Fuzzy random events and their corresponding conditional probability measures

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Abstract

The procedure of ordinary set splitting naturally makes possible to introduce the concept of a fuzzy random event. It is easy to calculate the a priori and conditional (ordinary condition) probabilities of a fuzzy random event. In the case of a fuzzy condition such calculations are not trivial. The present paper introduces a descriptive definition of such a kind of probability measure. Some properties of probability measures of fuzzy events are studied.

Keywords: probability measures, fuzzy sets.

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1. The splitting of an indicator

Let Ω be the universal set. Consider a subset $A \subseteq \Omega$ and its indicator $I_A \in \{0,1\}^\Omega$. Associate with it the pair $\left(I_{\tilde{A}}, I_{\tilde{A}^D}\right)$, where

$$I_{\tilde{A}}(\omega) = \mu_A(\omega) I_A(\omega), \quad I_{\tilde{A}^D}(\omega) = \left(1 - \mu_A(\omega)\right) I_A(\omega), \quad \omega \in \Omega$$

$$\mu_A : \Omega \rightarrow [0,1] \tag{1}$$

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Call such correspondence the splitting of indicator I_A . It is clear that $I_{\tilde{A}}, I_{\tilde{A}^D} \in [0,1]^{\Omega}$ and

$$I_A(\omega) = I_{\tilde{A}}(\omega) + I_{\tilde{A}^D}(\omega) \tag{2}$$

According to Zadeh [1] $I_{\tilde{A}}$ is the indicator of the fuzzy subset $\tilde{A}$. $I_{\tilde{A}^D}$ is considered as the indicator of the so called dual subset $\tilde{A}^D$. The relation between the dual subset and Zadeh's complement is evident:

$$I_{\neg \tilde{A}}(\omega) = 1 - I_{\tilde{A}}(\omega) = I_{A^c}(\omega) \vee I_{\tilde{A}^D}(\omega), \quad \omega \in \Omega \tag{3}$$

because for a given splitting of I_A the splitting of I_{A^c} is independent from $I_A, I_{A^c}(\omega) \cdot I_A(\omega) = 0, \ \omega \in \Omega$.

The splitting of the indicator of intersection $I_{A \cap B}$. Let for some $f, g \in [0,1]^{\Omega}$ we have:

$$I_A(\omega) \to \big(f(\omega) I_A(\omega), (1 - f(\omega)) I_A(\omega)\big)$$

and

$$I_B(\omega) \to \big(g(\omega) I_B(\omega), (1 - g(\omega)) I_B(\omega)\big), \ \omega \in \Omega \tag{4}$$

If for the splitted indicator one demands to fulfil the same natural condition as for nonsplited ones:

$$I_{\widetilde{A \cap B}}(\omega) \le I_{\tilde{A}}(\omega), \ I_{\tilde{B}}(\omega) \quad \omega \in \Omega \tag{5}$$

then, as it is easy to see for the splitted intersection indicator when (4) holds, the following two expressions are obtained:

$$I_{\widetilde{A \cap B}}(\omega) = \begin{cases} I_{\tilde{A}}(\omega) \wedge I_{\tilde{B}}(\omega) \\ I_{\tilde{A}}(\omega) \cdot I_{\tilde{B}}(\omega) \end{cases} \quad \omega \in \Omega \tag{6}$$

where $\wedge$ is the operation min and $\cdot$ the usual product. Representation

$$I_{A\cap B}(\omega)\to\left(f(\omega)g(\omega)I_A(\omega)I_B(\omega),\left(f(\omega)g(\omega)I_A(\omega)I_B(\omega)\right)^D\right), \tag{7}$$

where

$$\begin{aligned}\left(f(\omega)g(\omega)I_A(\omega)I_B(\omega)\right)^D &= f(\omega)\left(1-g(\omega)\right)I_A(\omega)I_B(\omega)+\\ &+\left(1-f(\omega)\right)g(\omega)I_A(\omega)I_B(\omega)+\\ &+\left(1-f(\omega)\right)\left(1-g(\omega)\right)I_A(\omega)I_B(\omega), \quad \omega\in\Omega\end{aligned} \tag{8}$$

We call the sequential splitting of intersection indicator. In fact, consider $I_{A\cap B}$ and split I_A, we obtain:

$$I_{A\cap B}(\omega)=\left(I_{\tilde{A}}(\omega)+I_{\tilde{A}^D}(\omega)\right)I_B(\omega)=\left(I_{\tilde{A}}(\omega)I_B(\omega)\right)+\left(I_{\tilde{A}^D}(\omega)I_B(\omega)\right).$$

Hence, in a given case

$$I_{\widetilde{A\cap B}}(\omega)=\left(I_{\tilde{A}}(\omega)I_B(\omega)\right)=f(\omega)I_{A\cap B}(\omega),$$

$$I_{\widetilde{A\cap B}^D}(\omega)=\left(I_{\tilde{A}^D}I_B(\omega)\right)=\left(1-f(\omega)\right)I_{A\cap B}(\omega)$$

Now repeat the splitting of $I_{\widetilde{A\cap B}}$ by the function $I_{\tilde{B}}(\omega)=g(\omega)I_B(\omega)$:

$$\begin{aligned}I_{\widetilde{A\cap B}}(\omega)&\to\left(g(\omega)I_B(\omega)I_{\widetilde{A\cap B}}(\omega),\ \left(1-g(\omega)\right)I_B(\omega)I_{\widetilde{A\cap B}}\right)=\\ &=\left(g(\omega)f(\omega)I_{A\cap B}(\omega)\right),\ \left(1-g(\omega)\right)f(\omega)I_{A\cap B}(\omega)\right).\end{aligned}$$

Similary,

$$I_{\widetilde{A\cap B}^D}(\omega)\to\left(g(\omega)\left(1-f(\omega)\right)I_{A\cap B}(\omega),\ \left(1-g(\omega)\right)\left(1-f(\omega)\right)I_{A\cap B}(\omega)\right)$$

Thus, splitting sequentially first I_A, and then I_B we can write

$$I_{A\cap B}(\omega)=\left(I_{\tilde{A}\tilde{\cap}\tilde{B}}(\omega),\ I_{\tilde{A}\tilde{\cap}\tilde{B}^D}(\omega)\right),$$

where

$$I_{\tilde{A}\tilde{\cap}\tilde{B}}(\omega)=I_{\tilde{A}}(\omega)\cdot I_{\tilde{B}}(\omega)=f(\omega)g(\omega)I_{A\cap B}(\omega),\quad \omega\in\Omega \tag{9}$$

and

$$I_{\tilde{A}\tilde{\cap}\tilde{B}^D}(\omega)=\left(I_{\tilde{A}}(\omega),I_{\tilde{B}}(\omega)\right)^D=$$

$$\left(I_{\tilde{A}}(\omega)I_{\tilde{B}^D}(\omega)+I_{\tilde{A}^D}(\omega)I_{\tilde{B}}(\omega)+I_{\tilde{A}^D}(\omega)I_{\tilde{B}^D}(\omega)\right)I_A(\omega)I_B(\omega)=$$

$$I_A(\omega)I_B(\omega)-I_{\tilde{A}}(\omega)I_{\tilde{B}}(\omega)=\left(1-f(\omega)g(\omega)\right)I_{A\cap B}(\omega),\quad \omega\in\Omega. \tag{10}$$

In contrast to (6) the representation

$$I_{A\cap B}(\omega)\to\left(I_{\tilde{A}}(\omega)\wedge I_{\tilde{B}}(\omega),\left(I_{\tilde{A}}(\omega)\wedge I_{\tilde{B}}(\omega)\right)^D\right)=$$

$$=\begin{cases}\left(f(\omega)I_{A\cap B}(\omega),\left(1-f(\omega)\right)I_{A\cap B}(\omega)\right), & f(\omega)\le g(\omega)\\ \left(g(\omega)I_{A\cap B}(\omega),\left(1-g(\omega)\right)I_{A\cap B}(\omega)\right), & g(\omega)\le f(\omega)\end{cases} \tag{11}$$

is called the simultaneous splitting of intersection indicator.

The splitting of indicator of a union $I_{A\cup B}$. We have:

$$I_{A\cup B}(\omega)=I_A(\omega)+I_B(\omega)-I_{A\cap B}(\omega),\quad \omega\in\Omega \tag{12'}$$

and

$$I_{\tilde{A}}(\omega)\vee I_{\tilde{B}}(\omega)+I_{\tilde{A}}(\omega)\wedge I_{\tilde{B}}(\omega)=I_{\tilde{A}}(\omega)+I_{\tilde{B}}(\omega), \tag{12''}$$

where $\vee$ is the max operation. If we demand, as in the case of intersection, to fulfil the natural condition:

$$I_{\tilde{A}\tilde{\cup}\tilde{B}}(\omega)\ge I_{\tilde{A}}(\omega),\ I_{\tilde{B}}(\omega),\quad \omega\in\Omega \tag{13}$$

then we get two possibilities again: simultaneous and sequential splitting of union indicator.

For simultaneous splitting:

$$I_{A\cup B}(\omega)\rightarrow\left(I_{\tilde{A}}(\omega)\vee I_{\tilde{B}}(\omega),\left(I_{\tilde{A}}(\omega)\vee I_{\tilde{B}}(\omega)\right)^{D}\right)=$$
$$=\left(f(\omega)I_A(\omega)\vee g(\omega)I_B(\omega),\left(f(\omega)I_A(\omega)\vee g(\omega)I_B(\omega)\right)^{D}\right),\ \omega\in\Omega;$$

and for sequential splitting:

$$I_{A\cup B}(\omega)\rightarrow\left(f(\omega)I_A(\omega)+g(\omega)I_B(\omega)-f(\omega)g(\omega)I_A(\omega)I_B(\omega),\right.$$
$$\left.\left(f(\omega)I_A(\omega)+g(\omega)I_B(\omega)-f(\omega)g(\omega)I_A(\omega)I_B(\omega)\right)^{D}\right),\ \omega\in\Omega.$$

Thus,

$$I_{\widetilde{A\cup B}}(\omega)=I_{\tilde{A}}(\omega)\vee I_{\tilde{B}}(\omega),\qquad \omega\in\Omega, \tag{14}$$

$$I_{\approx\atop{A\cup B}}(\omega)=I_{\tilde{A}}(\omega)+I_{\tilde{B}}(\omega)-I_{\tilde{A}}(\omega)I_{\tilde{B}}(\omega),\qquad \omega\in\Omega. \tag{15}$$

2. The lattice of splitted elements of ordinary indicators Boolean lattice

Consider the Boolean lattice $\mathbf{I}=\left(\{0,1\}^{\Omega},\vee,\wedge\right)$ with the natural ordering. The set of all splitted elements of this lattice with the natural ordering $\mathbf{I}^{\sim}=\left([0,1]^{\Omega},\vee,\wedge\right)$ is a lattice.

Theorem 2.1. $\mathbf{I}^{\sim}$ *is a Brouwer's lattice.*

The direct demonstration of this theorem (i.e., the demonstration that for any two elements $I_{\tilde{A}}$ and $I_{\tilde{B}}\in\mathbf{I}^{\sim}$ the set of all such $I_{\tilde{X}}\in\mathbf{I}^{\sim}$ that $I_{\tilde{A}}\wedge I_{\tilde{X}}\preceq I_{\tilde{B}}$ has the greatest elements $\left(I_{\tilde{B}}:I_{\tilde{A}}\right)$ called the relative pseudocomplement of $I_{\tilde{A}}$ in $I_{\tilde{B}}$) can be done according to [2]. It is easy to see that:

$$\left(I_{\tilde{B}}:I_{\tilde{A}}\right)(\omega)=\begin{cases}1, & I_{\tilde{A}}(\omega)\leq I_{\tilde{B}}(\omega)\\ I_{A^c}(\omega)\vee I_{\tilde{B}}(\omega), & I_{\tilde{A}}>I_{\tilde{B}}(\omega)\end{cases},\omega\in\Omega \tag{16}$$

where $I_{A^c}=\left(I_{\varnothing},I_{\tilde{A}}\right)$ is a pseudocomplement of $I_{\tilde{A}}$ and, as a function of ω, represents the indicator of the complement of the set A in Ω. Next two theorems are simply proved.

Theorem 2.2. *The following statements hold in the lattice* $\mathbb{I}^{\sim}$:

$$\begin{aligned}
&(i) && \text{If } I_{\tilde{A}}\preccurlyeq I_{\tilde{B}},\ \text{then } \left(I_{\varnothing}:I_{\tilde{B}}\right)\preccurlyeq\left(I_{\varnothing}:I_{\tilde{A}}\right),\\
&(ii) && I_{\tilde{A}}\preccurlyeq\left(I_{\varnothing}:\left(I_{\varnothing}:I_{\tilde{A}}\right)\right),\\
&(iii) && \left(I_{\varnothing}:I_{\tilde{A}}\right)=\left(I_{\varnothing}:\left(I_{\varnothing}:\left(I_{\varnothing}:I_{\tilde{A}}\right)\right)\right),\\
&(iv) && \left(I_{\varnothing}:\left(I_{\tilde{A}}\vee I_{\tilde{B}}\right)\right)=\left(I_{\varnothing}:I_{\tilde{A}}\right)\wedge\left(I_{\varnothing}:I_{\tilde{B}}\right),\\
&(v) && \left(I_{\varnothing}:\left(I_{\tilde{A}}\wedge I_{\tilde{B}}\right)\right)=\left(I_{\varnothing}:I_{\tilde{A}}\right)\vee\left(I_{\varnothing}:I_{\tilde{B}}\right).
\end{aligned} \tag{17}$$

Theorem 2.3. *The following statements hold in the lattice* $\mathbb{I}^{\sim}$:

$$\begin{aligned}
&(i) && \left(I_{\tilde{A}}:I_{\tilde{B}}\right)\wedge I_{\tilde{A}}=I_{\tilde{A}},\\
&(ii) && \left(I_{\tilde{A}}:I_{\tilde{B}}\right)\wedge I_{\tilde{B}}=I_{\tilde{A}}\wedge I_{\tilde{B}},\\
&(iii) && \left(\left(I_{\tilde{A}}:I_{\tilde{B}}\right):I_{\tilde{C}}\right)=\left(I_{\tilde{A}}:I_{\tilde{C}}\right)\wedge\left(I_{\tilde{B}}:I_{\tilde{C}}\right),\\
&(iv) && \left(I_{\tilde{A}}:\left(I_{\tilde{B}}\vee I_{\tilde{C}}\right)\right)=\left(I_{\tilde{A}}:I_{\tilde{B}}\right)\wedge\left(I_{\tilde{A}}:I_{\tilde{C}}\right).
\end{aligned} \tag{18}$$

3. The splitting of a set

The splitting of a set, which corresponds to the indicator splitting, as we actually have seen, is represented as:

$$\begin{aligned}
&\left(I_A\rightarrow\left(I_{\tilde{A}},I_{\tilde{A}^D}\right)\right)\leftrightarrows\left(A\rightarrow\left(\tilde{A},\tilde{A}^D\right)\right),\\
&\left(I_A=I_{\tilde{A}}+I_{\tilde{A}^D}\right)\leftrightarrows\left(A=\tilde{A}\oplus\tilde{A}^D\right).
\end{aligned} \tag{19}$$

Here $\oplus$ is the operation of set synthesis.

On the basis of (19) we can obtain a more general expression $\tilde{A} \oplus \tilde{B}$ which, obviously, will make sense under the condition that $\tilde{B} \subseteq \rceil \tilde{A}$, or $\tilde{A} \subseteq \rceil \tilde{B}$. We can obtain also the existence conditions for expressions $\tilde{A} \oplus \tilde{B} \oplus \tilde{C}$, etc.

Considering that such condition holds for expressions above, we can easily prove that

$$
\begin{aligned}
&(i)\quad \tilde{A} \oplus \tilde{B} = \tilde{B} \oplus \tilde{A},\\
&(ii)\quad \tilde{A} \oplus \left(\tilde{B} \oplus \tilde{C}\right) = \left(\tilde{A} \oplus \tilde{B}\right) \oplus \tilde{C},\\
&(iii)\quad \left(\tilde{A} \oplus \tilde{A}^D\right) \cap \left(\tilde{B} \oplus \tilde{B}^D\right) = (\widetilde{A \cap B}) \oplus (\widetilde{A \cap B})^D =\\
&\qquad = \left(\tilde{A} \cap \tilde{B}\right) \oplus \left[\left(A \cap \tilde{B}^D\right) \cup \left(\tilde{A}^D \cap B\right)\right],\\
&(iv)\quad \left(\tilde{A} \oplus \tilde{A}^D\right) \cup \left(\tilde{B} \oplus \tilde{B}^D\right) = (\widetilde{A \cup B}) \oplus (\widetilde{A \cup B})^D =\\
&\qquad = \left(\tilde{A} \cup \tilde{B}\right) \oplus \left[\left(\tilde{A}^D \cap \tilde{B}^D\right) \cup \left(A^C \cap \tilde{B}^D\right) \cup \left(\tilde{A}^D \cap B^C\right)\right],\\
&(v)\quad \tilde{A} \oplus \left(\tilde{B} \cap \tilde{C}\right) = \left(\tilde{A} \oplus \tilde{B}\right) \cap \left(\tilde{A} \oplus \tilde{C}\right),\\
&(vi)\quad \tilde{A} \oplus \left(\tilde{B} \cup \tilde{C}\right) = \left(\tilde{A} \oplus \tilde{B}\right) \cup \left(\tilde{A} \oplus \tilde{C}\right).
\end{aligned}
\tag{20}
$$

For example, to prove the last two formulas

$$
\begin{aligned}
&(v)\quad \tilde{A} \oplus \left(\tilde{B} \cap \tilde{C}\right) \leftrightarrows I_{\tilde{A}} + \left(I_{\tilde{B}} \wedge I_{\tilde{C}}\right) = \left(I_{\tilde{A}} + I_{\tilde{B}}\right) \wedge \left(I_{\tilde{A}} + I_{\tilde{C}}\right) \leftrightarrows\\
&\qquad \leftrightarrows \left(\tilde{A} \oplus \tilde{B}\right) \cap \left(\tilde{A} \oplus \tilde{C}\right).\\
&(vi)\quad \tilde{A} \oplus \left(\tilde{B} \cup \tilde{C}\right) \leftrightarrows I_{\tilde{A}} + \left(I_{\tilde{B}} \vee I_{\tilde{C}}\right) = \left(I_{\tilde{A}} + I_{\tilde{B}}\right) \vee \left(I_{\tilde{A}} + I_{\tilde{C}}\right) \leftrightarrows\\
&\qquad \leftrightarrows \left(\tilde{A} \oplus \tilde{B}\right) \cup \left(\tilde{A} \oplus \tilde{B}\right).
\end{aligned}
$$

We asume that in these formulas the following relations hold.

$$
\begin{aligned}
&\left(I_{\widetilde{A \cap B}} = I_{\tilde{A}} \wedge I_{\tilde{B}}\right) \leftrightarrows \left(\widetilde{A \cap B} = \tilde{A} \cap \tilde{B}\right),\\
&\left(I_{\widetilde{A \cup B}} = I_{\tilde{A}} \vee I_{\tilde{B}}\right) \leftrightarrows \left(\widetilde{A \cup B} = \tilde{A} \cup \tilde{B}\right).
\end{aligned}
\tag{21}
$$

which, because of (6), (14) and (19), are evident.

In the lattice of splitted sets almost all rules of Boolean lattice hold: 1) reflexivity, 2) antisymmetry, 3) transitivity, 4) idempotency, 5) commutativity, 6) associativity, 7) distributivity, 8) the annihilation law, 9) the involution law for fuzzy complement, 10) the indentity laws, 11) the order inversion laws, 12) De Morgan's laws.

In conection with the introduced notion of dual subsets we can prove the following laws: 13) involution law for the dual subset:

$$\left(\tilde{A}^D\right)^D = \tilde{A}. \tag{22}$$

14) Duality laws for union and intersection of splitted subsets:

$$\begin{gathered}\left(\tilde{A}\cup\tilde{B}\right)^D = \left(\tilde{A}^D\cap\tilde{B}^D\right)\cup\left(A^C\cap\tilde{B}^D\right)\cup\left(B^C\cap\tilde{A}^D\right)\\ \left(\tilde{A}\cap\tilde{B}\right)^D = \left(A\cap\tilde{B}^D\right)\cup\left(\tilde{A}^D\cap B\right)\end{gathered} \tag{23}$$

For example, to prove law 14). For the demonstration of the first law we can write:

$$\begin{gathered}\rceil\left(\tilde{A}\cup\tilde{B}\right) = (A\cup B)^C\cup\left(\tilde{A}\cup\tilde{B}\right)^D = \left(\tilde{A}\cup\tilde{B}\right)^D\cup\left(A^C\cap B^C\right) = \\ = \left(\tilde{A}\cup\tilde{B}\right)^D\oplus\left(A^C\cap B^C\right).\end{gathered}$$

On the other hand, according to (12)

$$\begin{gathered}\rceil\left(\tilde{A}\cap\tilde{B}\right) = \rceil\tilde{A}\cap\rceil\tilde{B} = \left(A^C\cup\tilde{A}^D\right)\cap\left(B^C\cup\tilde{B}^D\right) = \\ = \left(A^C\cap B^C\right)\cup\left(A^C\cap\tilde{B}^D\right)\cup\left(\tilde{A}^D\cap B^C\right)\cup\left(\tilde{A}^D\cap\tilde{B}^D\right) = \\ = \left(A^C\cap B^C\right)\oplus\left[\left(A^C\cap\tilde{B}^D\right)\cup\left(\tilde{A}^D\cap B^C\right)\cup\left(\tilde{A}^D\cap\tilde{B}^D\right)\right].\end{gathered}$$

Comparing these expressions we obtain the required proof. Now to prove the second 14). We have:

$$(\urcorner(\tilde{A}\cap\tilde{B}))\cap(A\cap B)=(\urcorner\tilde{A}\cup\urcorner\tilde{B})\cap(A\cap B)=$$
$$=\left(\left(A^C\cup\tilde{A}^D\right)\cup\left(B^C\cup\tilde{B}^D\right)\right)\cap(A\cap B)=$$
$$=\left(A^C\cap(A\cap B)\right)\cup\left(\tilde{A}^D\cap(A\cap B)\right)\cup\left(B^C\cap(A\cap B)\right)\cup\left(\tilde{B}^D\cap(A\cap B)\right)=$$
$$=\left(A\cap\tilde{B}^D\right)\cup\left(\tilde{A}^D\cap B\right).$$

In section 2 we considered some examples of the indicator splitting. They also are examples of splitting. They also are examples of splitting of the corresponding sets. Consider other examples.

Splitting of the set difference $A\setminus B$. Let Ω be the universal set, $A,B\subseteq\Omega$. The equality $A\setminus B=A\cap B^C$ holds. If we split the subsets A and B, then the splitting of this equality, according to (6), will be:

$$A\widetilde{\setminus}B=A\widetilde{\cap}B^C=\tilde{A}\cap B^C=\tilde{A}\cap\left(\varnothing:\tilde{B}\right). \tag{24}$$

For the splitting of the symmetric difference $A\Delta B$, we have:

$$A\Delta B=(A\setminus B)\cup(B\setminus A),$$
$$A\widetilde{\Delta}B=(A\widetilde{\setminus}B)\cup(B\widetilde{\setminus}A)=\left(\tilde{A}\cap B^C\right)\cup\left(\tilde{B}\cap A^C\right). \tag{25}$$

On the other hand

$$A\Delta B=(A\cup B)\setminus(A\cap B).$$

Thus for the splitted symmetric difference we also have the formula:

$$(A\widetilde{\Delta}B)=(A\widetilde{\cup}B)\cap(A\cap B)^C=\left(\tilde{A}\cup\tilde{B}\right)\cap\left(A^C\cap B^C\right). \tag{26}$$

According to (25):

$$\left(\tilde{A}\cap B^C\right)\cup\left(\tilde{B}\cap A^C\right)=\left(\tilde{A}\cup\tilde{B}\right)\cap\left(A^C\cup B^C\right). \tag{27}$$

Actually, taking into account law 7), we have:

$$\left(\tilde{A}\cup\tilde{B}\right)\cap\left(A^{C}\cup B^{C}\right)=\left(\tilde{A}\cap\left(A^{C}\cup B^{C}\right)\right)\cup\left(\tilde{B}\cap\left(A^{C}\cup B^{C}\right)\right)=$$
$$=\left(\left(A^{C}\cap\tilde{A}\right)\cup\left(\tilde{A}\cap B^{C}\right)\right)\cup\left(\left(\tilde{B}\cap A^{C}\right)\cup\left(\tilde{B}\cap B^{C}\right)\right)=$$
$$=\left(\tilde{A}\cap B^{C}\right)\cup\left(\tilde{B}\cap A^{C}\right).$$

(25) and (26) can be rewritten as:

$$A\tilde{\Delta}B=\left(\tilde{A}\cap\left(\varnothing:\tilde{B}\right)\right)\cup\left(\tilde{B}\cap\left(\varnothing:\tilde{A}\right)\right)=$$
$$=\left(\tilde{A}\cup\tilde{B}\right)\cap\left(\left(\varnothing:\tilde{A}\right)\cup\left(\varnothing:\tilde{B}\right)\right). \tag{28}$$

Let Ω be the universal set and $\Omega\supseteq\Lambda_1\supseteq\Lambda_2\supseteq\ldots$ By the equality:

$$(\Lambda_1\setminus\Lambda_2)\cup(\Lambda_2\setminus\Lambda_3)\cup\ldots=\Lambda_1\setminus\bigcap_{j=1}^{\infty}\Lambda_j; \tag{29}$$

the splitting of $\Lambda_1,\Lambda_2,\ldots$ according to above example lead to the equality:

$$\left(\tilde{\Lambda}_1\cap\Lambda_2^C\right)\cup\left(\tilde{\Lambda}_2\cap\Lambda_3^C\right)\cup\ldots=\tilde{\Lambda}_1\cap\left(\bigcap_{j=1}^{\infty}\Lambda_j\right)^C=$$
$$=\tilde{\Lambda}_1\cap\left(\bigcup_{j=1}^{\infty}\Lambda_j^C\right)=\tilde{\Lambda}_1\cap\left(\bigcup_{j=1}^{\infty}\left(\varnothing:\tilde{\Lambda}_j\right)\right) \tag{30}$$

Under the condition that $\forall I_{\tilde{\Lambda}_j}$ is a narrowing of $I_{\tilde{\Lambda}_1}$ on corresponding Λ_j, i.e., $\tilde{\Lambda}_j=\tilde{\Lambda}_1\cap\tilde{\Lambda}_j$.

Let again Ω be the universal set but now $\Lambda_1\subseteq\Lambda_2\subseteq\ldots\subseteq\Omega$. As it is well known

$$\bigcup_{j=1}^{\infty}\Lambda_j=\Lambda_1\cup(\Lambda_2\setminus\Lambda_1)\cup(\Lambda_3\setminus\Lambda_2)\cup\ldots \tag{31}$$

The splitting of $\Lambda_1,\Lambda_2,\ldots$ leads to formula:

$$\widetilde{\bigcup_{j=1}^{\infty}} \Lambda_j = \bigcup_{j=1}^{\infty} \tilde{\Lambda}_j = \tilde{\Lambda}_1 \cup \left(\tilde{\Lambda}_2 \cap \Lambda_1^C\right) \cup \left(\tilde{\Lambda}_3 \cap \Lambda_2^C\right) \cup \ldots =$$

$$= \tilde{A}_1 \cup \left(\tilde{\Lambda}_2 \cap \left(\varnothing : \tilde{\Lambda}_1\right)\right) \cup \left(\tilde{\Lambda}_3 \cap \left(\varnothing : \tilde{\Lambda}_2\right)\right) \cup \ldots \tag{32}$$

On condition that $\tilde{\Lambda}_j \supseteq \tilde{\Lambda}_{j+1} \cap \Lambda_j$.

Splitting of the element of the universal set (fuzzy point).

$$\omega_o \to \left(\tilde{\omega}_o, \tilde{\omega}_o{}^D\right), \quad \omega_o = \tilde{\omega}_o \oplus \tilde{\omega}_o{}^D \tag{33}$$

where

$$\tilde{\omega}_o \leftrightarrows I_{\{\tilde{\omega}_o\}}(\omega) = \begin{cases} a, & \omega = \omega_0 \\ & \qquad \omega_0 \in \Omega,\ a \in [0,1] \\ 0, & \omega \neq \omega_0 \end{cases} \tag{34}$$

Theorem 3.1. *Let Ω be the universal set, $\omega_o \in \Omega$ and $\tilde{\omega}_o$ is a corresponding splitted point, then the splitting of the universal set determined by the splitting of the ω_o point will be the relative pseudocomplement of $\tilde{\omega}_o{}^D$ in $\tilde{\omega}_o$:*

$$\tilde{\Omega} = \left(\tilde{\omega}_o, \tilde{\omega}_o{}^D\right). \tag{35}$$

Proof.

$$I_{\tilde{\Omega}} = I_{\{\tilde{\omega}_o\}} \vee I_{\{\omega_o\}^C} = I_{\neg\{\tilde{\omega}_0\}^D} \leftrightarrows \neg\tilde{\omega}_o{}^D = \left(\tilde{\omega}_o : \tilde{\omega}_o{}^D\right).$$

4. The probability measure splitting

Let $\left(\Omega, \mathcal{B}, p(\ )\right)$ be a given probability space. The probability of the event $\Lambda \in \mathcal{B}$ is calculated from the formula:

$$p(\Lambda) = \int_{\Omega} I_{\Lambda}(\omega) p(d\omega) \tag{36}$$

According to splitting procedure of the set Λ, rewrite this formula in the form:

$$p\left(\tilde{\Lambda} \oplus \tilde{\Lambda}^D\right) = \int_\Omega I_{\tilde{\Lambda}}(\omega)\, p(d\omega) + \int_\Omega I_{\tilde{\Lambda}^D}(\omega) p(d\omega), \tag{37}$$

where $I_{\tilde{\Lambda}}$ is a $\mathcal{B}$-measurable membership function (the corresponding subset $\tilde{\Lambda}$ is a fuzzy random event). Define $p\left(\tilde{\Lambda}\right)$ and $p\left(\tilde{\Lambda}^D\right)$ as follows

$$p\left(\tilde{\Lambda}\right) = \int_\Omega I_{\tilde{\Lambda}}(\omega) p(d\omega) \quad \text{and} \quad p\left(\tilde{\Lambda}^D\right) = \int_\Omega I_{\tilde{\Lambda}^D}(\omega) p(d\omega); \tag{38}$$

the probability of fuzzy event $\tilde{\Lambda}$ and the probability of dual fuzzy event $\tilde{\Lambda}^D$, correspondingly. Call the representation

$$p(\Lambda) = p\left(\tilde{\Lambda} \oplus \tilde{\Lambda}^D\right) = p\left(\tilde{\Lambda}\right) + p\left(\tilde{\Lambda}^D\right) \tag{39}$$

the procedure of probability measure splitting.

The main property of additivity

$$p\left(\bigcup_{j=1}^{\infty} \Lambda_j\right) = \sum_{j=1}^{\infty} p\left(\Lambda_j\right), \quad \Lambda_i \cap \Lambda_k = \varnothing, \quad i \neq k$$

can be splitted in the following way: the left hand side is

$$p\left(\bigcup_{j=1}^{\infty} \left(\tilde{\Lambda}_j \oplus \tilde{\Lambda}_j^D\right)\right) = \int_\Omega I_{\bigcup_{j=1}^{\infty}\left(\tilde{\Lambda}_j \oplus \tilde{\Lambda}_j^D\right)}(\omega) p(d\omega).$$

Because Λ_j are not intersected, then

$$\int_{\Omega} I_{\bigcup_{j=1}^{\infty}\left(\tilde{\Lambda}_j \oplus \tilde{\Lambda}_j{}^D\right)}(\omega) p(d\omega) = \sum_{j=1}^{\infty} \int_{\Omega} I_{\tilde{\Lambda}_j}(\omega) p(d\omega) +$$

$$+ \sum_{j=1}^{\infty} \int_{\Omega} I_{\tilde{\Lambda}_j{}^D}(\omega) p(d\omega) = \sum_{j=1}^{\infty} p\left(\tilde{\Lambda}_j\right) + \sum_{j=1}^{\infty} p\left(\tilde{\Lambda}_j{}^D\right). \quad (40')$$

And the right hand side

$$\sum_{j=1}^{\infty} p\left(\Lambda_j\right) = \sum_{j=1}^{\infty} p\left(\tilde{\Lambda}_j \oplus \tilde{\Lambda}_j^D\right) = \sum_{j=1}^{\infty} p\left(\tilde{\Lambda}_j\right) + \sum_{j=1}^{\infty} p\left(\tilde{\Lambda}_j^D\right) \quad (40'')$$

Finally, the property of additivity for the splitted events is written in the form:

$$p\left(\widetilde{\bigcup_{j=1}^{\infty} \Lambda_j}\right) = p\left(\sum_{j=1}^{\infty} \oplus \tilde{\Lambda}_j\right) = \sum_{j=1}^{\infty} p\left(\tilde{\Lambda}_j\right), \text{ or}$$

$$p\left(\bigcup_{j=1}^{\infty} \tilde{\Lambda}_j\right) = \sum_{j=1}^{\infty} p\left(\tilde{\Lambda}_j\right), \quad \Lambda_i \cap \Lambda_k = \varnothing, \quad i \neq k. \quad (41)$$

Thus, the denumerable additivity property holds for fuzzy subsets also. Similarly,

$$p\left(\left(\bigcup_{j=1}^{\infty} \tilde{\Lambda}_j\right)^D\right) = p\left(\bigcup_{j=1}^{\infty} \tilde{\Lambda}_j^D\right). \quad (42)$$

Actually, if $\tilde{A} \cap \tilde{B} = \varnothing$, then $\tilde{A}^D \cap \tilde{B}^D = \varnothing$ also, $\tilde{B}^D \subseteq A^C, \tilde{A}^D \subseteq B^C$. From these relations it follows that for nonintersecting subsets the law (23) can be represented as:

$$\left(\tilde{A} \cup \tilde{B}\right)^D = \tilde{A}^D \cup \tilde{B}^D. \quad (23')$$

In general, for any finite n

$$\left(\tilde{A}_1 \cup \ldots \cup \tilde{A}_n\right)^D = \tilde{A}_1^D \cup \ldots \cup \tilde{A}_n^D. \quad (43)$$

From $\Omega = \tilde{\Lambda} \oplus \rceil\tilde{\Lambda}$, where $\Lambda \subseteq \Omega$ is an arbitrary subset, it immediatelly follows:

$$p(\Omega) = p(\tilde{\Lambda}) + p(\rceil\tilde{\Lambda})$$
$$p(\rceil\tilde{\Lambda}) = 1 - p(\tilde{\Lambda}). \tag{44}$$

Further, because $(\varnothing : \tilde{A}) = A^C$, it is evident that:

$$p\left(\left(\varnothing : \tilde{A}\right)\right) = 1 - p(A) \tag{45}$$

From properties of Lebesgue-Stietjes integral the properties of the splitted probability measures follow:

1.- monotony: $\tilde{A} \subseteq \tilde{B} \Rightarrow p(\tilde{A}) \le p(\tilde{B})$,

2.- continuity with respect to monotonic sequences:

$$\begin{aligned} &\tilde{A}_n \uparrow \tilde{A} \Rightarrow p(\tilde{A}_n) \to p(\tilde{A}), \\ &\tilde{A}_n \downarrow \tilde{A} \Rightarrow p(\tilde{A}_n) \to p(\tilde{A}), \end{aligned} \tag{46}$$

3.- strong additivity:

$$p(\tilde{A} \cup \tilde{B}) = p(\tilde{A}) + p(\tilde{B}) - p(\tilde{A} \cap \tilde{B}),$$

4.- σ -semiadditivity:

$$p\left(\bigcup_{j=1}^{\infty} \tilde{A}_j\right) \le \sum_{j=1}^{\infty} p(\tilde{A}_j). \tag{47}$$

3 follows from the equality

$$\left(I_{\tilde{A}} \vee I_{\tilde{B}}\right)(\omega) + \left(I_{\tilde{A}} \wedge I_{\tilde{B}}\right)(\omega) = I_{\tilde{A}}(\omega) + I_{\tilde{B}}(\omega), \qquad \omega \in \Omega \tag{48}$$

and 4 from the inequality

$$\left(\bigvee_{j=1}^{\infty} I_{\tilde{A}_j}\right)(\omega) \leq \sum_{j=1}^{\infty} I_{\tilde{A}_j}(\omega). \tag{49}$$

5. Conditional probability (nonsplitted conditions)

The condition expressed by words "for a given event B" means that the initial probability space $(\Omega, \mathcal{B}, p())$ is replaced by the probability space $(\Omega, \mathcal{B}, p_B())$. As it is known, the conditional probability of some event A is the conditional mathematical expectation of indicator I_A :

$$E_B(A) = p_B(A) = \frac{1}{p(B)} \int_B I_A(\omega) p(d\omega) = \frac{1}{p(B)} \int_\Omega (I_A \wedge I_B)(\omega) p(d\omega). \tag{50}$$

In [3] this quantity is interpreted as the value of function

$$\varphi^{E_B(I_A)}(\omega) = E_B(I_A) I_B(\omega) + E_{B^c}(I_A) I_{B^c}(\omega), \quad \omega \in \Omega. \tag{51}$$

More generally, if $\mathcal{B}$ is the denumerable partition of Ω and $\mathcal{A}$ is the minimal σ-field induced by this partition, then the $\mathcal{A}$-measurable function

$$p^{\mathcal{A}}(\Lambda) = \sum_j \left(\frac{1}{p(B_j)} \int_{B_j} I_\Lambda(\omega) p(d\omega) \right) I_{B_j}(\omega), \quad \omega \in \Omega \quad \Lambda \in \mathcal{B} \tag{52}$$

is a value of the conditional probability for a given σ-field $\mathcal{A}$. The procedure of splitting of the indicator I_Λ leads to the notion of the conditional probability of fuzzy event $\tilde{\Lambda}$ for a given (nonsplitted) σ-field $\mathcal{A}$:

$$p^{\mathcal{A}}(\tilde{\Lambda}) = \sum_j \left(\frac{1}{p(B_j)} \int_{B_j} I_{\tilde{\Lambda}}(\omega) p(d\omega) \right) I_{B_j}(\omega), \tag{53}$$

and

$$p^{\mathcal{A}}\left(\tilde{\Lambda}^D\right)=\sum_j\left(\frac{1}{p\left(B_j\right)}\int_{B_j} I_{\tilde{\Lambda}_j^D}(\omega)p(d\omega)\right)I_{B_j}(\omega)=$$

$$=\sum_j\left(\frac{1}{p\left(B_j\right)}\int_{B_j} I_{\Lambda}(\omega)p(d\omega)\right)I_{B_j}(\omega)-\sum_j\left(\frac{1}{p\left(B_j\right)}\int_{B_j} I_{\tilde{\Lambda}}(\omega)p(d\omega)\right)I_{B_j}(\omega)=$$

$$=p^{\mathcal{A}}(\Lambda)-p^{\mathcal{A}}\left(\tilde{\Lambda}\right) \tag{54}$$

The rule of splitting:

$$p^{\mathcal{A}}(\Lambda)=p^{\mathcal{A}}\left(\tilde{\Lambda}\oplus\tilde{\Lambda}^D\right)=p^{\mathcal{A}}\left(\tilde{\Lambda}\right)+p^{\mathcal{A}}\left(\tilde{\Lambda}^D\right) \tag{55}$$

Formula (53) defines the conditional expectation for any nonnull crisp event $B\in\mathcal{A}$. Actually, B may be represented as a union over subclass of $\{B_j\}$:

$$B=\bigcup_j{}'B_j\ ,$$

and, according to (50), we may write:

$$p(B)E_B\left(I_{\tilde{A}}\right)=\int_{\bigcup_j{}'B_j} I_{\tilde{A}}(\omega)\,p(d\omega)=\sum_j{}'\int_{B_j} I_{\tilde{A}}(\omega)p(d\omega)=$$

$$=\sum_j{}'p\left(B_j\right)E_{B_j}\left(I_{\tilde{A}}\right). \tag{56}$$

We see that if $p^{\mathcal{A}}$ is known, then $E_B\left(I_{\tilde{A}}\right)$ can be evaluated.

Let $p_{\mathcal{A}}$ be a narrowing of P on $\mathcal{A}$, which is determined by the formula:

$$p_{\mathcal{A}}(B)=p(B),\qquad B\in\mathcal{A}.$$

Then, the right hand side of (50) can be represented as:

$$\sum_j{}'E_{B_j}\left(I_{\tilde{A}}\right)p\left(B_j\right)=\sum_j{}'E_{B_j}\left(I_{\tilde{A}}\right)\int_{B_j}p(d\omega)=$$

$$= \int_{\Omega} I_B(\omega) \sum_j{}' \left(E_{B_j}\left(I_{\tilde{A}}\right)\right) I_{B_j}(\omega) p(d\omega) = \int_B E^{\mathcal{A}}\left(I_{\tilde{A}}\right) dp_{\mathcal{A}} \,. \quad (57)$$

The left hand side is equal to $\int_B I_{\tilde{A}}(\omega) p(d\omega)$. We obtained the descriptive definition [3]. The conditional expectation $E^{\mathcal{A}}\left(I_{\tilde{A}}\right)$ for a given $\mathcal{A}$ (conditional probability of $\tilde{\mathcal{A}}$ for a given $\mathcal{A}$) is the $\mathcal{A}$-measurable function whose indefinite integral with respect to $p_{\mathcal{A}}$ is a narrowing on $\mathcal{A}$ of the definite integral of $I_{\tilde{A}}$ with respect to p:

$$\int_B E^{\mathcal{A}}(I_{\tilde{A}}) dp_{\mathcal{A}} = \int_B I_{\tilde{A}}(\omega) p(d\omega) \quad (58)$$

It is easy to see that

1.- $E\left(\left(E^{\mathcal{A}}\right)\right) = E\left(I_{\tilde{A}}\right) = p\left(\tilde{A}\right)$,

2.- If $\mathcal{A} = \mathcal{B}$, or $I_{\tilde{A}}$ is a $\mathcal{A}$-measurable, then $E^{\mathcal{A}}\left(I_{\tilde{A}}\right) = I_{\tilde{A}}$ a.s.

Note that (58) , makes sense also for non-denumerable partitions [3].

6. Conditional probability (splitted condition)

The constructive definition of $E_B(I_A)$ is such that the direct application of the splitting procedure for obtaining of conditional probability in the case of fuzzy condition is impossible. However, as it will be seen bellow, one can obtain a formula similar to (50) in the case of fuzzy condition (splitted condition). For this purpose when splitting the corresponding measure, we must retain some features of this formula. Let proceed from the notion of the mathematical expectation of a random event indicator for a given function [3]. If for such function take a function corresponding to the fuzzy condition (membership function of fuzzy condition) and then perform the convenient splitting, we can obtain a reasonable measure which has almost all basic properties of ordinary conditional probability.

Let $I_{\tilde{A}}$ induce the denumerable partition of

$\Omega\left(\bigcup_j A_j = \Omega,\ A_j \bigcap A_i = \varnothing,\ i \neq j\right)$. In this case

$$p^{I_{\tilde{A}}}(I_\Lambda)=\sum_j p_{A_j}(\Lambda)I_{A_j}(\omega)+p_{A^c}(\Lambda)I_{A^c}(\omega),\qquad \omega\in\Omega \tag{59}$$

Thus, for a function of conditional probability in the case of fuzzy condition we can take the expression:

$$p^{\tilde{A}}(I_\Lambda)=\sum_j \alpha_j p_{A_j}(\Lambda)I_{A_j}(\omega), \tag{60}$$

where the numbers

$$\alpha_j=\frac{1}{p\left(A_j\right)}\int_\Omega I_{\tilde{A}}(\omega)I_{A_j}(\omega)p(d\omega), \tag{61}$$

A similar expression is obtained for $p^{\tilde{A}^D}(I_\Lambda)$. It is clear that

$$\int_A p^{\tilde{A}}(I_\Lambda)p(d\omega)=p\left(\tilde{A}\cap\Lambda\right) \text{ and } \int_A p^{\tilde{A}^D}(I_\Lambda)p(d\omega)=p\left(\tilde{A}^D\cap\Lambda\right) \tag{62}$$

Now consider any measurable indicator $I_{\tilde{A}}$. If, for any natural n, we define the function

$$I_{\tilde{A}}^{(n)}(\omega)=\sum_{n=0}^{\infty}\frac{k}{2^n}I_{\left\{\frac{k}{2^n}\le I_{\tilde{A}}(\omega)<\frac{k+1}{2^n}\right\}}(\omega) \tag{63}$$

then the sequence $\left\{I_{\tilde{A}}^{(n)}(\omega)\right\}\uparrow$ and in $\forall\omega\in\Omega\to I_{\tilde{A}}(\omega)$. We have

$$p^{\tilde{A}^{(n)}}(I_\Lambda)=\sum_{k=0}^{\infty}p_{\left\{\frac{k}{2^n}\le I_{\tilde{A}}(\omega)<\frac{k+1}{2^n}\right\}}(\Lambda)\frac{k}{2^n}I_{\left\{\frac{k}{2^n}\le I_{\tilde{A}}(\omega)<\frac{k+1}{2^n}\right\}}(\omega) \tag{64}$$

and

$$\int_A p^{\tilde{A}^{(n)}}(I_\Lambda)p(d\omega)=\sum_{k=0}^{\infty}\frac{k}{2^n}p\left(\left\{\frac{k}{2^n}\le I_{\tilde{A}}(\omega)<\frac{k+1}{2^n}\right\}\cap\Lambda\right) \tag{65}$$

It is clear that

$$\lim_{n\to\infty}\sum_{k=0}^{\infty}\frac{k}{2^n}p\left(\left\{\frac{k}{2^n}\le I_{\tilde{A}}(\omega)<\frac{k+1}{2^n}\right\}\cap\Lambda\right)=p\left(\tilde{A}\cap\Lambda\right) \tag{66}$$

These definitions make sense because of the generalized Radon-Nikodym's theorem [3] . Comparing formulas above we conclude that $p^{\tilde{A}}$ can be considered as the conditional probability function in the case of the measurable fuzzy condition. All considered formulas were related with measures of crisp events in the case of fuzzy condition. If, in right hand side of basic formula (65), we perform the splitting of indicator I_Λ, then we obtain the definition of $p^{\tilde{A}}\left(I_{\tilde{\Lambda}}\right)$:

$$\int_A p^{\tilde{A}}\left(I_{\tilde{\Lambda}}\right)dp_{\tilde{A}} = \int_\Omega I_{\tilde{\Lambda}}(\omega)I_{\tilde{A}}(\omega)p(d\omega) = p(A\widetilde{\widetilde{\cap}}\Lambda) \tag{67}$$

where $A\widetilde{\widetilde{\cup}}\Lambda = \tilde{A}\oplus\tilde{\Lambda}$ (see (9)).

The version of conditional probability in the case of fuzzy condition almost surely has all properties of ordinary probabilities except the condition $p^{\tilde{A}}(I_\Omega)=1$, which must be replaced by

$$p^{\tilde{A}}(I_\Omega)\frac{d\,p_{I_{\tilde{A}}}}{d\,p} + p^{\tilde{A}^D}(I_\Omega)\frac{d\,p_{I_{\tilde{A}^D}}}{d\,p} = \frac{d\,p_{I_A}}{d\,p} \quad \text{a.s.} \tag{68}$$

We have

$$\begin{gathered} p^{\tilde{A}}\left(I_\varnothing\right)=0 \quad \text{a.s.,} \qquad p^{\tilde{A}}\left(I_\Lambda\right)\geq 0 \quad \text{a.s.,} \qquad \Lambda\neq\varnothing \\ p^{\tilde{A}}\left(\Lambda_1\cup\Lambda_2\right)=p^{\tilde{A}}\left(\Lambda_1\right)+p^{\tilde{A}}\left(\Lambda_2\right) \quad \text{a.s.} \qquad \Lambda_1\cap\Lambda_2=\varnothing \end{gathered} \tag{69}$$

Formulas (68) and (69) are evident. For example in the case (68) we have (taking into account that $p_{I_{\tilde{A}}} \prec\!\prec p$ and $p_{I_{\tilde{A}^D}} \prec\!\prec p$):

$$\int_A \left[p^{\tilde{A}}(I_\Lambda)\frac{dp_{I_{\tilde{A}}}}{dp} + p^{\tilde{A}^D}(I_\Lambda)\frac{dp_{I_{\tilde{A}^D}}}{dp}\right]dp = \int_\Omega I_\Lambda(\omega)\left[I_{\tilde{A}}(\omega)+I_{\tilde{A}^D}(\omega)\right]dp =$$

$$= \int_\Omega I_\Lambda(\omega)I_A(\omega)dp = \int_A p^A(I_\Lambda)\frac{dp_{I_A}}{dp} \Rightarrow$$

$$\Rightarrow p^{\tilde{A}}(I_\Lambda)\frac{dp_{I_{\tilde{A}}}}{dp} + p^{\tilde{A}^D}(I_\Lambda)\frac{dI_{\tilde{A}^D}}{dp} = p^A(I_\Lambda)\frac{dp_{I_A}}{dp} \quad \text{a.s.} \tag{70}$$

From this for $\Lambda = \Omega$ follows (68), because $p^{\Lambda}(I_{\Omega}) = 1$. Adduce some formulas connected with conditional probability functions in the case of fuzzy conditions.

$$\int_A p^A(I_\Lambda)dp_{I_A} = \int_A p^{\tilde{A}}(I_\Lambda)dp_{I_{\tilde{A}}} + \int_A p^{\tilde{A}^D}(I_\Lambda)dp_{I_{\tilde{A}^D}} \tag{71}$$

$$\int_\Omega p^{\urcorner\tilde{A}}(I_\Lambda)dp_{I_{\urcorner\tilde{A}}} = p(\Lambda) - \int_A p^{\tilde{A}}(I_\Lambda)dp_{I_{\tilde{A}}} \tag{72}$$

$$\int_\Omega p^{\urcorner\tilde{A}}(I_\Lambda)dp_{I_{\urcorner\tilde{A}}} = \int_A p^{\tilde{A}^D}(I_\Lambda)dp_{I_{\tilde{A}^D}} + \int_{A^C} p^{A^C}(I_\Lambda)dp_{I_{A^C}}, \tag{73}$$

$$\int_\Omega p^{(A:\tilde{A})}(I_\Lambda), dp_{I_{(A:\tilde{A})}} = p(\Lambda) \tag{74}$$

$$\int_\Omega p^{(\tilde{A}:A)}(I_\Lambda)dp_{I_{(\tilde{A}:A)}} = \int_A p^{\tilde{A}}(I_\Lambda)dp_{I_{\tilde{A}}} + \int_{A^C} p^{A^C}(I_\Lambda)dp_{I_{A^C}} \tag{75}$$

All these formulas are simple results of above definitions.

References

[1] ZADEH, L.A., *Fuzzy sets*, Inform. and Control **8** (1965), 338.

[2] BIRKHOFF, G., *Lattice theory*, Rev. Ed. American Mathematical Society (1948), N.Y.

[3] LOÈVE, M., *Probability Theory* 3 *ed.* Van Nostrand, Princenton, N.Y., 1960.