results when $f_2 \not\equiv 0$.

We say that System (1) satisfies a *Maximum Principle* if any nonnegative data: $f_i \ge 0$, $\forall 1 \le i \le N$, implies that any solution $u := (u_1, \ldots, u_i, \ldots u_N)$ is nonnegative: $u_i \ge 0$. The proofs here are often analoguous to that of [5], [6]; we do not use the decoupling method as in [3, 4, 15] or [16].

Our paper is organized as follow: In Section 2 we recall some results on Schrödinger equations, $N = 1$; then we study linear systems of N equations in Section 3, and semilinear systems in Section 4. For all these cases we study necessary or sufficient conditions for having the Maximum Principle and for existence of solutions. Note that, generally our necessary conditions are different from the sufficient ones, and only for the case $q_1 = q_2 = \ldots = q_N$, we have been able to obtain necessary and sufficient conditions (e.g. as in [5, 6, 7]).

2. RECALLS ON THE SCALAR CASE

We first recall some results on the scalar, case.

Let $h \in L^2(\mathbb{R}^n)$; a is a given number and q is a continuous function such that:

$$\exists c > 0, \quad 0 < c < q(x), \quad q(x) \to +\infty \text{ as } |x| \to \infty. \tag{4}$$

We study here the equation, in its variational form:

$$L_q u := (-\Delta + q)u = au + h \quad \text{in } \mathbb{R}^n \tag{E}$$

The associated variational space is $V_q(\mathbb{R}^n)$: the completion of $\mathcal{D}(\mathbb{R}^n)$ with respect to the norm

$$\|u\|_q := \left[\int_{\mathbb{R}^n} |\nabla u|^2 + q\,|u|^2\right]^{1/2}.$$

Since, e.g. by [18], the embedding of $V_q(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$ is compact, L_q considered as an operator in $L^2(\mathbb{R}^n)$ is positive, selfadjoint, with compact inverse. Hence its spectrum is discrete; it consists in an infinite sequence of positive eigenvalues, tending to $+\infty$; moreover, the smallest one, denoted by $\lambda(q)$ is simple and is associated with an eigenfunction φ which does not change sign in $\mathbb{R}^n$:

$$\begin{cases} L_q\,\varphi = \lambda(q)\varphi \quad \mathbb{R}^n; \\ \varphi(x) \to 0 \text{ as } |x| \to \infty;\ \varphi > 0 \quad \text{in } \mathbb{R}^n. \end{cases}$$

Such an eigenvalue is usually referred as *«principal eigenvalue»*. It is characterized by:

$$\lambda(q) \int_{\mathbb{R}^n} |u|^2 \leq \int_{\mathbb{R}^n} (|\nabla u|^2 + q\,|u|^2) \quad \forall u \in V_q(\mathbb{R}^n). \tag{5}$$

The equality in (5) holds if and only if $u = k\varphi$, $k \in \mathbb{R}$.

Proposition 1 (Maximum Principle) *For any $a < \lambda(q)$, $L_q - a$ is with compact inverse and therefore, for any $h \in L^2(\mathbb{R}^n)$, there exists a unique $u \in V_q(\mathbb{R}^n)$ solution of (E); $0 \leq h \in L^2(\mathbb{R}^n)$, implies $u \geq 0$ if and only if $a < \lambda(q)$. Moreover $u > 0$ if $h \not\equiv 0$.*

This is the (strong) Maximum Principle, that can be found in [19].

3. LINEAR SYSTEMS

We consider here system (1) with N equations, $N \geq 2$. We assume that Hypothesis (2), and (3) are verified; set: $q := \max(q_i)$, $1 \leq i \leq N$ and $\lambda(q)$ the associated principal eigenvalue. Note that, since $q \to \lambda(q)$ increases with q, $\lambda(q) \geq \lambda(q_i)$, $\forall i \in \{1,..., N\}$. We seek weak solutions in $V_{q1}(\mathbb{R}^n) \times ... \times V_{qn}(\mathbb{R}^n)$. Indeed, if the potentials are such that:

$$|q_1 - q_i| \leq A.q_1 \text{ with } 0 < A < 1, \tag{3'}$$

the variational spaces $V_{q1}(\mathbb{R}^n) = V_{qn}(\mathbb{R}^n) = V_q(\mathbb{R}^n)$ are the same.

First let us recall some notions about nonsingular M-matrices.

Definition 1 *Any matrix M of the form*

$$M = sI - B, \quad s > 0, \quad B \geq 0$$

for wich $s > \rho(B)$, the spectral radius of B, is called a nonsingular M-matrices.

Proposition 2 *If M is with negative entries outside the diagonal, each of the following conditions is equivalent to the statement: «M is a nonsingular M-matrix»:*

(P1) M is semipositive; that is there exists $X >> 0$ with $MX >> 0$.

(P2) M is inverse-positive; that is, M^{-1} exists and $M^{-1} \geq 0$.

(P3) There exists a positive diagonal matrix D such that $MD + DM'$ is positive definite.

(P4) All the principal minors of M are positive.

For more details about M-matrices and the proof of the proposition see [20].

The system (1) could be written:

$LU = AU + F$
with $A = (a_{ij})_{1\leq i,j\leq N}$ and

$$U = \begin{pmatrix} u_1 \\ \cdot \\ u_N \end{pmatrix}, F = \begin{pmatrix} f_1 \\ \cdot \\ f_N \end{pmatrix}, L = \begin{pmatrix} -\Delta+q_1 & 0 & \cdot & \cdot & 0 \\ 0 & \cdot & 0 & \cdot & \cdot \\ \cdot & 0 & \cdot & 0 & \cdot \\ \cdot & \cdot & 0 & \cdot & 0 \\ 0 & \cdot & \cdot & 0 & -\Delta+q_N \end{pmatrix}$$

For all $1 \leq i \leq N$, denote by φ_i the principal positive eigenfunction of $L_{qi} = -\Delta + q_i$ wich is associated with the principal eigenvalue $\lambda(q_i)$ and Λ the matrix:

$$\Lambda = \begin{pmatrix} \lambda(q_i) & 0 & \cdot & \cdot & 0 \\ 0 & \cdot & 0 & \cdot & 0 \\ \cdot & 0 & \cdot & 0 & \cdot \\ \cdot & \cdot & 0 & \cdot & 0 \\ 0 & \cdot & \cdot & 0 & \lambda(q_N) \end{pmatrix}.$$

Note that since the system is cooperative, the matrix $\Lambda - A$ is with negative entries outside the diagonal.

Theorem 1. *We suppose that* (2), (3), *and* (3') *are satisfied; assume that for all $1 \leq i \leq N$, f_i* are in $L^2(\mathbb{R}^n)$; *if the Maximum Principle holds for (1), then neccesarly*

$$\begin{cases} (a) \quad \forall 1 \leq i \leq N \quad a_{ii} < \lambda(q_i) \\ (b) \quad \lambda(q)I - A \text{ is a nonsingular M-matrix} \end{cases} \tag{6}$$

Remark: For $N = 2$, (6.b) is $(\lambda(q) - a_{11})(\lambda(q) - a_{22}) > a_{12}a_{21}$.

Proof of Theorem 1: Assume that $\forall 1 \leq i \leq N$, $f_i \geq 0$, $f_i \not\equiv 0$, and that any solution $(u_1,..., u_N)$ is such that $u_i \geq 0$, $\forall 1 \leq i \leq N$.

Multiplying (1.i) by the principal eigenfunction φ_i associated with the principal eigenvalue $\lambda(q_i)$, integrating over $\mathbb{R}^n$, we obtain:

$$\left\{(\lambda(q_i) - a_{ii}) \int u_i \varphi_i - \sum_{j\neq i} a_{ij} \int u_j \varphi_i \geq 0 \ (> 0 \text{ if } f_i \not\equiv 0)\right.$$

Since by hypothesis, the Maximum Principle holds, $\int_{u_i\varphi_i}$, $\int_{u_j\varphi_i}$, are nonnegative; moreover $a_{ij} > 0$, $\forall i \neq j$; therefore it follows that $a_{ii} < \lambda(q_i)$.

Then we rewrite (1.*i*) as follows:

(1')

$$\left\{(1'.i) \quad L_q u_i = \sum_{j=1}^{j=N} a_{ij}\, u_j + (q-q_i)u_i + f_i\right.$$

Denote by φ the principal eigenfunction, associated with the principal eigenvalue $\lambda(q)$. Multiplying (1'.*i*) by φ and integrating over $I\!R^n$, we get:

$$\left\{\lambda(q) \int u_i\, \varphi - \sum_{j=1}^{j=N} a_{ij} \int u_j\, \varphi = \int [(q-q_i)u_i + f_i]\varphi \geq 0 \; (> 0 \text{ if } f_i \not\equiv 0)\right.$$

Since by hypothesis, the Maximum Principle holds, we get $(\lambda(q)I - A)X = G >> 0$ with $X >> 0$, where X_i, the i-th component of X, is: $X_i = \int u_i\varphi$. By *(P1)*, $(\lambda(q)I - A)$ is a nonsingular M-matrix.

Theorem 2. *Assume that for all* $1 \leq i \leq N$, f_i *are in* $L^2(I\!R^n)$ *and that (2) and (3) are satisfied. If* $(\Lambda - A)$ *is a nonsingular M-matrix then the Maximum Principle holds for (1).*

Proof of Theorem 2: *Assume that for all* $1 \leq i \leq N$, $f_i \geq 0$. For any solution $(u_1,..., u_N)$ of (1), we multiply (1.*i*) by u_i^- := max $(0, -u_i)$ and integrate. We have:

$$\int (\nabla u_i)\left(\nabla u_i^-\right) + q_i\, u_i\, u_i^- = -\int \left[\left|\nabla u_i^-\right|^2 + q_i\left|u_i^-\right|^2\right] =$$

$$a_{ii} \int u_i\, u_i^- + \sum_{j\neq i} a_{ij} \int \left(u_j^+ - u_j^-\right) u_i^- + \int f_i\, u_i^-,$$

so that:

$$\int \left[\left|\nabla u_i^-\right|^2 + q_i\left|u_i^-\right|^2\right] =$$

$$a_{ii} \int \left|u_i^-\right|^2 - \sum_{j\neq i} a_{ij} \int u_j^+\, u_i^- + \sum_{j\neq i} a_{ij} \int u_j^-\, u_i^- - \int f_i\, u_i^-$$

$$\leq a_{ii} \int \left|u_i^-\right|^2 + \sum_{j\neq i} a_{ij} \int u_j^-\, u_i^-$$

$$\leq a_{ii} \int \left|u_i^-\right|^2 + \sum_{j\neq i} a_{ij} \left[\int \left|u_j^-\right|^2 \int \left|u_i^-\right|^2\right]^{1/2}.$$

It follows from the variational characterization of $\lambda(q_i)$ (5), that

$$\left\{\left(\lambda(q_i) - a_{ii}\right)\left[\int \left|u_i^-\right|^2\right]^{1/2} \leq \sum_{j\neq i} a_{ij} \left[\left|u_j^-\right|^2\right]^{1/2}.\right.$$

or equivalently

$$(\Lambda - A)X \leq 0 \text{ with } X = \begin{pmatrix} \left[\int \left|u_1^-\right|^2\right]^{1/2} \\ \left[\int \left|u_N^-\right|^2\right]^{1/2} \end{pmatrix} \geq 0$$

Since $(\Lambda - A)$ is a nonsingular M-matrix, by *(P2)* we have

$X = (\Lambda - A)^{-1}\, [(\Lambda - A)X] \leq 0$

It follows that $u_i \equiv 0$, for all $1 \leq i \leq N$.

We are now concerned with the proof of existence of positive solutions. We use Lax-Milgram theorem for proving:

Theorem 3. *Assume that* (2) *and* (3) *are satisfied. If* $(\Lambda - A)$ *is a nonsingular M-matrix, and if* $f_i \geq 0$ *for all* $1 \leq i \leq N$, *then System* (1) *has a unique solution which is nonnegative, Conversely, if, in addition,* (3') *is satisfied and if System* (1) *has a unique solution which is nonnegative for all* $F \geq 0$, *then* (6) *holds.*

Proof of Theorem 3: Note that, if $(\Lambda - A)$ is a nonsingular M-matrix, by *(P2)* $(\Lambda - A)^t$ is a nonsingular M-matrix too. So by *(P3)* there exists a positive diagonal matrix D such that $(\Lambda - A)^t D + D(\Lambda - A)$ is positive definite.

$$D = \begin{pmatrix} d_1 & 0 & \cdot & \cdot & 0 \\ 0 & \cdot & 0 & \cdot & \cdot \\ \cdot & 0 & \cdot & 0 & \cdot \\ \cdot & \cdot & 0 & \cdot & 0 \\ 0 & \cdot & \cdot & 0 & d_N \end{pmatrix} \text{ and } \forall 1 \leq i \leq N \quad d_i > 0$$

We first rewrite (1.*i*) and (1.2) as:

(7)

$$\left\{(7.i)\; d_i\left(L_{q_i} + m\right)u_i = d_i(a_{ii} + m)u_i + \sum_{j\neq i} d_i\, a_{ij}\, u_j + d_i f_i \text{ in } I\!R^n\right.$$

where m is large enough so that $a_{ii} + m > 0$, $\forall 1 \leq i \leq N$.

We define a map l from $[(V_{q_1}(I\!R^n))\times...\times(V_{q_N}(I\!R^n))]\times[(V_{q_1}(I\!R^n))\times...\times(V_{q_N}(I\!R^n))$ in $I\!R$, by

$$l(U, V) = \sum_{i=1}^{i=N} d_i \int_{I\!R^n} \left(\nabla u_i \cdot \nabla v_i + (m+q_i)u_i\, v_i\right)$$

$$-\sum_{i=1}^{i=N} d_i(a_{ii} + m) \int_{I\!R_n} u_i\, v_i - \sum_{j\neq i} d_i\, a_{ij} \int_{I\!R_n} u_j\, v_i$$

It is easy to check that l is a continuous bilinear form. Moreover we prove that if $\Lambda - A$ is a nonsingular M-matrix, then l is coercive. First, by Cauchy-Schwarz inequality and by (5), we have:

$l(U, U) =$

$$= \sum_{i=1}^{i=N} d_i \int[|\nabla u_i|^2 + (m + q_i)|u_i|^2] - \sum_{i=1}^{i=N} d_i(a_{ii} + m)\int u_i^2 - \sum_{j\neq i} d_i\, a_{ij} \int u_j\, u_i$$

$$\geq \sum_{i=1}^{i=N} d_i\left(1 - \frac{a_{ii} + m}{\lambda(q_i) + m}\right)\int[|\nabla u_i|^2 + (m + q_i)u_i^2] - \sum_{j\neq i} d_i\, a_{ij}\left(\int|u_j|^2 \int|u_i|^2\right)^{1/2}$$

$$\geq \sum_{i=1}^{i=N} d_i(\lambda(q_i) - a_{ii})\frac{\|u\|_{q_i}^2 + m}{\lambda(q_i) + m} - \sum_{j\neq i} d_i\, a_{ij}\frac{\|u_j\|_{q_j+m}\|u_i\|_{q_i+m}}{[(\lambda(q_j) + m)(\lambda(q_i) + m)]^{1/2}}$$

where $\|u\|_{q+m}^2 := \int(|\nabla u|^2 + (m+q)|u|^2)$.

Setting $X = (x_i)_{1\leq i\leq N}$ with $x_i = \dfrac{\|u_i\|_{q_i+m}}{\sqrt{\lambda(q_i)+m}}$, we get

$$l(U, U) \geq X^t D(\Lambda - A) X$$

Since $X^t D(\Lambda - A) X = \frac{1}{2} X^t ((\Lambda - A))^t D + D(\Lambda - A)) X$ and $(\Lambda - A)^t D + D(\Lambda - A)$ is positive definite, l is coercive and this implies that (7) has a unique solution for any f_i, $1 \leq i \leq N$ in $L^2(IR^n)$. This solution is positive whenever, $f_i \geq 0$, for all $1 \leq i \leq N$, by the Maximum Principle.

4. SEMINILEAR SYSTEMS

Now we prove existence of solutions when f_i depends on x, u_i and u_j by using sub and super solutions.

Hypotheses: Assume that for all $1 \leq i \leq N$ there exist $\vartheta_i \in L^2(IR^n)$, $\vartheta_i > 0$, $\exists M > 0$ such that

$(\Lambda - A - MI)$ is a nonsingular M-matrix (8)

$\{0 \leq f_i(x,(u_1,..., u_N)) \leq M u_i + \vartheta_i$ for $u_1 \geq 0,..., u_N \geq 0$ (9)

f_i are lipschitz w.r.t. $(u_1,..., u_N)$ (10)

Theorem 4. *If (2), (3) and (8) to (10) are satisfied, then (1) has at least a positive solution.*

Proof of Theorem 4: We use the method of sub-super solutions.

Construction of a super-solutions: consider

$$\{(L_{q_i} + m)u = (m + a_{ii})u_i + \sum_{j\neq i} a_{ij}\, u_j + Mu_i + \vartheta_i \text{ in } IR^n \tag{11}$$

It follows from the section 2 and from (8) that (11) admits a positive solution $U^0 := (u_1^0,..., u_N^0)$. By (9), $(u_1^0,..., u_N^0)$ is a super solution, i.e.:

$$\left\{L_{q_i} u_i^0 := (-\Delta + q_i)u_i^0 \geq a_{ii}\, u_i^0 + \sum_{j\neq i} a_{ij}\, u_j^0 + f_i(x, (u_1^0,.., u_N^0)) \text{ in } IR^n\right. \tag{12}$$

Obviously by (9), $U_0 := (0,..., 0)$ is a subsolution, i.e.:

$$\left\{L_{q_i} u_{i0} := (-\Delta + q_i)u_{i0} \leq a_{ii}\, u_{i0} + \sum_{j\neq i} a_{ij}\, u_{j0} + f_i(x, (u_{10},.., u_{N0})) \text{ in } IR^n\right. \tag{13}$$

Definition of a compact operator: choose $m > 0$ so that for all $1 \leq i \leq N$, $m + a_{ii} > 0$.

Then we define $T : U \in [U_0; U^0] \rightarrow W = \begin{bmatrix} w_1 \\ \cdot \\ w_N \end{bmatrix} := TU \in (L^2(IR^n))^N$ by:

$$\{(L_{q_i} + m)w_i = (m + a_{ii})u_i + \sum_{i\neq j} a_{ij}\, u_j + f_i(x,(u_1,..., u_N)) \text{ in } IR^n \tag{14}$$

which can also be written:

$$\left\{w_i = (L_{q_i} + m)^{-1}\left[(m + a_{ii})u_i + \sum_{j\neq i} a_{ij}\, u_j + f_i(x,(u_1,..., +u_N))\right]\right.$$

It follows from (9) and from the scalar case that TU is well defined; we prove now that it is continuous and hence by (10) and the compactness of the operators $(L + m)^{-1}$, T is compact.

Let $(U_k)_{k\in IN}$, be convergent sequences in $(L^2(IR^n))^N$: $\forall 1 \leq i \leq N$, $u_{ik} \rightarrow u_i$. Set $T(U_k) = (W_k)$ and $T(U) = (W)$.

$$(L_{q_i} + m)(w_{ik} - w_i) = (m + a_{ii})(u_{ik} - u_i) + \sum_{j\neq i} a_{ij}(u_{uj} - u_j) + f_i(x,(u_{1k},.., u_{Nk})) - f_i(x,(u_1,.., u_N)).$$

We multiply each equation by $(w_{ik} - w_i)$ and integrate over IR^n; we get:

$$\|w_{ik} - w_i\|_{q_{i+m}}^2 \leq \|w_{ik} - w_i\|_{L^2}[(m + a_{ii})\|u_{ik} - u_i\|_{L^2} + \sum_{j\neq i} a_{ij}\|u_{jk} - u_j\|_{L^2} + \|f_i(x,(u_{1k},.., u_{Nk})) - f_i(x,(u_1,.., u_N))\|_{L^2}].$$

By (10):

$$\|w_{ik} - w_i\|_{L^2} \leq C\left[\sum_{i=1}^{i=N}\|u_{ik} - u\|_{L^2}\right] \rightarrow 0 \text{ as } k \rightarrow +\infty.$$

$[U_0; U^0]$ is an invariant set for $T : T([U_0; U^0]) \subseteq [U_0; U^0]$. It follows from (9) that $1 \leq i \leq N$, $f_i(x,(u_1,..., u_N)) \geq 0$ for $u_1 \geq, 0,..., u_N \geq 0$. Therefore, for $U \geq U_0$ we have

$$\left[(m+a_{ii})u_i + \sum_{j\neq i} a_{ij}\, u_j + f_i\big(x,(u_1,\ldots,u_N)\big)\right] \geq 0$$

Then applying the Maximum Principle for the scalar case we obtain that $TU \geq U_0 = (0,\ldots,0)$.

We show now that $0 \leq U \leq U^0$ implies that $TU \leq U^0$. We substract (14) from (11):

$$\left(L_{q_i}+m\right)\left(u_i^0-w_i\right) = (m+a_{ii})\left(u_i^0-u_i\right) + \sum_{j\neq i} a_{ij}\left(u_j^0-u_j\right)$$

$$+Mu_i^0 + \vartheta_i - f_i\big(x,(u_1,\ldots,u_n)\big).$$

Of course $(m + a_{ii})(u_i^0 - u_i) + \sum_{j\neq i} a_{ij}(u_j^0 - u_j) \geq 0$. By (9), $0 \leq f_i(x,(u_1,\ldots,u_N)) \leq M\, u_i + \vartheta_i$; therefore $(L_{qi} + m)(u_i^0 - w_i) \geq 0$. By the scalar case, $u_i^0 - w_i \geq 0$. Hence $[U_0; U^0]$ is invariant by T.

Since $[U_0; U^0]$ is convex, bounded and closed in $(L^2(I\!R^n))^N$, we can apply Schauder Fixed Point Theorem which gives the existence of at least one solution in $[U_0; U^0]$.

To prove uniqueness, we assume there exists a concave function

$$(u_1,\ldots,u_N) \rightarrow H(x, u_1,\ldots,u_N)$$

such that:

$$\left\{ f_i(x, u_1,\ldots,u_N) = \frac{\partial H}{\partial u_i}(x, u_1,\ldots,u_N) \right. \tag{15}$$

Theorem 5. *If (2), (3), (8) to (10) and (15) are satisfied, then (1) has a unique positive solution.*

Proof of Theorem 5: By (P1), if $(\Lambda - A - M\,I)$ in a nonsingular M-matrix, $(\Lambda - A)$ is a nonsingular M-matrix too. So, as in the proof of the theorem 3, we consider the positive diagonal matrix D such that $(\Lambda - A)^t D + D(\Lambda - A)$ is positive definite. Assume that $U = (u_1,\ldots,u_N)$ and $V = (v_1,\ldots,v_N)$ are solutions of (1); set $W := U - V$. Then:

$$\left\{(-\Delta+q_i)w_i = a_{ii}\, w_i + \sum_{j\neq i} a_{ij}\, w_j + \left[\frac{\partial H}{\partial u_i}(u_1,\ldots,u_N) - \frac{\partial H}{\partial v_i}(v_1,\ldots,v_N)\right] \text{in } I\!R^n\right.$$

Multiplying each equation by $d_i\, w_i$ and adding the N equations, we obtain

$$\sum_{i=1}^{i=N} d_i \int\left(|\nabla w_i|^2 + q_i\, w_i^2\right) = \sum_{i=1}^{i=N} d_i\, a_{ii} \int w_i^2 + \sum_{j\neq i} d_i\, a_{ij} \int w_i\, w_j$$

$$+ \sum_{i=1}^{i=N} d_i \int\left[\frac{\partial H}{\partial u_i}(u_1,\ldots,u_N) - \frac{\partial H}{\partial v_i}(v_1,\ldots,v_N)\right](w_i)$$

By (15), we have:

$$\sum_{i=1}^{i=N} d_i\big(\lambda(q_i) - a_{ii}\big)\int w_i^2 \leq \sum_{j\neq i} d_i\, a_{ij} \int w_i\, w_j \leq \sum_{j\neq i} d_i\, a_{ij} \int w_i^2 \int w_j^2 \tag{16}$$

Setting $X = (x_i)_{1\leq i\leq N}$ with $x_i = \int_{I\!R^n} w_i^2$, we get $X^t D(\Lambda - A)\, X \leq 0$, or equivalently

$$X^t\, ((\Lambda - A)^t D + D(\Lambda - A))\, X \leq 0.$$

Since $(\Lambda - A)^t D + D(\Lambda - A)$ is positive definite, it follows $X = 0$ $(w_i = 0\ \forall 1 \leq i \leq N)$.

This work was supported by European network-contract ERBCHRXCT 930409. the authors thank M. and T. Hoffman-Ostenhoff for remarks on Schrödinger operators.

BIBLIOGRAFÍA

1. Cosner, C., Hernández, J. & Mitidieri, E. *Maximum principles and application to reaction-diffusion systems* Birkhauser, Boston, (in preparation).
2. Figueiredo, D.G. de & Mitidieri, E. (1986) *A maximum principle for an elliptic system and applications to semilinear problems. S.I.A.M. J. Math. Anal.*, **17**, 836-849.
3. Figueiredo, D.G. de & Mitidieri, E. (1990) *Maximum principle for cooperative elliptic systems Comptes Rendus Acad. Sc. Paris*, **310**, 49-52.
4. Figueiredo, D.G. de & Mitidieri, E. (1988) *Maximum principle for linerar elliptic systems. Quaterno Matematico* **177**, Dip. Sc. Mat., Univ. Trieste.
5. Fleckinger, J., Hernández, J. & Thélin, F. de. (1992) *Principe du maximum pour un sytème elliptique non linéaire. Comptes Rendus Acad. Sc. Paris*, **314**, Ser. I, 665-668.
6. Fleckinger, J., Hernández, J. & Thélin, F. de. (1995) *On maximum principles and existence of positive solutions for some cooperative elliptic systems. Diff and Int Eq.*, **8**, (1) 69-85.
7. Fleckinger, J., Hernández, J. & Thélin, F. de. (1992) *A maximum principle for linear cooperative elliptic systems. Mathematics in Science and Engineering.* Edited by: William F. Ames, Vol. **192.** *Differential Equations with Applications to Mathematical Physics*, pp. 79-86.
8. Hernández, J. (1981) Some existence and stability results for solutions of reaction-diffusion systems with nonlinear boundary conditions. *Eds.:* Mottoni, P. de & Salvadori, L. In: *Non linear differential equations : Invariance, Stability and Bifurcation.* New York, Acad. Press, pp. 161-173.
9. Hernández, J. (1990) *Maximum principles and decoupling for positive solutions of reaction-diffusion systems with.* In: *Reaction-Diffusion Equations.* Eds.: Brown, K.J. & Lacey, A.A. Clarendon Press, Oxford, pp. 199-224.
10. Sweers, G. (1992) Strong positivity in C(IRn) for elliptic systems. *Math. Z.*, **209**, 251-271.
11. Boccardo, L., Fleckinger, J. & Thélin, F. de. (1994) Existence of solutions for some nonlinear cooperative systems and some applications. *Diff and Int Eq.*, **7**, (3), 689-698.

12. López-Gómez, J. & Molina, M. (1994) The maximum principle for cooperative weakly coupled elliptic systems and some applications. *Diff and Int Eq.*, **7,** (2).

13. Caristi, G. & Mitidieri, E. (1990) Maximum principles for a class of non coperative elliptic systems. *Delft Progress Rep.*, **14,** 33-56.

14. Clément, P. & Egberst, P. (1990) On the sum of two maximal monotone operators. *Diff. and Int. Eqs.* **3,** 1127-1138.

15. Abakhti-Mchachti, A. (1993) Systèmes semilinéaires d'équations de Schrödinger. *Thèse numéro* **1338,** Université Toulouse 3.

16. Abakhti-Mchachti, A. & Fleckinger-Pellé, J. Existence of solutions for non cooperative semilinear elliptic systems defined on an unbounded domain. *Pitman Research Notes in Maths,* **266,** 92-106.

17. Fleckinger, J. & Serag, H. (1995) Semilinear cooperative systems on ($\mathbb{R}^n$). *Rendiconti di Matematica,* serie VII, Volume **15,** Roma, 89-108.

18. Fleckinger, J. (1981) Estimate of number of eigenvalues for an operator of Schrödinger type. *Proceedings of the Royal Society of Edinburgh,* **89A,** 355-361.

19. Reed, M. & Simon, B. (1979) Methods of modern mathematical physics. Volume IV, *Analysis of Operators,* Academic Press, New York.

20. Bermann, A. & Plemmons, R.J. (1979) Nonnegative Matrices in the Mathematical Sciences. Academic Press, New York.