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# ON SOME MATRIX INEQUALITIES IN BANACH SPACES

**(operators ideals/tensor products of operators/summning and factorable operators)**

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## ABSTRACT

For $1 \leq p,\ q \leq \infty$ with $1/p + 1/q \geq 1$ we prove a natural factorization theorem for those linear and continuous $T: X \to Y$ between Banach spaces which have the property that $A \otimes T: l_{q'} \otimes_\varepsilon X \to l_p \otimes_\pi Y$ is continuous for each linear and continuous $A: l_{q'} \to l_p$ (here $\varepsilon$ and $\pi$ denote the injective and projective norm, respectively). Our result is a special case of a more general factorization theorem formulated in the abstract framework of so-called weights on Banach spaces -this setting grew out of some nowadays standard ideas which can be traced back to S. Kwapień's and A. Pietsch's important work on $s$-summing and $L_s$-factorable operators.

## RESUMEN

Para $1 \leq p,\ q \leq \infty$, con $1/p + 1/q \geq 1$, se demuestra un teorema de factorización natural para aquellas aplicaciones lineales y continuas $T: X \to Y$ entre espacios de Banach que tienen la propiedad de que $A \otimes T: l_{q'} \otimes_\varepsilon X \to l_p \otimes_\pi Y$ es continua para cada $A: l_{q'} \to l_p$ lineal y continua (donde $\varepsilon$ y $\pi$ denotan, respectivamente, la norma inyectiva y proyectiva). Nuestro resultado es un caso particular de un teorema de factorización más general formulado en la estructura abstracta de los llamados pesos sobre espacios de Banach -este surgió de algunas ideas hoy estándar que pueden encontrarse en la importante obra de S. Kwapień y A. Pietsch sobre operadores $s$-sumables y $L_s$-factorizables.

## 1. INTRODUCTION AND PRELIMINARIES

Our aim is to characterize operators between Banach spaces which satisfy certain abstract *matrix inequalities* (in terms of $s$-summing and $L_s$-factorable operators) -see [2], [3] and [9] for all needed information on operator ideals and tensor products. This is done by a standard separation argument within the framework of so-called weights on Banach spaces.

The original motivation was to produce a —in a certain sense natural— factorization theorem for (linear and continuous) operators $T: X \to Y$ between Banach spaces which for fixed $1 \leq p,\ q \leq \infty$ with $1/p + 1/q \geq 1$ fulfill the following inequality:

(I) There is a constant $c \geq 0$ such that for all $n$, all $n \times n$ matrices $(a_{k,l})$ and all vectors $x_1,\dots, x_n \in X$, $y'_1,\dots, y'_n \in Y'$

$$\left|\sum_{k,l} a_{k,l}\, y'_k(Tx_l)\right| \leq$$

$$\leq c \left\| (a_{k,l}) : l_{q'}^n \to l_p^n \right\| \sup_{\|x'\|\leq 1} \left( \sum_k \left|x'(x_k)\right|^{q'} \right)^{1/q'} \sup_{\|y\|\leq 1} \left( \sum_k \left|y'_k(y)\right|^{p'} \right)^{1/p'}.$$

By a standard closed graph argument (I) has an equivalent formulation in terms of tensor products:

(I') For all operators $A: l_{q'} \to l_p$ the tensor product operator

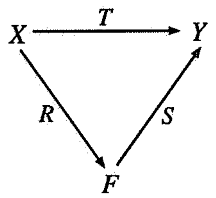
$$A \otimes T: l_{q'} \otimes_\varepsilon X \to l_p \otimes_\pi Y$$

is continuous; here as usual $\varepsilon$ denotes the injective and $\pi$ the projective norm.

The most spectacular characterization of such a matrix inequality can be given in the case $p = q = 1$: Grothendieck's famous inequality from [4] states that $T$ satisfies (I) iff it factorizes through a Hilbert space. The case $1/p + 1/q = 1$ is completely covered by results from Kwapień's important paper [5]: For $p = q'$ it is obvious that $T$ fulfills (I') if it satisfies (I') for $A = \mathrm{id}: l_p \to l_p$ only, and in this case (I) reduces to

$$\left|\sum_k y'_k(Tx_k)\right| \leq c \sup_{\|x'\|\leq 1} \left( \sum_k \left|x'(x_k)\right|^p \right)^{1/p} \sup_{\|y\|\leq 1} \left( \sum_k \left|y'_k(y)\right|^{p'} \right)^{1/p'}$$

Such $T$ form the Banach operator ideal $D_p$ of all $p$-*dominated operators*, and by Kwapień's factorization theorem from [5] this ideal consists of all compositions



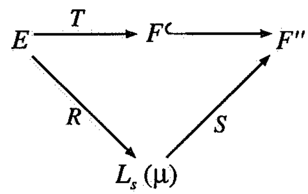

where $R$ is $p$-summing and $S'$ is $p'$-summing. Recall that an operator $U: E \to F$ is $s$-summing$(1 \le s < \infty)$ whenever

$$\pi_s(U) := \sup\left\{ \left(\sum_{k=1}^{n} \|Ux_k\|^s\right)^{1/s} \,\middle|\, \sup_{\|x'\|\le 1}\left(\sum_{k=1}^{n} |x'(x_k)|^s\right)^{1/s} \le 1 \right\} < \infty$$

(for $s = \infty$ put $\pi_\infty(U) := \|U\|$). Moreover, Kwapień in [5] showed that $D_p$ is in trace duality with the Banach operator ideal $L_{p'}$ of all $p'$-factorable operators:

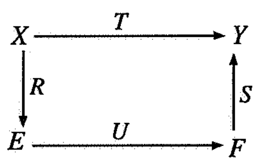
$$D_p = L_{p'}^*,$$

for $1 \le s \le \infty$ an operator $U: E \to F$ is $s$-factorable (in short $T \in L_s$) whenever $T$ for some measure $\mu$ allows a factorization




In [2, p. 373] this latter duality result is extended to the case $1/p + 1/q > 1$: Every operator $T$ satisfies (I) if and only if $T \in (L_q \circ L_{p'})^*$, the adjoint of the ideal of all compositions $UV$ with $U \in L_q$, $V \in L_{p'}$. But it remained open how a Kwapień-like factorization theorem for $1/p + 1/q > 1$ could look like.

The following consideration shows that there is a natural candidate for such a factorization theorem: Assume that $T: X \to Y$ allows a factorization




such that

(II) $\mathrm{id}\otimes R: l_{q'} \otimes_\varepsilon X \to l_{q'} \otimes \Delta_{q'} E$ is continuous (i.e. $R$ is $q'$-summing, see [2, p. 128]),

(II') $\mathrm{id}\otimes S: l_p \otimes \Delta_p F \to l_p \otimes_\pi Y$ is continuous (by duality this means that $S'$ is $p'$-summing),

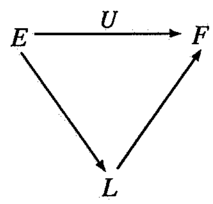
(III) $A \otimes U: l_{q'} \otimes \Delta_{q'} E \to l_p \otimes \Delta_p F$ is continuous for each $A: l_{q'} \to l_p$; here $\Delta_s$ stands for the norm on $l_s \otimes X$ induced by $l_s(X)$, the Banach space of all absolutely $s$-summable sequences in $X$. Then it is obvious that for each $A: l_{q'} \to l_p$

$$A \otimes T: l_{q'} \otimes_\varepsilon X \to l_p \otimes_\pi Y$$

is continuous.

Our main application states that the converse of this result holds: If $T: X \to Y$ satisfies the matrix inequality (I), then it can be written as a product $T = SUR$ with $R$, $S$ as in (II) and (II'), and $U$ as in (III).

The Banach operator ideal of all $U$ which satisfy (III) is well-understood: Again it was shown by Kwapień in [5] that for $1/p + 1/q = 1$ every operartor $U: E \to F$ fulfills (III) if and only if it factorizes through a subspace $L$ of a quotient of some $L_p(\mu)$ (= quotient of a subspace...):




(for $p = q = 2$ this is an important characterization of 2-factorable operators due to Lindenstrauss and Petczýnsky [6]). In [1] (see [2, p. 369]) Kwapień's result was extended to the case $1/p + 1/q > 1$: An operator $U$ satisfies (III) iff there are a probability measure $\mu$ and closed subspaces

$$\begin{array}{ccccc} K & \subset & L & \subset & L_{q'}(\mu) \\ \cap & & \cap & & \cap \\ M & \subset & N & \subset & L_p(\mu) \end{array}$$

such that $U$ factorizes through the canonical mapping

$$L/K \to N/M, \quad f + K \to f + M.$$

In the language of Banach operator ideals these $U$ form the injective and surjective hull $L_{p,q}^{\mathrm{inj\,sur}}$ of the ideal $L_{p,q}$ of all $(p, q)$-factorable operators (see [2, sec. 18]).

## 2. KWAPIEŃ'S SEPARATION ARGUMENT

For a normed space $X$ denote by $H_1(X)$ the set of all positive homogeneous mappings $h: X \to \mathbb{R}_{\ge 0}$ such that

$$\sup_{\|x\|\le 1} h(x) \le 1.$$

For $1 \le p \le \infty$ and $C \subset H_1(X)$ we call

$$w_{p,C}: \bigoplus_{\mathbb{N}} X \to \mathbb{R}_{\ge 0}, \quad w_{p,C}(x_k) := \sup_{h \in C}\left(\sum_k h(x_k)^p\right)^{1/p}$$

a *weight on X* (with the obvious modification for $p=\infty$). Standard examples are the *weak p-weight*

$$w_p(x_k) := \sup_{\|x'\| \le 1} \left( \sum_k |x'(x_k)|^p \right)^{1/p}$$

and the *strong p-weight*

$$\Delta_p(x_k) := \left( \sum_k \|x_k\|^p \right)^{1/p}.$$

Another interesting example was investigated by Matter [7] and López-Sánchez [8]: For $0 \le \theta < 1$ and $1 \le p \le \infty$

$$w_{p,\theta}(x_k) := \left( \sup_{\|x'\| \le 1} \sum_k \left( |x'(x_k)|^{1-\theta} \|x_k\|^{\theta} \right)^{p/1-\theta} \right)^{1-\theta/p}$$

defines a weight on $X$ which for $\theta = 0$ obviously equals $w_p$.

We remark that if $H_1(X)$ is endowed with the topology $\tau_s$ of pointwise convergence on $X$, then by Tychonoff's theorem the $\tau_s$-closure $\overline{C}$ of each $C \subset H_1(X)$ is compact, and moreover

$$w_{p,C} = w_{p,\overline{C}}.$$

The space of all Borel probability measures $\mu$ on $\overline{C}$ is denoted by $M_1^+(\overline{C})$.

Dealing with weights the following result is central:

**Lemma.** *For $k=1,\dots,n$ let $w_{p_k,C_k}$ be a weight on the normed space $X_k$ and $\sum_k 1/p_k = 1$. Then for each function*

$$\varphi : \prod_{k=1}^{n} X_k \to \mathbb{R}_{\ge 0},$$

*wich is positive homogeneous in each coordinate, the following are equivalent:*

(1) *For all m and $x_1 \in X_1^m, \dots, x_n \in X_n^m$*

$$\sum_{k=1}^{m} \varphi(x_1(k), \dots, x_n(k)) \le \prod_{k=1}^{n} w_{p_k,C_k}(x_k).$$

(2) There are $\mu_k \in M_1^+(\overline{C}_k)$ such that for all $x \in \prod_{k=1}^{n} X_k$

$$\varphi(x) \le \prod_{k=1}^{n} \left( \int_{\overline{C}_k} h(x_k)^{p_k} d\mu_k(h) \right)^{1/p_k}.$$

For $n = 2$, the weak $p_k$-weights $w_{p_k}$, and $\varphi = |\phi|$ with $\phi$ a bilinear form on $X_1 \times X_2$, this result - at least essentially - is due to Kwapień [5]. Our (only formally) more general version is proved exactly in the same way using a Hahn-Banach separation argument - for example copy word by word the proofs of [9, Th. 17.4.2], or [2, Th. 19.2.]. Sometimes we will refer to this lemma as "Kwapień's separation argument".

If $w_{p,C}$ is a weight on $X$, then we call an operator $T: X \to Y$ *$(p,C)$-summing* whenever

$$\pi_{p,C}(T) := \sup \left\{ \left( \sum_k \|Tx_k\|^p \right)^{1/p} \,\middle|\, w_{p,C}(x_k) \le 1 \right\} < \infty.$$

The class of $(p,C)$-summing operators $T: X \to Y$ will be denoted by

$$\Pi_{p,C}(X,Y).$$

Together with $\pi_{p,C}$ this is a seminormed space which for the weak $p$-weight equals $\Pi_p(X,Y)$, all $p$-summing operators, and for the strong $p$-weight the space $L(X,Y)$ of all operators.

Obviously, for each $\mu \in M_1^+(\overline{C})$

$$I_X : X \to L_p(\mu, \overline{C}), \quad (I_X x)h := h(x)$$

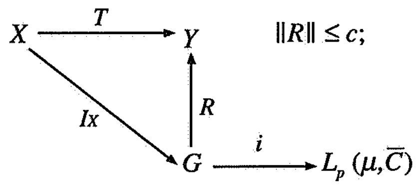
is $(p,C)$-summing and $\pi_{p,C}(I_X) \le 1$. It is not hard to guess from the Grothendieck-Pietsch cycle of ideas that this mapping is the *prototype* of a $(p,C)$-summing operator: Assume that $T \in L(X,Y)$ is $(p,C)$-summing with $\pi_{p,C}(T) \le c$. Then for all $x_1, \dots, x_n \in X$ and $y_1', \dots, y_n' \in Y'$

$$\sum_k |y_k'(Tx_k)| \le c\, w_{p,C}(x_k) \Delta_{p'}(y_k').$$

Hence by Kwapień's separation argument there is $\mu \in M_1^+(\overline{C})$ such that for all $x \in X$

$$\|Tx\| \le c \left( \int_{\overline{C}} h(x)^p d\mu(h) \right)^{1/p}.$$

Clearly, this inequality gives a factorization




put $G := \text{range} I_X$ and $R(I_X x) := Tx$. Vice versa, it is obvious that any operator $T$ which factors in this way has $(p,C)$-summing norm $\le c$:

$$\pi_{p,C}(T) \le \|R\| \pi_{p,C}(I_X) \le c.$$

For the weights $w_{p,\theta}$ these results were discovered by Matter [7] and López Sánchez [8].

### 3. A MATRIX VERSION OF KWAPIEŃ'S SEPARATION ARGUMENT

For each $n$ let $Z_n$ be a subspace of all $n \times n$ matrices, and $\|\cdot\|_n$ a norm on $Z_n$. We will say that the sequence $((Z_n, \|\cdot\|_n))$ of such "matrix spaces" satisfies the so-called *r-condition* $(1 \le r \le \infty)$ whenever for all $a \in Z_n$, $b \in Z_m$

$$a \oplus b := \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in Z_{n+m}$$

$$\|a+b\|_{n+m} \le \left(\|a\|_n^r + \|b\|_m^r\right)^{1/r}$$

(obvious modification for $r = \infty$).

The following variant of Kwapień's separation argument is basic to our applications.

**Proposition.** *Let $w_{p,C}$ on X and $w_{q,D}$ on Y be weights such that $1/p + 1/q \ge 1$ and assume that $((Z_n, \|\cdot\|_n))$ for r defined by $1/p' + 1/q' + 1/r = 1$ satisfies the r-condition. Then for every function $\varphi : X \times Y \to \mathbb{R}_{\ge 0}$, which is positive homogeneous in each coordinate, the following are equivalent:*

(1) *For all n, all $a \in Z_n$ and all $x \in X^n, y \in Y^n$*

$$\left|\sum_{k,l} a(k,l)\varphi(x(l), y(k))\right| \le w_{q',C}(x) w_{p',D}(y) \|a\|_n .$$

(2) *There are $\mu \in M_1^+(\overline{C})$ and $\nu \in M_1^+(\overline{D})$ such that for all n, all $a \in Z_n$, and all $x \in X^n$, $y \in Y^n$*

$$\left|\sum_{k,l} a(k,l)\varphi(x(l), y(k))\right| \le$$

$$\le \left(\sum_l \int_{\overline{C}} h(x(l))^{q'} d\mu(h)\right)^{1/q'} \left(\sum_k \int_{\overline{D}} g(y(k))^{p'} d\nu(g)\right)^{1/p'} \|a\|_n .$$

PROOF: Clearly, only the implication (1) $\Rightarrow$ (2) needs a proof. We assume without loss of generality that $C$ and $D$ are $\tau_s$-closed. Moreover, we will indentify each $a = (a_n) \in \oplus_{\mathbb{N}} Z_n$ with the block diagonal matrix

$$(a(i,j)) := \oplus a_l = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \in Z_{|a|},$$

where $|a| := \sum_l$ (number of rows of $a_l$). The idea of the proof is to apply Kwapien's séparation argument to the modulus of the mapping

$$\phi : \oplus_{\mathbb{N}}{}_{q'} X \times \oplus_{\mathbb{N}}{}_{p'} Y \times \oplus_{\mathbb{N}}{}_{r} Z_n \to \mathbb{R}$$

$$\phi(x,y,a) := \sum_{k,l} a(k,l)\varphi(x(l), y(k))$$

(assume that the "length" of $x$ and $y$ equal $|a|$, otherwise add zeros). Define for $h \in C$

$$\hat{h}(x) := \left(\sum_k h(x(l))^{q'}\right)^{1/q'}, \quad x \in \oplus_{\mathbb{N}} X$$

and

$$\hat{C} := \{\hat{h} \mid h \in C\} \subset \mathcal{H}_1\left(\oplus_{\mathbb{N}}{}_{q'} X\right).$$

For each $m$ and $(x_j) \in (\oplus X)^m, (y_j) \in (\oplus Y)^m$, $(a_j) \in (\oplus Z_n)^m$ we have

$$\sum_j \sum_{k,l} a_j(k,l)\varphi(x_j(l), y_j(k)) = \sum_{k,l} (\oplus a_j)(k,l)\varphi((\oplus x_j)(l), (\oplus y_j)(k))$$

(again the "length" of the $x_j$ and $y_j$ equal $|a_j|$). By assumption and since $((Z_n, \|\cdot\|_n))_n$ satisfies the $r$-condition

$$\sum_j |\phi(x_j, y_j, a_j)| \le w_{q',C}(\oplus x_j) w_{p',D}(\oplus y_j) \|\oplus a_j\|_{\Sigma|a_j|}$$

$$\le w_{q',\hat{C}}(x_j) w_{p',\hat{D}}(y_j) \left(\sum_j \|a_j\|_{|a_j|}^r\right)^{1/r} .$$

Hence Kwapień's separation argument gives some $\hat{\mu} \in M_1^+(\hat{C})$ and $\hat{\nu} \in M_1^+(\hat{D})$ such that for all $x \in \oplus X$, $y \in \oplus Y$, $a = (a_n) \in \oplus Z_n$

$$|\phi(x,y,a)| \le \left(\int_{\hat{C}} \hat{h}(x)^{q'} d\hat{\mu}(\hat{h})\right)^{1/q'} \left(\int_{\hat{D}} \hat{g}(y)^{p'} d\hat{\nu}(\hat{g})\right)^{1/p'} \left(\sum \|a_n\|_n^r\right)^{1/r};$$

note that $\hat{C}$ is $\tau_s$-closed since the bijection

$$\hat{\cdot} : C \to \hat{C}$$

is a homeomorphism. Define $\mu$ and $\nu$ to be the image measure of $\hat{\mu}$ and $\hat{\nu}$ with respect to the inverse of this mapping. Then the conclusion follows if we apply the preceding inequality to all $x \in X^n$, $y \in Y^n$ and "single" matrices $a \in Z_n$. ■

## 4. WEIGHTED SEMINORMS ON TENSOR PRODUCTS

If $\varphi$ in the preceding matrix version of Kwapień's separation argument is bilinear, then it turns out to be convenient to reformulate this result in terms of tensor products.

Again we start with some notation. Let $w_{q',C}$ be a weight on $X, w_{p',D}$ a weight on $Y$ and $Z=\left(\left(Z_n,\|\cdot\|_n\right)\right)$ a sequence of matrix spaces which satisfies the $r$-condition with $r$ defined by $1/p'+1/q'+1/r=1$. Define for each $z\in X\otimes Y$

$$\alpha(z):=\inf\|a\|_n w_{q',C}(x_l)w_{p',D}(y_k),$$

the infimum taken over all finite representations

$$z=\sum_{k,l=1}^{n} a(k,l)x_l\otimes y_k$$

with $a\in Z_n$, $x\in X^n$ and $y\in Y^n$. Then it is not hard to see that $\alpha$ is a seminorm on $X\otimes Y$ which we will call the *weighted seminorm generated by* $w_{q',C}, w_{p',D}$ and $Z$ (for the proof ot the $\Delta$-inequality mimic the proof of [2, Ex. 12.8]).

**Examples.** Let $1/p'+1/q'+1/r=1$.

(1) Recall that Lapreste's tensor norms $\alpha_{p,q}$ on $X\otimes Y$ are defined by

$$\alpha_{p,q}(z):=\inf\left\{\left\|(\lambda_k)\right\|_r w_{q'}(x_k)w_{p'}(y_k)\ \middle|\ z=\sum_k \lambda_k x_k\otimes y_k\right\}$$

which generalize Saphar's and Chevet's tensor norms $g_p:=\alpha_{p,1}$ and $d_p:=\alpha_{1,p}$ (see [2, sec. 12]). Hence, if $Z_n$ consists of all $n\times n$ diagonal matrices $D_\lambda$ normed by $\|D_\lambda\|_n:=\left(\sum_{k=1}^{n}|\lambda_k|^r\right)^{1/r}$, then by definition (and Hölder's inequality)

| $\alpha$ | $w_{p'}$ | $\Delta_{p'}$ |
|---|---|---|
| $w_{q'}$ | $\alpha_{p,q}$ | $d_{q'}$ |
| $\Delta_{q'}$ | $g_{p'}$ | $\pi$ |

read: $\pi$ is the weighted (semi)norm generated by $\Delta_{q'},\Delta_{p'}$ and $\left(\left(Z_n,\|\cdot\|_n\right)\right)$,...

(2) Take for $Z_n$ all $n\times n$ matrices normed by

$$\|a\|_n:=\left\|a:l_{q'}^n\to l_p^n\right\|.$$

The tensor norms $\gamma_{p,q},\beta_{p,q}$, and $\delta_{p,q}$ defined via the table

| $\alpha$ | $w_{p'}$ | $\Delta_{p'}$ |
|---|---|---|
| $w_{q'}$ | $\beta_{p,q}$ | $\delta^t_{q,p}$ |
| $\Delta_{q'}$ | $\delta_{p,q}$ | $\gamma_{p,q}$ |

were studied in [2, sec. 28] - here $\gamma_{p,q}$ is of particular interest since it is the projective associate of the dual $\alpha'_{p,q}$ of $\alpha_{p,q}$.

The following theorem is our main result on weighted seminorms $\alpha$ on $X\otimes Y$ generated by $w_{q',C}, w_{p',D}$ and $Z$, and an analogue of the Grothendieck-Pietsch factorization theorem for $\alpha$-continuous functionals on $X\otimes Y$. It shows in particular that *the seminorm*

$$\alpha_{p,q,Z} \text{ on } X\otimes Y$$

*generated by the strong-weights* $\Delta_{q'},\Delta_{p'}$ *and Z plays an exeptional role among all such* $\alpha$.

**Theorem.** *For* $1\le p,q,r\le\infty$ *with* $1/p'+1/q'+1/r=1$ *let* $\alpha$ *be a weighted seminorm on* $X\otimes Y$ *generated by the weight* $w_{q',C}$ *on X, the weight* $w_{p',D}$ *on Y and the sequence* $Z=\left(\left(Z_n,\|\cdot\|\right)\right)$ *satisfying the r-condition. Then for each linear functional* $\varphi$ *on* $X\otimes Y$ *the following are equivalent:*

(1) $\varphi\in\left(X\otimes_\alpha Y\right)'$

(2) *There is a factorization*

$$\begin{array}{ccc} X\otimes Y & \xrightarrow{\ \varphi\ } & \mathbb{K} \\ \downarrow{\scriptstyle R\otimes S} & \nearrow{\scriptstyle \psi} & \\ E\otimes F & & \end{array}$$

*with* $R\in\Pi_{q',C}(X,E)$, $S\in\Pi_{p',D}(Y,F)$ *and* $\psi\in\left(E\otimes_{\alpha_{p,q,Z}}F\right)'$.

*In this case:* $\|\varphi\|=\inf\left\{\pi_{q',C}(R)\pi_{p',D}(S)\|\psi\|\right\}$.

Let us reformulate this result in terms of operators.

*For each operator T: X → Y' the following are equivalent:*

(1') There is a constant $c\ge 0$ such that for all $n$, all $a\in Z_n$ and all $x_1,\dots,x_n\in X$, $y_1,\dots,y_n\in Y$

$$\left|\sum_{k,l} a_{k,l}\left(Tx_l\right)\left(y_k\right)\right|\le c\|a\|_n w_{q',C}(x_l)w_{p',D}(y_k).$$

(2') *There is a factorization*

$$\begin{array}{ccc} X & \xrightarrow{T} & Y' \\ \downarrow R & & \uparrow S' \\ E & \xrightarrow{U} & F' \end{array}$$

*with R and S as in* (1) *and U such that for some* $d \geq 0$

$$\left\| a \otimes U : l_{q'}^n \otimes_{\Delta_{q'}} E \to l_p^n \otimes_{\Delta_p} F' \right\| \leq d \| a \|_n$$

*holds for all n and* $a \in Z_n$.

*In this case*: $\inf c = \inf\{\pi_{q',C}(R)\pi_{p',D}(S) \inf d\}$

PROOF: The implication (2) $\Rightarrow$ (1) follows by some straight forward estimations using the definitions only. The converse is an immediate consequence of the matrix version of Kwapień's separation argument: Obviously $\varphi \in (X \otimes_\alpha Y)'$ is equivalent to statement (1) of the proposition in section 3 $(w.l.o.g.: \|\varphi\| \leq 1)$. Hence there are probability measures $\mu$ and $\nu$ such that

$$\left| \sum_{k,l} a(k,l)\varphi(x(l) \otimes y(k)) \right| \leq$$

$$\leq \left( \sum_l \int_{\bar{C}} h(x(l))^{q'} d\mu(h) \right)^{1/q'} \left( \sum_k \int_{\bar{D}} g(y(k))^{p'} d\nu(g) \right)^{1/p'} \|a\|_n.$$

Define (for the definition of $I_X$ see section 1)

$$Rx := I_X x \in \text{range } I_X =: E \to L_{q'}(\mu, \bar{C}),\ x \in X$$

$$S_y := I_Y y \in \text{range } I_Y =: F \to L_{p'}(\nu, \bar{D}),\ y \in Y$$

$$\psi(Rx \otimes Sy) := \varphi(x \otimes y)$$

Then $\pi_{q',C}(R) = \pi_{p',D}(S) = 1$, and by the preceding inequality

$$\psi \in \left(E \otimes_{\alpha_{p,q,Z}} F\right)' \text{ with } \|\psi\| \leq 1,$$

which proves (1). Finally, in order to see (1') $\Leftrightarrow$ (2') define

$$\varphi : X \times Y \to \mathbb{K},\quad \varphi(x,y) := (Tx)(y).$$

Then it is obvious that (1) $\Leftrightarrow$ (1') and (2) $\Leftrightarrow$ (2'). ■

## 5. MATRIX INEQUALITIES

We now deal with more specialized situations - again it will always be assumed that $1 \leq p,q,r \leq \infty$ and $1/p' + 1/q' + 1/r = 1$.

In the first two examples we apply the theorem to the weak and strong weights, and take for $Z_n$ the space $M_n$ of all $n \times n$ matrices normed by

$$\|a\|_n := \left\| a : l_{q'}^n \to l_p^n \right\|.$$

The following result was already announced in the introduction.

**Example 1.** *For every operator* $T: X \to Y$ *the following are equivalent*:

(1) *For all operators* $A : l_{q'} \to l_p$

$$A \otimes T : l_{q'} \otimes_\varepsilon X \to l_p \otimes_\pi Y$$

*is continuous.*

(2) *There is a* $c \geq 0$ *such that for all n, all* $n \times n$ *matrices* $(a_{k,l})$ *and all* $x_1,\ldots,x_n \in X$, $y_1',\ldots,y_n' \in Y'$

$$\left| \sum a_{k,l} y_k'(Tx_l) \right| \leq c \left\| a : l_{q'}^n \to l_p^n \right\| w_{q'}(x_l) w_{p'}(y_k').$$

(3) *There is a factorization*

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ \downarrow R & & \uparrow S \\ E & \xrightarrow{U} & F \end{array}$$

*with* $R \in \Pi_{q'}(X,E)$, $S' \in \Pi_{p'}(Y',F')$ *and U such that for all operators* $A : l_{q'} \to l_p$

$$A \otimes U : l_{q'} \otimes_{\Delta_{q'}} K \to l_p \otimes_{\Delta_p} L'$$

*is continuous.*

*In this case:*

$$\sup_{\|A\| \leq 1} \|A \otimes T\| = \inf c = \inf\left\{ \pi_{q'}(R)\pi_{p'}(S') \sup_{\|A\| \leq 1} \|A \otimes U\| \right\}.$$

PROOF: A direct argument for the implication (3) $\Rightarrow$ (1) was given in the introduction. Recall that the equalities

$$l_{q'}^n \otimes_\varepsilon X = \left(X^n, w_{q'}\right)$$

$$l_p^n \otimes_\pi Y'' = \left(l_{p'}^n \otimes_\varepsilon Y'\right)' = \left((Y')^n, w_{p'}\right)'$$

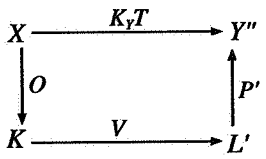
hold isometrically, and that

$$\text{id} \otimes \kappa_Y : l_p^n \otimes_\pi Y \to l_p^n \otimes_\pi Y''$$

is an isometric embedding ($\kappa_Y : Y \to Y''$ the canonical embedding). Hence (1) $\Rightarrow$ (2) follows by a simple closed graph argument. For the proof of (2) $\Rightarrow$ (3) we apply the theorem to the weights $w_{q'} = w_{q',C}$ on $X$, $w_{p'} = w_{p',D}$ on $Y'$ and $M_n$ (as above), and the operator

$$\kappa_Y T : X \to Y''.$$

This way we obtain a factorization



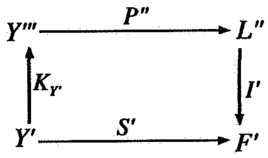

with $O \in \Pi_{q'}$, $P \in \Pi_{p'}$ and $V$ such that for all $A$

$$A \otimes V : l_{q'} \otimes_{\Delta_{q'}} K \to l_p \otimes_{\Delta_p} L'$$

is continuous. In order to produce a factorization as in (3) define

$$R : X \to \text{range } O =: E,\ Rx := Ox$$
$$U : E \to \text{range } V =: F,\ Ux := Vx$$
$$S : F \to Y, Sx := P'x$$

It remains to show that $S' \in \Pi_{p'}$. But since $P'' \in \Pi_{p'}(Y''', L'')$, this follows from the fact that




commutes $(I : F \to L'$ the canonical embedding). ■

It is interesting to reformulate this result in terms of $s$-summing and $s$-factorable operators. It was already mentioned in the introduction that (1) is equivalent to the fact that $T$ is in the adjoint ideal of the composition $L_q \circ L_{p'}$. Hence (1) $\Leftrightarrow$ (3) gives the formula

$$\left(L_q \circ L_{p'}\right)^* = \Pi_{p'}^{\text{dual}} \circ L_{p,q}^{\text{injsur}} \circ \Pi_{q'}$$

here $\Pi_{p'}^{\text{dual}}$ stands for the ideal of all opertors with $p'$-summing duals. Moreover, it can easily be seen that this equality even holds isometrically if all involved ideals are given their natural norms. By [2,p. 337] we know that

$$L_{p,q}^{\text{injsur}} = \left(I_{q'}^{\text{dual}} \circ I_{p'}\right)^*$$

($I_s$ the ideal of all $s$-integral operators), hence we can also write

$$\left(L_q \circ L_{p'}\right)^* = \Pi_{p'}^{\text{dual}} \circ \left(I_{q'}^{\text{dual}} \circ I_{p'}\right)^* \circ \Pi_{q'}.$$

Finally, we remark that

$$\left(L_q \circ L_\infty\right)^* = \begin{cases} \left(\Pi_q^{\text{dual}} \circ \Pi_{q'}\right)^{\text{inj}} & 2 \le q \le \infty \\ \Pi_2 & 1 < q \le 2 \\ L_2 & q = 1 \end{cases}$$

and

$$\left(L_q \circ L_{p'}\right)^* = \begin{cases} \Pi_q^{\text{dual}} \circ \Pi_{q'} & q' = p \\ \Pi_2^{\text{dual}} \circ \Pi_2 & 1 < p,q \le 2 \end{cases}$$

(see [2, p.373]), and that each of these formulas gives information on the matrix inequalities (2) of example 1.

Exactly the same way —replace $w_{p',D}$ by $\Delta_p$— we obtain

**Example 2.** *For every operator $T: X \to Y$ the following are equivalent:*

(1) *For all operators $A : l_{q'} \to l_p$*

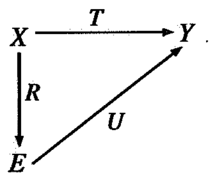
$$A \otimes T : l_{q'} \otimes_\varepsilon X \to l_p \otimes \Delta_p Y$$

*is continuous.*

(2) *There is $c \ge 0$ such that for all n, all $n \times n$ matrices $(a_{k,l})$ and all $x_1, \ldots, x_n \in X$*

$$\left(\sum_l \left\| \sum_k a_{k,l} T(x_l) \right\|^p \right)^{1/p} \le c \left\| a : l_{q'}^n \to l_p^n \right\| w_{q'}(x_l).$$

(3) *There is a factorization*




*with $R \in \Pi_{q'}(X,E)$, and $U$ such that for all $A : l_{q'} \to l_p$*

$$A \otimes U : l_{q'} \otimes_{\Delta_{q'}} E \to l_p \otimes_{\Delta_p} F$$

*is continuous.*

*In this case*: $\sup_{\|A\| \le 1} \|A \otimes T\| = \inf c = \inf \left\{ \pi_{q'}(R) \sup_{\|A\| \le 1} \|A \otimes U\| \right\}.$

In terms of operator ideals (see [2,p. 374]) this reads as follows:

$$\left(L_q \circ I_{p'}\right)^* = L_{p,q}^{\text{injsur}} \circ \Pi_{q'}.$$

We finish looking at two subspaces of $M_n$: the subspace of all $n \times n$ diagonal matrices and the subspace of all regular $n \times n$ matrices (differences of two in the lattice sense positive operators).

**Example 3**. *For every operator $T: X \to Y$ the following are equivalent*:

(1) *For all diagonal operators $D_\lambda : l_{q'} \to l_p$*

$$D_\lambda \otimes T : l_{q'} \otimes_\varepsilon X \to l_p \otimes_\pi Y$$

*is continuous, or equivalently: there is $c \geq 0$ such that for all n and all $x_1,\dots,x_n \in X$, $y'_1,\dots,y'_n \in Y'$*

$$\left(\sum_k |y'_k(Tx_k)|^{r'}\right)^{1/r'} \leq c w_{q'}(x_k) w_{p'}(y'_k).$$

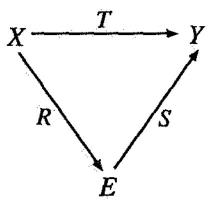
(2) *For all regular operators $A : l_{q'} \to l_p$*

$$A \otimes T : l_{q'} \otimes_\varepsilon X \to l_p \otimes_\pi Y$$

*is continuous, or equivalently: there is $c \geq 0$ such that for all n, all regular $n \times n$ matrices and all $x_1,\dots,x_n \in X$, $y'_1,\dots,y'_n \in Y'$*

$$\left|\sum_{k,l} a_{k,l} y'_k(Tx_l)\right| \leq c \left\| a : l^n_{q'} \to l^n_p \right\| w_{q'}(x_l) w_{p'}(y'_k).$$

(3) *There is a factorization*



$$R \in \Pi_{q'}(X,E)$$
$$S' \in \Pi_{p'}(Y',E')$$

*In this case*: $\sup_{\|\lambda\|_r \leq 1} \|D_\lambda \otimes T\| = \inf c = \inf\{\pi_{p'}(S')\pi_{q'}(R)\}$.

Operators as in (1) are known under the name $(q',p')$-dominated operators, and - as mentioned in the introduction - the equivalence (1) ⇔ (3) for $1/p+1/q=1$ is due to Kwapien [5], and in the general case $1/p+1/q>1$ to [9, 17.4.2] (see also [2, sec. 19]).

PROOF: For regular (in particular, diagonal) operators $A : l_{q'} \to l_p$

$$\left\| A \otimes U : l_{q'} \otimes_{\Delta_{q'}} E \to l_p \otimes_{\Delta_p} F \right\| = \|A\| \, \|U\|$$

(see e.g. [2, p.80]). Hence the proof of (1) ⇔ (3) is an easy modification of the proof given for example 1 (in (1) the equivalence of both statements follows by the closed graph theorem). Moreover, (3) implies both statements in (2) - use again the argument from the introduction. Finally, we remark that it is obvious that each of the statements in (2) implies (1). ■

Clearly, analoguous results hold if $w_{p'}$ is replaced by $w_{p',\sigma}$ and $w_{q'}$ by $w_{q',\nu}$ with $0 \leq \sigma,\nu < 1$ and $1 \leq p,q,r \leq \infty$ such that

$$\frac{1-\sigma}{p'} + \frac{1-\nu}{q'} + \frac{1}{r} = 1;$$

in this setting equivalence (1) ⇔ (3) of the preceding result was observed in [8].

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