Rev.R.Acad.Cien.Exact.Fis.Nat. (Esp)
Vol. 90, N.º 3, pp 141-147, 1996
Matemáticas

A TOPOLOGY OVER A SET OF SYSTEMS

(Linear inequalities systems/Hausdorff metrics/strict-strongly inconsistent systems/quasi-strongly inconsistent systems)

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Presentado por Pedro Jiménez Guerra, 10 de Abril de 1996

ABSTRACT

The systems of an arbitrary number of linear inequalities over a real locally convex space have been classified in three classes, namely: consistent, weakly inconsistent and strongly inconsistent, i.e. having ordinary solutions, weak solutions or not solutions respectively. In this paper, the third type is divided in two classes: strict-strongly and quasi-strongly inconsistent and is given a topology over a quotient space of the set of systems over finite- dimensional spaces, that yields a set of results in accordance with the theorem of classification of such systems, based upon their associated wedges, given in $[Go,2]$.

RESUMEN

Los sistemas con un número arbitrario de desigualdades lineales en un espacio real localmente convexo se clasifican en tres clases, a saber: consistentes, debilmente inconsistentes y fuertemente inconsistentes, i.e., teniendo soluciones ordinarias, soluciones débiles o no teniendo solución, respectivamente. En este artículo, el tercer tipo de sistemas, lo hemos dividido, a su vez, en dos: stricta-fuertemente inconsistentes y quasi-fuertemente inconsistentes y damos una topología sobre un espacio cociente del conjunto de los sistemas, en espacios de dimensión finita, que da lugar a una serie de resultados en consonancia con el teorema de clasificación de tales sistemas, basado en sus cuñas asociadas, dado en $[Go,2]$.

1. INTRODUCTION

If X is a real locally convex space and J is an arbitrary non-empty index set, a system σ is defined by a pair of mappings x: $J \to X$ and c: $J \to \mathbb{R}$ such that for each $j \in J$, $x(j) = x_j$, $c(j) = c_j$ and on try to find an element φ in the topological dual of X, such that

$$\varphi\left(x_j\right) \geq c_j \quad \text{for all } j \in J \qquad (1)$$

Formally, a system σ is written:

$$\sigma = \left\{\langle x_j,\varphi \rangle \geq c_j, j \in J\right\} \qquad (2)$$

In our case, since X will be a finite dimensional space, φ is a vector belonging to X and φ (x_j) as well as $\langle x_j,\varphi\rangle$, which appears in (2), will denote the usual inner product.

When there exists a vector φ that holds expression (1), σ is said to be consistent, if not, it is called inconsistent system. An inconsistent system is called weakly-inconsistent if it has asymptotic solution, i.e. there is a net $\{\varphi_d : d \in D\}$ in X such that,

$$\lim \inf_{d\in D} \langle x_j,\varphi_d\rangle \geq c_j \text{ for every } j \in J.$$

Finally, an inconsistent system that it is not weakly-inconsistent, is said to be strongly-inconsistent system.

All through this paper we shall frequently do allusion to Theorem of Classification $[Go,2]$, which statement is the following:

Theorem of classification. *In a real locally convex space X, a system*

$$\sigma = \left\{\langle x_j,\varphi \rangle \geq c_j, j \in J\right\}$$

is:

(a) Consistent if and only if (0,1) $\notin$ *cl* M_σ

(b) Weakly-inconsistent if and only if (0,1) $\in$ *cl* $M_\sigma \backslash M_\sigma$

(c) Strongly-inconsistent if and only if (0,1) $\in$ M_σ,

being

(0,1) the pair defined by the zero element of vector space X and the identity element of R,

$$M_\sigma = cone\ \{(x_j, c_j), j \in J\}$$

i.e. the set of non-negative linear combinations of pairs $(x_j, c_j) \in X \times R$, *cl* M_σ *the closure of* M_σ *in the topological product space* $X \times R$.

Given the finite dimensional space X and a certain index set J, we will denote by L_c the set of consistent systems, L_w the set of weakly-inconsistent systems and L_s the set of strongly-inconsistent systems. Now, a natural question arises:

Is it possible to find a topology over the family Θ of all systems defined in X with index set J such that, L_c and L_s are open sets and their boundaries are equal to L_w?

The main goal of this paper is to give a positive answer to this question, after some considerations about the set Θ itself and a refinement of the previous theorem of classification of systems. The results that we shall obtain can have applications in the theory of stability as well as to some optimization problems of linear functions in the general theory of continuous linear semi-infinite programming.

2. THE CHARACTERISTIC SECTOR OF A SYSTEM

All through this paper, X will be the euclidean space R^{k-1} where $k > 1$ and the index set $J = R$.

In contrast with $[Go,2]$, where the main tool is the wedge M_σ, in our analysis plays a crucial role the characteristic sector S_σ of a system, defined as

$$S_\sigma = \mathrm{cl}\ (M_\sigma \cap B),$$

being B the closed unit ball in R^k.

Then, the cone M_σ, as well as, the sector S_σ associated to a given system σ are subsets of R^k and in this space we shall consider the norm $\| \ \|$, defined as usual

$$\|x\| = (x_1^2 + x_2^2 + ... + x_k^2)^{1/2}.$$

In many reasonings all through this paper we shall frequently use, although without explicit mention, the following proposition:

Proposition 2.1. *Let* σ *be a system, its characteristic sector* S_σ *may also be defined as*

$$S_\sigma = (cl\ M_\sigma) \cap B.$$

Proof: Since $cl\ (M_\sigma \cap B) \subseteq (cl\ M_\sigma) \cap B$, we shall show the converse. If x is a vector belonging to $(cl\ M_\sigma) \cap B$, we may consider two cases:

Case 1. $\|x\| < 1$. Then, let ε be a positive number such that $\varepsilon < 1 - \|x\|$; the open ball $B(x, \varepsilon)$ meets M_σ at a point y. Thus $y \in B(x, \varepsilon) \cap M_\sigma \cap$ B, since $\|y\| \leq \|y - x\| + \|x\| < 1$.

Case 2. $\|x\| = 1$. For each $n = 1,2,...$, let y_n be a point belonging to M_σ such that $\|x - y_n\| < \frac{1}{n}$. Then, $\frac{n}{n+1} y_n \in M_\sigma \cap$ B since $\|y_n\| < \frac{n+1}{n}$.

On the other hand, the sequence $\left(\frac{n}{n+1} y_n\right)_n$ converges to x. Indeed,

$$\left\| \frac{n}{n+1} y_n - x \right\| = \frac{n}{n+1} \left\| y_n - x - \frac{1}{n} x \right\| \leq \|y_n - x\| + \frac{1}{n} < \frac{2}{n},$$

therefore $x \in cl\ (M_\sigma \cap B)$. ■

Definition 2.2. *Two systems* σ, ω *are said to be equivalent if and only if their characteristic sectors are equal.*

Obviously, the above definition is a equivalent relation and yields in the set Θ of all systems a quotient set Θ^*,on which we are going define a topology induced by a metrics.

3. A METRIC STRUCTURE ON Θ^*

Definition 3.1. *Given two classes* $\bar{\sigma}$, $\bar{\omega}$ *of* Θ^*, *we define*

$$d\ (\bar{\sigma}, \bar{\omega}) = d_H\ (S_\sigma, S_\omega),$$

being S_σ, S_ω *the characteristic sectors of their representative systems* σ, ω *and* $d_H\ (S_\sigma, S_\omega)$ *the Hausdorff distance between them.*

The foregoing definition not depend of their representative systems and, obviously, is a distance upper bounded by 1. For simplicity, we shall continue using the notation of their representative systems and we point out, to do easy understanding the reasonings that we shall explain,that the definition of Hausdorff metrics we shall use is the following:

$$d_H\ (S_\sigma, S_\omega) = inf\ \{\varepsilon > 0: S_\sigma \subset S_\omega + \varepsilon B, S_\omega \subset S_\sigma + \varepsilon B\}$$

Proposition 3.2. *The set* L_c *of consistent systems is open in* (Θ^*, d).

Proof: Let σ be an element of L_c, noticing the Theorem of Classification, the vector $u = (0,1) \notin cl\ M_\sigma$, so $u \notin S_\sigma$ and we can take then a positive number

$$\delta = \min\ \{\|u - z\|: z \in S_\sigma\},$$

such that the open ball $B(\sigma, \delta)$ lies in L_c. Indeed, if $\omega \in B(\sigma, \delta)$ is a weakly-inconsistent or strongly-inconsistent system, thus $u \in S_\omega$. But, on the other hand, for $\varepsilon = d(\omega, \sigma) < \delta$, we have that $S_\omega \subset S_\sigma + \varepsilon B$. Therefore there exist $z \in S_\sigma$ and $v \in B$ such that $u = z + \varepsilon v$. Then $\|u-z\| \leq \varepsilon$, but noticing definition of number δ, we would have that $\delta \leq \varepsilon$, being a contradiction since $\varepsilon < \delta$. ■

Proposition 3.3. *The set L_w of weakly-inconsistent systems is not open in (Θ^*, d).*

Proof: We shall consider two cases:

Case 1. $k = 2$, i.e., $X = R$. Let us define the system

$$\sigma = \{\langle x_j, \varphi \rangle \geq c_j, j \in J\},$$

being

$$x_j = j,\ c_j = j^{-1} \text{ if } j > 0$$

$$x_j = c_j = 0 \text{ if } j \leq 0.$$

The characteristic cone of this system $M_\sigma = \{(x,y): x > 0,\ y > 0\} \cup \{(0,0)\}$, therefore $(0,1) \in cl\ M_\sigma \setminus M_\sigma$. Taking into account the Theorem of Classification, σ is a weakly-inconsistent system.

For each $n = 1,2,\ldots$, we define a system

$$\sigma_n = \{\langle x_{n_j}, \varphi \rangle \geq c_{n_j}, j \in J\}$$

being,

$$x_{n_j} = j - \frac{1}{n},\ c_{n_j} = j^{-1} \text{ if } j > 0,$$

$$x_{n_j} = c_{n_j} = 0 \text{ if } j \leq 0,$$

Noticing that for $j < \frac{1}{n}$, the slope $m_j = \dfrac{1}{j(j-\frac{1}{n})}$ attains its maximum value when $j = \frac{1}{2n}$, the characteristic cone M_{σ_n} is the interior of the area limited by the rays $r_0 = \{(x,0): x \geq 0\}$, $r_n = \{(x, -4n^2x): x \leq 0\}$ and the origin (0,0), for each $n = 1,2,\ldots$

Therefore, $(0,1) \in M_{\sigma_n}$ and noticing again the Theorem of Classification, the systems σ_n are strongly-inconsistents.

On the other hand,it is easy to check that

$$\lim_n d(\sigma_n, \sigma) = \lim_n d_H(S_{\sigma_n}, S_\sigma) = \lim_n \frac{1}{\sqrt{1+16n^4}} = 0,$$

showing that L_w is not an open set.

Case 2. $k > 2$. Let us consider the system

$$\sigma = \{\langle x_j, \varphi \rangle \geq c_j, j \in J\}$$

defined as

$$x_j = (0,\ldots,0,\ j),\ c_j = j^{-1} \text{ if } j > 0,$$

$$x_j = (0,\ldots,0),\ c_j = 0 \text{ if } j \leq 0,$$

where the vectors $x_j \in R^{k-1}$.

We may easily check, in the same way that Case 1, this system is weakly-inconsistent. Now, the systems

$$\sigma_n = \{\langle x_{n_j}, \varphi \rangle \geq c_{n_j}, j \in J\},$$

defined as

$$x_{n_j} = (0,\ldots,\ 0,\ j - \frac{1}{n}),\ c_{n_j} = j^{-1} \text{ if } j > 0,$$

$$x_{n_j} = (0,\ldots,\ 0),\ c_{n_j} = 0 \text{ if } j \leq 0$$

are strongly-inconsistents and, again as Case 1, the sequence (σ_n) converges to σ. ■

Remark 3.4. The above proposition has been showed defining a weakly-inconsistent system σ and constructing a sequence $(\sigma_n)_n$ of strongly-inconsistent systems that converges to σ. In the same way, given the system σ, would be able define a sequence (σ_n) of consistent systems, likewise converging to σ, as follows:

Case 1. $k = 2$. For each $n = 1,2,\ldots$, let us define $A_n = \{1, 1/2, \ldots, 1/n, 2, 3, \ldots, n\}$ and the system

$$\sigma_n = \{\langle x_{n_j}, \varphi \rangle \geq c_{n_j}, j \in J\}$$

as

$$x_{n_j} = j,\ c_{n_j} = j^{-1} \text{ if } j \in A_n$$

$$x_{n_j} = c_{n_j} = 0 \text{ if } j \notin A_n$$

Case 2. $k > 2$. Analogously, for each $n = 1,2,\ldots$, the system σ_n may be generalized in the same way that the Case 2 of Proposition 3.3., i.e., on defining

$$\sigma_n = \{\langle x_{n_j}, \varphi \rangle \geq c_{n_j}, j \in J\}$$

where

$$x_{n_j} = (0,\ldots,0,\ j),\ c_{n_j} = j^{-1} \text{ if } j \in A_n$$

$$x_{n_j} = (0,\ldots,\ 0),\ c_{n_j} = 0 \text{ if } j \notin A_n$$

Proposition 3.5. *The set L_s of the strongly-inconsistent systems is not open in (Θ^*, d).*

Proof: Let us consider two cases:

Case 1. $k = 2$. Let

$$\sigma = \{\langle x_j, \varphi \rangle \geq c_j, j \in J\}$$

be the system defined as

$$x_j = 0 \text{ for all } j \in J$$

$$c_j = 0 \text{ for all } j \in J, \text{ excep for certain index } j_0$$

$$c_{j_0} = 1.$$

σ is obviously a strongly-inconsistent system. Now, we are going to construct a sequence (σ_n) that converges to σ. Indeed, for each positive integer n, we define

$$\sigma_n = \{\langle x_{n_j}, \varphi \rangle \geq c_{n_j}, j \in J\}$$

as follows:

$$x_{n_j} = c_{n_j} = 0 \text{ if } j \neq j_0$$

$$x_{n_j} = \frac{1}{n},\ c_{n_j} = n \text{ if } j = j_0.$$

The systems σ_n are consistent, as we may easily check, since for each n, $\varphi = n^2$ is a solution of each of them. Moreover, we have that

$$\lim_n d(\sigma_n, \sigma) = \lim_n d_H(S_{\sigma_n}, S_\sigma) = \lim_n \frac{1}{\sqrt{1+n^4}} = 0.$$

Case 2. $k > 2$. In this case, the system σ would be defined as

$$x_j = (0,...,0) \text{ for all } j \in J$$

$$c_j = 0 \text{ for all } j \in J, \text{ excep for certain index } j_0$$

$$c_{j_0} = 1.$$

As for the systems

$$\sigma_n = \{\langle x_{n_j}, \varphi \rangle \geq c_{n_j}, j \in J\},$$

they may be generalized on the following way:

$$x_{n_j} = (0,...,0),\ c_{n_j} = 0 \text{ if } j \neq j_0,$$

$$x_{n_j} = (0,...,0, \frac{1}{n}),\ c_{n_j} = n \text{ if } j = j_0.$$

Analogously, we would show that the sequence $(\sigma_n)_n$ of consistent systems converges to the strongly-inconsistent system σ. ■

Proposition 3.6. *If $bd(L_c)$ and $bd(L_s)$ are the boundaries of L_c and L_s respectively. Then*

$$L_w \subset bd(L_c) \cap bd(L_s)$$

Proof: We shall do it by induction over the dimension of the space.

If $k = 2$, i.e. $X = R$. Let us consider a weakly-inconsistent system

$$\sigma = \{\langle x_j, \varphi \rangle \geq c_j, j \in J\}.$$

Noticing the Theorem of Classification, we have that vector $u = (0,1) \in cl\ M_\sigma \setminus M_\sigma$, therefore there exists a sequence (u_n), n = 1,2,... of vectors in M_σ such that converges to $u = (0,1)$ and, without loss of generality, we may suppose that $\|u_n\| = 1$.

Let v be the vector belonging to S_σ, $\|v\| = 1$, such that:

$$\|u - v\| = \max\{\|u - t\| \{: t \in S_\sigma, \|t\| = 1\},$$

if we denote $v = (y_0, d_0)$ and $u_n = (y_n, d_n)$, $n = 1,2,...$, we may define another system

$$\sigma^* = \{\langle x_j^*, \varphi \rangle \geq c_j^*, j \in J\}$$

by means of

$$x_j^* = y_j,\ c_j^* = d_j \text{ if } j \in \{0,1,2,...\},$$

$$x_j^* = c_j^* = 0 \text{ if } j \notin \{0,1,2,...\},$$

and we obtain, on this way, a system σ^* equivalent to σ.

For each $N = 1,2,...$, we define a consistent system

$$\sigma_N^* = \{\langle x_{N_j}^*, \varphi \rangle \geq c_{N_j}^*, j \in J\},$$

being

$$x_{N_j}^* = y_j,\ c_{N_j}^* = d_j \text{ if } j \in \{0,1,..., N\},$$

$$x_{N_j}^* = c_{N_j}^* = 0 \text{ if } j \notin \{0,1,..., N\}$$

and we have, obviously, that the sequence (σ_N^*) converges to σ^*.

For the construction of a sequence (ω_N^*) of strongly-inconsistent systems that converges to σ^*, we shall consider two cases:

Case 1. The vector $v = (y_0, d_0) \neq (0, -1) = -u$.

We define the number

$$\varepsilon = min\{\|v - u\|, \|v + u\|\};$$

now, since $\lim_n u_n = u$, there is a positive integer p such that $\|u_n - u\| < \varepsilon$ for all $n \geq p$. For each $N = 1,2,...$, we define a consistent system

$$\omega_N^* = \{\langle y_{N_j}^*, \varphi \rangle \geq d_{N_j}^*, j \in J\},$$

being

$$y^*_{N_j} = y_j,\ d^*_{N_j} = d_j \text{ if } j = 0.$$

$$y^*_{N_j} = -y_j,\ d^*_{N_j} = d_j \text{ if } j = p + N - 1$$

$$y^*_{N_j} = d^*_{N_j} = 0 \text{ if } j \neq 0 \text{ and } j \neq p + N - 1.$$

Case 2. The vector $v = (y_0, d_0) = (0, -1) = -u$.

In this case, the characteristic sector of the system σ^* is the closed unit semi-disc on the right or on the left. If it is on the right (on the left is analogous), we define, for each $N = 1,2,...$ the system

$$\omega_N{}^* = \left\{\left\langle z^*_{N_j}, \varphi\right\rangle \geq e^*_{N_j}, j \in J\right\},$$

being,

$$z^*_{N_j} = \frac{1}{N},\ e^*_{N_j} = -1 \text{ if } j > 0$$

$$z^*_{N_j} = 1,\ e^*_{N_j} = 0 \text{ if } j = 0$$

$$z^*_{N_j} = -\frac{1}{N},\ e^*_{N_j} = 1 \text{ if } j < 0$$

In both cases, may easily chek it, we obtain sequences $(\omega_N{}^*)$ of strongly-inconsistent systems that converge, each one of them, to σ^*.

If $k > 2$, i.e. $X = R^{k-1}$, let us suppose that the statement of Proposition is true as far as dimension $k - 2$; we must show that to be also correct for $k - 1$:

Let us consider

$$\sigma = \left\{\langle x_j, \varphi \rangle \geq c_j, j \in J\right\}$$

a weakly-inconsistent system in R^{k-1}. The closure of its characteristic cone $cl\ M_\sigma$ is a closed convex cone in R^k, then there exists a supporting hyperplane H that goes through the point $u = (0_{k-1}, 1)$, where 0_{k-1} is the vector zero of R^{k-1}. Let us suppose that H has a characteristic vector η (we may assume it has unit norm), thus we may define it as

$$H = \left\{x \in R^k : \langle \eta, x\rangle = 0\right\}$$

with $\langle \eta, x\rangle \geq 0$ for all $x \in cl\ M_\sigma$.

If $\langle \eta, x\rangle = 0$ for all $x \in M_\sigma$, would have that $M_\sigma \subset H$, and this means that it would be in a lower dimension, therefore, noticing the hypothesis of induction, the proposition would be showed. Thus, we may suppose, without loss of generality, that there exists a vector a $\in M_\sigma$ such that $\langle \eta, a\rangle > 0$. Let q be a positive integer so that

$$\left\langle \eta - \frac{1}{q}u, a\right\rangle = \langle \eta, a\rangle - \frac{1}{q}\langle u, a\rangle > 0;\ (1)$$

for each $N \geq q$, we define the hyperplane

$$H_N^- = \left\{x \in R^k : \left\langle x, \eta - \frac{1}{N}u\right\rangle = 0\right\}$$

and the index set

$$J_N^- = \left\{j \in J : (x_j, c_j) = s_j \text{ such that } \left\langle s_j, \eta - \frac{1}{N}u\right\rangle \geq 0\right\},$$

where (x_j, c_j) have been defined by means the system given σ. The relation (1) implies that $J_N^- \neq \varnothing$, then we may define, for each $N \geq q$, a system

$$\sigma_N^- = \left\{\left\langle x_{N_j}, \varphi\right\rangle \geq c_{N_j}, j \in J\right\}$$

by means of

$$x_{N_j} = x_j,\ c_{N_j} = c_j \text{ if } j \in J_N^-$$

$$x_{N_j} = 0_{k-1},\ c_{N_j} = 0 \text{ if } j \notin J_N^-$$

These systems σ_N^-, are obviously consistent since the vector u is such that

$$\left\langle \eta - \frac{1}{N}u, u\right\rangle = -\frac{1}{N},$$

hence $u \notin clM_{\sigma_N^-}$ for all $N \geq q$. On the other hand, since the euclidean distance from u to each hyperplane H_N^- is given by

$$d_e(u, H_N^-) = \frac{\left|\left\langle \eta - \frac{1}{N}u, u\right\rangle\right|}{\left\|\eta - \frac{1}{N}u\right\|} = \frac{1}{\sqrt{1+N^2}},$$

we have that

$$lim_N d(\sigma_N^-, \sigma) = lim_N d_H\left(S_{\sigma_N^-}, S_\sigma\right) = 0$$

and therefore, the sequence $\left(\sigma_N^-\right)_{N \geq q}$ of consistent systems converges to σ.

To end the proof, it is enough notice that, given the weakly-inconsistent system

$$\sigma = \left\{\langle x_j, \varphi\rangle \geq c_j, j \in J\right\},$$

the vector $u \in clM_\sigma$, hence we may define an equivalent strongly-inconsistent system

$$\omega = \left\{\langle z_j, \varphi\rangle \geq e_j, j \in J\right\}$$

as:

$$z_j = x_j,\ e_j = c_j \text{ if } j \neq j_0$$

$$z_j = 0_{k-1},\ e_j = 1 \text{ if } j = j_0$$

for certain index j_0. ■

4. REFINEMENT OF THE PREVIOUS CLASSIFICATION OF LINEAR INEQUALITIES SYSTEMS

In many phases of the proofs in above results, we have been able to note the scarce difference between some strongly-inconsistent and weakly-inconsistent systems; in fact, it is enough to add the vector $u = (0_{k-1},1)$ to a weakly-inconsistent to obtain a strongly-inconsistent system. This question generates the following definitions:

Definition 4.1. *A strongly-inconsistent system σ is said to be strict-strongly inconsistent if and only if $u \in \overset{0}{M}_\sigma$.*

The set of strict-strongly inconsistent systems will be denoted by L_{ss}.

Definition 4.2. *A strongly-inconsistent system σ is said to be quasi-strongly inconsistent if and only if $u \in M_\sigma \cap bd(M_\sigma)$.*

The set of quasi-strongly inconsistent systems will be denoted by L_{qs}. Then, the set of strongly-inconsistent systems $L_s = L_{ss} \cup L_{qs}$.

Proposition 4.3. *The set L_{ss} is open in (Θ^*, d).*

Proof: Let σ be an element belonging to L_{ss}, then u $\in \overset{0}{M}_\sigma$ and therefore, there exists a positive number r such that the open ball, for the euclidean topology in R^k,

$$B(u,r) \subset M_\sigma \quad (1)$$

Let us consider, in the Hausdorff topology, the open ball $B(\sigma, r)$ and we will see that it is contained in $\overset{0}{L}_{ss}$. Indeed, if $\sigma' \in B(\sigma, r)$, then $d(\sigma', \sigma) = \varepsilon < r$. If $u \in \overset{0}{M}_{\sigma'}$ the proposition will be showed, if not, i.e. if $u \notin \overset{0}{M}_{\sigma'}$, we have two cases:

Case 1: $u \in bd(M_{\sigma'})$. Let us consider the supporting hyperplane H' of characteristic vector η' (we may assume $\|\eta'\| = 1$) that goes through the point u, of equation

$$H' = \{x \in R^k : \langle \eta', x\rangle = 0\},$$

fulfiling

$$\langle \eta', u_j'\rangle \geq 0 \, for.all \, u_j' = (x_j', c_j') \in M_{\sigma'} \quad (2)$$

Now, noticing relation (1), there exists $t_0 \in S_\sigma$ such that

$$r \leq |\langle \eta', t_0\rangle| = \max\{|\langle \eta', t\rangle| : \langle \eta', t\rangle < 0, t \in S_\sigma\},$$

therefore

$$\langle \eta', t_0\rangle \leq -r \quad (3)$$

On the other hand, since $d(\sigma', \sigma) = \varepsilon < r$, we have $S_\sigma \subset S_{\sigma'} + \varepsilon B$, then given $t_0 \in S_\sigma$ there exists $t' \in S_{\sigma'}$ such that $\|t_0 - t'\| \leq \varepsilon$. Now, from (2) and (3), we obtain

$$\langle \eta', t' - t_0\rangle = \langle \eta', t'\rangle - \langle \eta', t_0\rangle \geq r \quad (4)$$

and by Cauchy-Schwarz inequality

$$|\langle \eta', t' - t_0\rangle| \leq \|\eta'\| . \|t' - t_0\| \leq \varepsilon < r,$$

but this is a contradiction with (4).

Case 2: $u \notin clM_{\sigma'}$. We consider, in the same way, a supporting hyperplane G' of d $M_{\sigma'}$ with characteristic vector ξ'' of unit norm (in this case G' does not go through u) and reasoning analogously, we may find a point $t_0 \in S_\sigma$ such that

$$|\langle \xi', t_0\rangle| \geq r,$$

then we are exactly in the case 1, and developping the same argument, we come to the same contradiction. ■

Theorem 4.4. *The consistent and strongly-inconsistent systems have equal boundary and this is the union of weakly-inconsistent and quasi-strongly inconsistent systems, i.e.*

$$bd(L_c) = bd(L_s) = L_w \cup L_{qs}.$$

Proof: If $\sigma \in bd(L_c)$, implies that $\sigma \notin L_c$ since L_c is open by proposition 3.2. Then, if $\sigma \in L_w$, by proposition 3.6, we have that

$$L_w \subset bd(L_c) \cap bd(L_s),$$

hence $\sigma \in bd(L_s)$. If $\sigma \in L_s = L_{ss} \cup L_{qs}$, then necessary $\sigma \in L_{qs}$, since L_{ss} is open by proposition 4.3.

Let us consider two cases:

Case 1: $M_\sigma = C$ (degenerated cone) $= \{x \in R^k : x = \lambda u, \lambda \geq 0\}$, where $u = (0_{k-1}, 1)$. This is, exactly the associated cone to system σ which we had defined to do the proof of proposition 3.5., where we showed that L_s was not open, showing that $\sigma \in bd(L_s)$.

Case 2: $M_\sigma \neq C$. Since $\sigma \in L_{qs}$, we have that $u \in bd(M_\sigma) \cap M_\sigma$ and as $M_\sigma \backslash C \neq \varnothing$, we may define a system σ' such that $M_{\sigma'} = M_\sigma \backslash C$. Obviously σ' is a weakly-inconsistent system which is moreover equivalent to σ and noticing proposition 3.6. we obtain again $\sigma \in bd(L_s)$.

Conversely, if $\sigma \in bd(L_s)$ then $\sigma \notin L_{ss}$ since L_{ss} is open by proposition 4.3. and $\sigma \notin L_c$ since L_c is also open by proposition 3.2., then $\sigma \in L_w \cup L_{qs}$.

If $\sigma \in L_w$, by proposition 3.6 we have that $\sigma \in bd(L_c)$ and if $\sigma \in L_{qs}$ we have already seen that $M_\sigma = C$ or we may define an equivalent weakly-inconsistent system σ'.

In the first case, from proposition 3.5. we have that $\sigma \in bd(L_c)$ and as the second case, from proposition 3.6. we may also deduce that $\sigma' \equiv \sigma \in bd(L_c)$. ■

It is enough to apply the above theorem to obtain

Corollary 4.5. $cl(L_c) = L_c \cup L_w \cup L_{qw}$, $cl(L_{ss}) = L_{ss} \cup L_w \cup L_{qs}$ ■

Corollary 4.6. *L_w and L_{qs} are nowhere dense, hence are first category sets.*

Proof: it is enough notice that $cl(L_w) = cl(L_{qs}) = L_w \cup L_{qs}$ and apply theorem 4.4. to deduce that $cl(\overset{0}{L}_w) = cl(\overset{0}{L}_{qs}) = \varnothing$. ■

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