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Matemáticas

A SEQUENTIAL ANALOGUE OF THE GROTHENDIECK-PTÁK TOPOLOGY

SUNDAY GLUYEMI

(topology/metrizable space/angelic spaces)

Department of Pure and Applied Mathematics. Ladoke Akintola University of Techonology. P.M.B. 4000 Ogbomoso, Nigeria.

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ABSTRACT

A topology, coarser than that of Grothendieck-Pták, is introduced and its behaviour in various classes of locally convex spaces is studied.

RESUMEN

Se introduce una topología menos fina que la de Grothendieck-Pták y se estudia su comportamiento en varias clases de espacios localmente convexos.

INTRODUCTION

In Grothendieck's completeness theorem and in Pták's closed graph and open mapping theorems a certain topology on the topological dual of a locally convex space plays an important role.

Let (X, T) be a locally convex Hausdorff topological vector space. Following the terminology of A. Wilansky [8, Definition 12-2-1, p. 184] a subset A of the continuous dual *X'* of *(X,T)* is almost weak* closed (or briefly: aw*-closed) if $A \cap E$ is σ(X', X)-compact for every T-equicontinuous, σ(X', X)-closed subset E of *X'*. A linear form f on X' is aw*-continuous if its kernel Ker f=$\{x' \in X' : \langle f, x' \rangle = 0\}$ is aw*-closed [8, Definition 12-2-8, p. 185]. One form of Grothendieck's completeness theorem [8, Corollary 12-2-16, p. 187] states that X is complete if and only if every aw*-continuous linear form on X' is σ(X', X)-continuous, i.e. belongs to the image $\tilde{X}$ of X in the algebraic dual $X'^{\#}$ of X' (briefly: «belongs to X»).

Wilansky deduces Grothendieck's theorems from a lemma [8, Lemma 12-2-9, p. 185] which contains various characterizations of aw*-continuity. In the present note we introduce two concepts weaker than aw*-closedness and aw*-continuity, respectively, and prove for them an Analogue of Wilansky's lemma.

Let f be a linear form on X'. S.F. Bellenot and E.G. Ostling [1, p. 26] introduced the following condition:

$$(*) \begin{cases} \text{for every } T\text{-equicontinuous sequence } (x'_n) \text{ in } X' \text{ that} \\ \text{converges to zero for } \sigma(X', X) \text{ one has } f(x'_n) \to 0. \end{cases}$$

They say that (X, T) has property WC (or that (X, T) is a WC space) if (*) implies that f belongs to $\tilde{X}$ (or simply: to X). Using the Analogue we shall prove that a large class of locally convex spaces are complete if and only if they have property WC.

1. THE ANALOGUE

In what follows (X, T) will be a Hausdorff locally convex space, X' its continuous (topological) dual, and σ(X', X) the weak* topology on X'.

Definition 1. A subset A of X' is sequentially almost weak* closed (briefly: saw*-closed) if $A \cap E$ is $\sigma(X', X)|_E$-closed for every σ*(X', X)*-metrizable, T-equicontinuous subset E of X'.

A linear form f on X' is saw*-continuous if ker f is saw*-closed.

By [8, Lemma 12-2-3, p. 184] every aw*-closed set is saw*-closed and so every aw*-continuous linear form is saw*-continuous.

Recall that a Hausdorff topological space S is said to be angelic [6, p. 30], or to have countably determined compactness, if for every relatively countably compact subset B of S the following two conditions are satisfied:

(i) B is relatively compact,

(ii) for each x belonging to the closure of B there exists a sequence of points of B that converges to x.

Every metrizable space is angelic. Any subspace of an angelic space is angelic. A subset of a Hausdorff topological space is said to be angelic if it is an angelic space with respect to the iduced topology.

Definition 2. A subset A of X' is angelically almost weak* closed (briefly: aaw*-closed) if $A \cap E$ is $\sigma(X', X)|_E$-closed for every $\sigma(X', X)$-angelic, T-equicontinuous subset E of X'.

A linear form f on X' is aaw*-continuous if Ker f is aaw*-closed.

Clearly every aaw*-closed subset of X' is saw*-closed, hence every aaw*-continuous linear form on X' is saw*-continuous.

Theorem 1 (The Analogue). Let f be a linear form on the continuous dual X' of the Hausdorff locally convex space (X, T). The following are equivalent:

a) f satisfies condition (*).
b) f is aaw*-continuous.
c) f is saw*-continuous.
d) The restriction of f to E is $\sigma(X', X)$-continuous for every $\sigma(X', X)$-metrizable, T-equicontinuous subset E of X'.

Proof. a) $\Rightarrow$ b): Assume that E is a $\sigma(X', X)$-angelic, T-equicontinuous subset of X', and let $y' \in E$ be a point in the $\sigma(X', X)$-closure of $\ker f \cap E$ in E. There exists a sequence (y'_n) of points of $\ker f \cap E$ which converges to y' for the topology $\sigma(X', X)$. The sequence $(y'_n - y')$ is equicontinuous and converges to zero for $\sigma(X', X)$ [8, Problem 4-1-1, p. 39]. By hypothesis $f(y'_n - y') \to 0$ and so $f(y'_n) \to f(y')$. But $y'_n \in \ker f$, i.e. $f(y'_n) = 0$ and so $f(y') = 0$. Thus $y' \in \ker f$ and therefore $\ker f \cap E$ is $\sigma(X', X)$-closed in E.

b) $\Rightarrow$ c): This has been observed above.

To prove the implication c) $\Rightarrow$ a) we need the following simple

Lemma. If (x'_δ) is an equicontinuous net in the dual X' of a locally convex space (X, T) and $\varepsilon_\delta \geqslant \varepsilon > 0$, then $(\varepsilon_\delta^{-1} x'_\delta)$ is also equicontinuous.

Proof. Given $\eta > 0$, there exists a neighbourhood V of 0 in X such that $|\langle x'_\delta, x\rangle| \leqslant \varepsilon\eta$ for $x \in V$ and all δ. But then $|\langle \varepsilon_\delta^{-1} x'_\delta, x\rangle| = \varepsilon_\delta^{-1} |\langle x'_\delta, x\rangle| \leqslant \varepsilon^{-1} \varepsilon\eta = \eta$ for $x \in V$ and all δ. ■

c) $\Rightarrow$ a): The proof given here is patterned after the proof of (a) $\Rightarrow$ (b) in [8, Lemma 12-2-9, p. 185]. We assume that $f \in X'^{\#}$ does not satisfy a) and deduce that ker f is not saw*-closed. There exists an equicontinuous sequence (x'_n) in X' which converges to 0 for $\sigma(X', X)$ but $f(x'_n)$ does not converge to 0. Hence there exists $\varepsilon > 0$ and a subsequence (x'_{n_k}) of $x'_n)$ such that $|f(x'_{n_k})| \geqslant \varepsilon$ for all k. Since $f \neq 0$, we can choose $u' \in X'$ such that $f(u') = 1$. Let

$$v'_k = u' - x'_{n_k}/f(x'_{n_k}).$$

By the Lemma the set $E = \{u'\} \cup \{v'_k\}$ is equicontinuous and by [7, Corollary on p. 402] it is metrizable for $\sigma(X', X)$.

However, $\ker f \cap E$ is not $\sigma(X', X)$-closed in E. Indeed, $v'_k \in \ker f \cap E$ and v'_k converges to u' for $\sigma(X', X)$ since

$$|\langle x'_{n_k}, x\rangle / f(x'_{n_k})| \leqslant \varepsilon^{-1} |\langle x'_{n_k}, x\rangle| \to 0$$

as $k \to \infty$ for every $x \in X$. But $u' \notin \ker f$.

a) $\Rightarrow$ d): Let E be an equicontinuous and $\sigma(X', X)$-metrizable subset of X'. Let (x'_n) be a sequence in E which converges for the topology $\sigma(X', X)|_E$ to an element $x' \in E$. Then (x'_n) converges to x' for $\sigma(X', X)$ and so $x'_n - x' \to 0$ for $\sigma(X', X)$. Also $\{x'_n\}$ is equicontinuous, hence so is $\{x'_n - x'\}$. If f satisfies (*), then $f(x'_n - x') \to 0$, i.e. $f(x'_n) \to f(x')$. Since E is $\sigma(X', X)$-metrizable, this proves that f is $\sigma(X', X)$-continuous on E.

d) $\Rightarrow$ a): By [7, Corollary on p. 402] and [8, Problem 9-1-5, p. 131], if f satisfies d) then it satisfies a).

2. APPLICATIONS

Let us denote by AW*(X) the space of aw*-continuous linear forms on X' and by SAW*(X) the space of saw*-continuous (i.e. aaw*-continuous) linear forms on X'. We always have $AW^*(X) \subset SAW^*(X)$. Grothendieck's completeness theorem states that (X, T) is complete if and only if $AW^*(X) = \tilde{X}$. By Theorem 1 the space X has property WC if and only if $SAW^*(X) = \tilde{X}$.

Theorem 2. (i) If every equicontinuous subset of X' is $\sigma(X', X)$-angelic, the $AW^*(X) = SAW^*(X)$.

(ii) Let X be such that every equicontinuous subset of X' is angelic. Then X is complete if and only if it is a WC space.

Proof. (i) If $f \in SAW^*(X)$, then $\ker f \cap E$ is $\sigma(X', X)$-closed in E for every angelic equicontinuous set $E \subset X'$. But in X' every equicontinuous set is angelic, so $f \in AW^*(X)$.

(ii) We have $AW^*(X) = SAW^*(X)$. The space X has property WC if and only if $SAW^*(X) = \tilde{X}$, and X is complete if and only if $AW^*(X) = \tilde{X}$. ■

Examples. (1) In the dual of a separable locally convex space X every equicontinuous set is $\sigma(X', X)$-metrizable [8, Theorem 9-5-3, p. 143]. The fact that a separable locally convex space is complete if and only if it is a WC space also follows from equivalence (2) $\Leftrightarrow$ (6) in Theorem 3.1 of [1, p. 27].

(2) Buchwalter [3, p. 13] says that a Hausdorff locally convex space (X, T) is semi-weak if every equicontinuous subset of X' is contained in the balanced, convex, $\sigma(X', X)$-closed hull of an equicontinuous sequence converging to 0 for $\sigma(X', X)$. He proves [3, Proposition (1.2), p. 14] that every equicontinuous subset of the dual of a semi-weak space is $\sigma(X', X)$-metrizable. The fact that a semi-weak space is com-

plete if and only if it is a WC space also follows from equivalence (1) $\Leftrightarrow$ (6) in Theorem 3.1 of [1, p. 27].

The following three examples were kindly pointed out to us by Professor Manuel Valdivia:

(3) In the dual of a reflexive Fréchet space every equicontinuous subset is $\sigma(X', X)$-angelic. Indeed, B. Cascales and J. Orihuela proved that any *(DF)*-space Y is angelic with respect to $\sigma(Y, Y')$ [4, pp. 370 and 374]. If X is reflexive Fréchet space, then X' equipped with the Mackey topology is a *(DF)*-space, and the topologies $\sigma(X', X'')$ and $\sigma(X', X)$ coincide.

(4) A Hausdorff locally convex space X is said to be weakly compactly generated if theres exists a balanced, convex, $\sigma(X, X')$-compact subset in X whose span is dense in X. In the dual of a weakly compactly generated space X every equicontinuous subset is $\sigma(X', X)$-angelic. Indeed, let A be a balanced, convex, $\sigma(X, X')$-compact subset of X whose span is dense in X. Equip X' with the gauge q of the polar A^0 and denote it X'_q. It is a Banach space. Every $\sigma(X', X)$-compact subset M of X' is $\sigma(X'_q, (X'_q)')$-compact and angelic with respect to the latter topology. If $B \subset M$ and u belongs to the $\sigma(X'_q, (X'_q)')$-closure of B, then there exists a sequence in B which converges to u with respect to $\sigma(X'_q, (X'_q)')$, hence also with repect to $\sigma(X', X)$ since the two topologies coincide on M.

(5) Let Z be a separable Banach space with does not contain a copy of l^1, and let $X = Z'$ be its strong (normed) dual. Then in the dual X' of X every equicontinuous subset is $\sigma(X', X)$-angelic. Indeed, let Z be a separable Banach space which does not contain l^1, and let H be the unit ball of $Z' = X$. On H consider the topology $\sigma(Z', Z)$ for which it is compact and metrizable. Let B_1 (H) be the space of all functions of the first Baire class defined on H, equipped with the topology of pointwise convergence. For any u in Z'' there exists by a theorem of H.P. Rosenthal [5, p. 215] a sequence (u_n) in Z which converges to u with respect to the topology $\sigma(Z'', Z')$. The restrictions of the u_n to H are continuous, hence $u \in B_1$ (H). Thus $Z'' = X'$ with the topology $\sigma(Z'', Z')$ is a subspace of B_1 (H). J. Bourgain, D.H. Fremlin and M. Talagrand proved that B_1 (H) is angelic [2, Theorem 3F, pp. 845 and 860], hence X' is angelic with respect to $\sigma(X', X)$.

3. µlc SPACES

Denote by WSC* (X) the space of sequentially $\sigma(X', X)$-continuous linear forms on X. Clearly $\tilde{X} \subset$ WSC* (X) $\subset$ SAW* (X). Wilansky [9] calls (X, T) a µlc space if WSC* (X) = $\tilde{X}$. Clearly, if X is a WC space then it is a µlc-space.

Theorem 3. Let X be such that every equicontinuous subset of X' is $\sigma(X', X)$-angelic. If X is complete, then it is a µlc space.

Proof. It follows from Theorem 2 (ii) that X is a WC space, i.e. SAW*(X) = $\tilde{X}$. A fortiori WSC*(X) = $\tilde{X}$.

Theorem 4. A sequentially barrelled sapce is a µlc space if and only if it has property WC.

Proof. We only have to prove that a sequentially barrelled µlc space has property WC. Now µlc means that $\tilde{X}$ = WSC* (X). On the other hand, if X is sequentially barrelled, then every sequence in X' which tends to 0 for $\sigma(X', X)$ is equicontinuous, hence WSC*(X) = SAW*(X). ■

Remark. Fréchet-Montel spaces are µlc spaces. Indeed, they are barrelled, hence sequentially barrelled, complete by definition, and separable by Dieudonné's theorem.

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