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# FRÉCHET VALUED CO-ECHELON SPACES

**(co-echelon spaces/spaces of linear/continuons functions)**

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## ABSTRACT

Let $A$ be a Köthe matrix on the natural numbers such that the Köthe echelon space $\lambda^1(A)$ is distinguished. Let $E$ be a Fréchet space. The Köthe co-echelon space $K_\infty(E)$ associated with $A$ and $E$ is barrelled if and only if the pair $(\lambda^1(A),E)$ satisfies condition $(S_2)^*$ of Vogt. This solves an open problem of Bierstedt and Bonet.

## RESUMEN

Sea $A$ una matriz de Köthe tal que el espacio $\lambda^1(A)$ es distinguido. Sea $E$ un espacio de Fréchet. El espacio Köthe coescalonado $K_\infty(E)$ asociado a $A$ y $E$ es tonelado si y solo si el par $(\lambda^1(A),E)$ cumple la condición $(S_2)^*$ de Vogt. Se resuelve así un problema abierto de Bierstedt y Bonet.

In the last section of [1] a list of open problems dealing with weighted inductive limits of vector valued functions was presented. In this note we observe that the new developments on the vanishing of the functors $\mathrm{Ext}^1$ and $\mathrm{Proj}^1$ (see [3], [5], [6]) permit to solve problem (4) of [1], Section 4, in a complete and satisfactory way. Namely, it is possible to characterize (under a reasonable assumption) the Fréchet valued co-echelon spaces of infinite order whose associated projective hull is barrelled, by using the condition $(S_2)^*$ that appears in the study of the functor $\mathrm{Ext}^1(E,F)$ for pairs of Fréchet spaces $(E,F)$. This is established in Theorem 1. As an application, Fréchet valued co-echelon spaces that concide algebraically and topologically with their associated projective hull are studied, thus improving some results contained in the Ph. D. thesis of A. Galbis [7].

Let $E$ be a Fréchet space with an increasing fundamental sequence of seminorms $(\|\cdot\|_k)_{k\in\mathbf{N}}$. For every $k\in\mathbf{N}$ and for every $u\in E'$, we define $\|u\|_k^* := \sup\{|u(x)| : x\in E, \|x\|_k\le 1\}$. If $V=(v_n)_{n\in\mathbf{N}}$ is a decreasing sequence of strictly positive functions ("weights") on $\mathbf{N}$, then the associated system of weights is

$$\bar{V} := \left\{\bar{v}:\mathbf{N}\to\,]0,+\infty[\,;\forall n\in\mathbf{N}\sup_{i\in\mathbf{N}}\frac{\bar{v}(i)}{v_n(i)}<\infty\right\}.$$

As in [1], we consider $k_\infty(E):=ind_n l^\infty(v_n,E)$, $k_o(E):=ind_n c_o(v_n,E)$ and $K_\infty(E):=\mathrm{proj}_{\bar{v}\in\bar{V}}\, l^\infty(\bar{v},E)$, $K_0(E):=\mathrm{proj}_{\bar{v}\in\bar{V}}\, c_0(\bar{v},E)$. The space $K_\infty(E)$ (resp. $K_0(E)$) is called the projective hull associated with the vector valued co-echelon space $k_\infty(E)$ (resp. $k_0(E)$).

For every $n,i\in\mathbf{N}$ we set $a_n(i)=v_n(i)^{-1}$ and we let $\lambda^1(A)$ denote the Köthe echelon space of order one associated with the Köthe matrix $A=(a_n)_{n\in\mathbf{N}}$.

If $F$ and $G$ are locally convex spaces, $L_b(F,G)$ stands for the space of linear continuous maps $L(F,G)$ between $F$ and $G$, endowed with the topology of the uniform convergence on the bounded sets of $F$ and $LB(F,G)$ denotes the space of all bounded linear maps between $F$ and $G$.

In problem (4) of [1], Section 4, the authors ask if the barrelledness of $K_\infty(E)$ is equivalent to the property $(S_2)^*$ (see e.g.. [14]), at least when $\lambda^1(A)$ is distinguished. Our main result provides a positive answer.

We refer to [14, 15, 16, 17] for information about the functors $\mathrm{Proj}^1$, $\mathrm{Ext}^1$ and the property $(S_2)^*$.

**Theorem 1.** *Let $V$ be a decreasing sequence of weights on $\mathbf{N}$ such that $\lambda^1(A)$ is distinguished and let $E$ be a Fréchet space. The following conditions are equivalent:*

(i) *$K_\infty(E)$ is ultrabornological,*

(ii) *$K_\infty(E)$ is barrelled,*

(iii) *$K_0(E)$ is barrelled,*

(iv) *the pair $(\lambda^1(A),E)$ satisfies the property $(S_2)^*$:*

$$\forall N\ \exists n, M\ \forall L, m\ \exists l, S>0\ \forall i\in\mathbb{N}\ \forall u\in E':$$

$$a_m(i)\|u\|_M^* \le S\,max\left(a_n(i)\|u\|_N^*, a_L(i)\|u\|_l^*\right)$$

*Proof.* The implication (i) $\Rightarrow$ (ii) holds in general, while (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are proved in [1], 3.10. Assume that (iv) holds. In order to prove that (i) holds, it is enough to show that $L_b(\lambda^1(A),E)$ is ultrabornological (see [1], 2.2). To this end, consider $E$ as the reduced projective limit of Banach spaces $(E_n)_{n\in\mathbb{N}}$. By [6], 3.1 (see also [3]), $(S_2)^*$ implies that $\mathrm{Proj}^1\left(L(\lambda^1(A),E)\right)_{n\in\mathbb{N}} = \mathrm{Ext}^1(E,F)=0$. Since $L_b(\lambda^1(A),E)=\mathrm{proj}_n L_b(\lambda^1(A),E)$ and since the spaces $(\lambda^1(A)E_n)$ are complete (LB)-spaces (see [2], Cor. 7), the vanishing of the functor $\mathrm{Proj}^1$ for the projective sequence $\left(L(\lambda^1(A)E_n)\right)_{n\in\mathbb{N}}$ implies that $L_b(\lambda^1(A),E)$ is ultrabornological (see [16], 5.7). □

**Remark.** Theorem 4.9 in [15] provides examples of pairs $(\lambda^1(A),E)$ such that $K_\infty(E)$ is barrelled. For instance, if $\lambda^1(A)$ is a nuclear stable power series space of infinite type and $E$ is a nuclear Fréchet space, then $K_\infty(E)$ is barrelled if and only if $E$ has the property $(\Omega)$.

**Corollary1.** *Let $\lambda^1(A)$ be a distinguished Köthe echelon space and let E be a Fréchet space. Consider the following conditions:*

(i) *$L_b(\lambda^1(A),E)$ is barrelled,*

(ii) *the pair $(\lambda^1(A),E)$ satisfies $(S_2)^*$,*

(iii) *$\lambda^1(A)_b' \hat{\otimes}_\varepsilon E$ is barrelled,*

(iv) *$\lambda^1(A)_b' \hat{\otimes}_\pi E$ is barrelled.*

*Then* (i) *is equivalent to* (ii). *If $\lambda^1(A)$ is a Montel space, then* (iii) *is equivalent to* (ii); *and if moreover $\lambda^1(A)$ is nuclear, then* (iv) *is equivalent to* (ii).

*Proof.* The equivalence of (i) and (ii) follows immediately from the previous theorem. To get the other assertions, simply observe that if $\lambda_1(A)$ is a Montel space, then $L_b(\lambda^1(A),E)=\lambda^1(A)\hat{\otimes}_\varepsilon E$. The other assertion follows by well-known properties of tensor products and nuclear spaces (see e.g.[9]). □

**Remark.** The equivalence of (ii) and (iv) in the previous corollary should be compared with Grothendieck's investigations on the complete projective tensor product of a Fréchet space with a (DF)-space (see [8], Ch. II, par. 4). also see [10]).

According to [1], 2.3, the equality $k_\infty(E)=K_\infty(E)$ holds algebraically if and only if $L(\lambda^1(A),E)= LB(\lambda^1(A),E)$. If $\lambda^1(A)$ is a Schwartz space, then the equalities $k_\infty(E)=k_0(E)$ and $K_\infty(E)=K_0(E)$ hold algebraically and topologically. Therefore, by [1], 3.6, $k_\infty(E)$ is a topological subspace of $K_\infty(E)$ if and only if $L(E,\lambda^\infty(A))= LB(E,\lambda^\infty(A))$. This is in turn equivalent to the identity $L(E,\lambda^1(A))=LB(E,\lambda^1(A))$, if $\lambda^1(A)$ is nuclear. In [4] (see also [12], [13] there are examples of nuclear power series spaces $\Lambda_r(\alpha)$ and $\Lambda_r(\beta)$ with $r=1,\infty$, such that $L(\Lambda_r(\alpha),\Lambda_r(\beta))=LB(\Lambda_r(\alpha),\Lambda_r(\beta))$ and $L(\Lambda_r(\beta),\Lambda_r(\alpha))= LB(\Lambda_r(\beta),\Lambda_r(\alpha))$. According to the previous observations, if $\lambda^1(A)=\Lambda_r(\alpha)$ and $E=\Lambda_r(\beta)$, then the equality $k_\infty(E)=K_\infty(E)$ holds algebraically and topologically. Corollary 2 characterizes this behaviour.

**Corollary 2.** *Let V be a decreasing sequence of weights on $\mathbb{N}$ and E a Fréchet space such that the equality $K_\infty(E)=k_\infty(E)$ holds algebraically. T.f.a.e.:*

(i) *The identity $K_\infty(E)=k_\infty(E)$ holds topologically,*

(ii) *$\lambda^1(A)$ is distinguished and the pair $(\lambda^1(A),E)$ satisfies $(S_2)^*$.*

*Proof.* The implication (i) $\Rightarrow$ (ii) follows from [1], 3.10. Conversely, if $(S_2)^*$ holds and $\lambda^1(A)$ is distinguished, Theorem 1 implies that $K_\infty(E)$ is ultrabornological. Therefore, the equality $K_\infty(E)=k_\infty(E)$ holds topologically since the two spaces have the same bounded sets by [1], 3.1. □

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