Positive and negative approximate controllability results for semilinear parabolic equations

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Abstract

The approximate controllability of solutions of a large class of semilinear parabolic problems is studied. We extend previous results in the literature under a sublinear asymptotic condition on the nonlinearities. On the other hand, we show that this property fails for strictly superlinear nonlinearities.

Resumen

En este trabajo se estudia la controlabilidad aproximada de una amplia clase de problemas semilineales parabólicos. En una primera parte se muestra esta propiedad bajo una hipótesis de comportamiento asintótico sublineal sobre las no linealidades extendiendo resultados conocidos en la literatura. En una segunda parte se muestra que la propiedad no se verifica si los términos no lineales son estrictamente superlineales.

1. Introduction

Let Ω a bounded regular subset of $\mathbb{R}^n, T>0$, $\mathcal{O}$ an open subset of $Q:=\Omega\times(0,T)$, f a continuous function, $A(x,t)\in L^\infty(Q)$ and $\beta(\cdot)$ a *bounded* maximal monotone graph of $\mathbb{R}^2$ such that $D(\beta)=\mathbb{R}$. The main goal of this article is the study of "the controllability" of the parabolic problem:

$$(\mathcal{P})\begin{cases} y_t-\Delta y+f(y)+A(x,t)\beta(y)\ni u\chi_{\mathcal{O}} & \text{in } Q,\\ y(x,t)=0 & \text{on } \Sigma=\partial\Omega\times(0,T),\\ y(x,0)=y_0(x) & \text{on } \Omega.\end{cases}$$

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Problem $(\mathcal{P})$ arises in the modelling of many different applications. When $A \equiv 0$ the semilinear equation of $(\mathcal{P})$ is relevant, for instance, in chemical kinetics (see e.g. Aris [1975] and Díaz [1991a]). If $A \equiv 0$ the equation of $(\mathcal{P})$ may become multivalued. So, if $A > 0$ in Q, problem $(\mathcal{P})$ includes a large class of parabolic variational inequalities arising in the study of many different contexts (see e.g. Duvant-Lions [1972], Brézis [1973], Benilan [1978] and Díaz [1980a] [1980b] for details about modelling, existence, uniqueness and some qualitative properties). The case $A < 0$ in Q have been studied recently in the context of some combustion problems (Gianni-Hulshof [1982]) and also in Climatology (Díaz [1983], [1994a], [1994b]).

The main goal of this paper is the study of the controllability of solutions y of $(\mathcal{P})$ by means of controls u acting on the open set $O \subset Q$: we say that problem $(\mathcal{P})$ has the *exact controllability property with respect to the states space X and controls space* $\mathcal{U}$ if for each $y_d \in X$ fixed there exists a control $u \in \mathcal{U}$ and a solution $y = y(\cdot, u)$ of the associated problem $(\mathcal{P})$ such that $y(T : u) = y_d$.

This property usually holds for conservative problems (such as, for instance, the wave equation). Nevertheless, in the case of parabolic problems the smoothing effect is an impediment for this property except for very special choices of X. Due to this, a weaker formulation is introduced: we say that problem $(\mathcal{P})$ has the approximate controllability property with respect to the states space X and controls space $\mathcal{U}$ if for each $y_d \in X$ fixed there exist a sequence of controls $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{U}$ and solutions $y_n = y(\cdot, u_n)$ of the associated problems such that $y_n(T) \to y_d$ in X as $n \to \infty$.

A first general answer on the approximate controllability for linear parabolic problems was given in Lions [1968]. Concerning the nonlinear case, the first type of results assumed $O = Q$ and $A \equiv 0$. (See Henry [1978] and a recent simpler proof in Díaz-Fursikov [1993]). Another interesting question related to the case $O = Q$ and $A \equiv 0$ appears under a restriction condition on the sign of the controls. For example, if we assume $y_0 \equiv 0$ and $y_d \in L^2_+(\Omega)$ it is possible to choose the control u in the space $L^2_+(O)$. This question was already studied in Díaz [1990] for the linear case with controls on the boundary. The study of this property for the nonlinear case was carried out in Díaz-Henry-Ramos [1994].

A new method for the study of the controllability of the linear case $(f = 0,\ A = 0)$ was introduced in Lions [1990], [1991] and later extended to a special formulation of $(\mathcal{P})$ by Fabre-Puel-Zuazua [1992] [1995]. In both cases the domain of controllability is restricted to open subsets of the form $O = \omega \times (0,T)$, where ω is an open regular subset of Ω and the nonlinear equation considered corresponds to the case $A \equiv 0$ and f satisfying

(1) f is a globally Lipschitz continuous function and

(2) $|f(s)| \leq a + b|s|$ if $|s| > M$ for some positive numbers *a, b* and *M*.

Our main goal is to study the approximate controllability property when assumption (1) fails. More precisely, in Section 2 we will show that the regularity on f given by (1) can be improved once that the *sublinear asymptotic condition* (2) is assumed. As an special case of our results (see Theorem 1 below) we can make sure that the approximate controllability holds, if, for instance, $A \equiv 0$ and

(3) $$f(s) = \lambda |s|^{r-1} s$$

with $0 < r < 1$ (notice that such a function f does not satisfy (1)). Our controllability result replaces condition (1) by the differentiability of f in merely one point (see condition (6)). This result generalizes and develops the one presented in Díaz [1994a] for $A \equiv 0$. Here we also consider the multivalued case $A \equiv 0$ under an additional assumption on β more general than the boundedness condition.

Section 3 is devoted to show that the sublinear asymptotic condition (2) is fundamental in the study of the approximate controllability property on strictly included subdomains $O \subset Q$. More precisely we shall show that if we assume $A \equiv 0$ (only for simplicity) and f is given by (3), with $r > 1$, then a uniforme estimate holds making impossible the approximate controllability property. In fact, the results of this section are the motivation of a further work Díaz [1994b] where the approximate controllability of problem $(\mathcal{P})$ is proved for functions f having a superlinear asymptotic behaviour and under suitable restrictions on the desired state y_d .

2. Approximate controllability under a sublinear growth

In order to make clear our results we start by considering the case in which f is given by (3). The case $r=1$ corresponds to the linear model. The approximate controllability can be obtained by different methods: using the Hahn-Banach Theorem (Lions [1968]), by some constructive methods (Lions [1990]) and by a duality argument (Lions [1991]).

Concerning the case $0<r<1$, it is interesting to mention the work of Seidman [1974] where an abstract result is presented and whose application to the problem $(\mathcal{P})$ was already pointed out in Díaz [1990]. Nevertheless, this point of view is very sophisticated and we shall follow a different method.

The main goal of this section is to obtain a more general result applicable to the case $0<r\leq 1$. We shall follow the duality method introduced in Lions [1991], latter improved in Fabre-Puel-Zuazua [1992] [1995], relative to the case $\beta\equiv 0$, $\mathcal{O}=\omega\times(0,T)$ with $\omega\subset\Omega$ and *f* satisfying (1) and (2).

Our result is the following

Theorem 1 Assume $f(s)$ such that

$$f\in C(\mathbb{R}) \tag{4}$$

$$\begin{cases} \text{there exists } M>0, c_1>0, \text{ and } c_2>0 \text{ such that} \\ \quad |f(s)|\leq c_1+c_2|s|, \text{ if } |s|>M \end{cases} \tag{5}$$

$$\text{the derivative } f'(s_o) \text{ exists, for some } s_o\in\mathbb{R}. \tag{6}$$

Then problem $(\mathcal{P})$ has the approximate controllability property with state space $X=L^p(\Omega)$, $1<p<\infty$ and control space $\mathcal{U}=L^\infty(Q)$.

Remark 1 Condition (6) holds when *f* is Lipschitz continuous on some interval $(a,b)\subset\mathbb{R}$. Indeed, by a well known result (see, for instance, page 145 of Brezis [1973]), there exists $f'(s)$ for almost every $s\in(a,b)$. □

Corollary 1 *The conclusion of Theorem 1 holds in each one of the following special cases:*

i) *f is a globally Lipschitz continuous function,*

ii) *f is a locally Lipschitz continuous function satisfying (5),*

iii) $f(s)=\lambda|s|^{r-1}s$ *and* $0<r\leq 1$. □

Before starting with the proof of Theorem 1 we shall give some previous results and definitions.

Proposition 1 *If M is an open subset of* Q, $1<p'<\infty$, $a\in L^{\infty}(Q)$ *and* φ *satisfies*

$$\begin{cases} -\varphi_t-\Delta\varphi+a(x,t)\varphi=0 & \text{in } Q, \\ \varphi=0 & \text{on } \Sigma, \end{cases}$$

$\varphi(T)\in L^{p'}(\Omega)$ *and*

$$\varphi=0 \quad \text{in } M,$$

then $\varphi=0$ *in* Q.

Proof. Let $t^*=\sup\{t\leq T:\ \exists x\in\Omega \text{ such that } (x,t)\in M\}$. Then, by the unique continuation theorem (see Mizohata [1958] and Saut-Scheurer [1987]) and the uniqueness of solutions we deduce that $\varphi\equiv 0$ in $Q^*=\Omega\times(0,t^*)$. Further, by backward uniqueness results (see page 173 of Friedman [1964]), we conclude that $\varphi\equiv 0$ in the whole domain Q. ■

Proposition 2 (Fabre-Puel-Zuazua [1992], [1995]) *The result of Theorem 1 is true for the linear case with a potential (i.e. with* $A\equiv 0$ *and replacing* $f(y)$ *by* $a(x,t)y$*, with* $a(x,t)\in L^{\infty}(Q)$*). Furthermore, the controls can be taken of "quasi bang-bang type"*[1] □

For the sake of the exposition we recall here the idea of the proof of the Proposition 2. Those authors obtain the result for $O=\omega\times(0,T)$ but the proof for a general open set O of Q follows with easy modifications. They obtain the result by minimizing the functional

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[1] We say that v is of "quasi bang-bang type" if there exist a constant C and a function $\varphi(\cdot)$ such that $v(\cdot)\in C\, sig\, \varphi(\cdot)$

$$J(\varphi^0)=\frac{1}{2}\left(\int_O |\varphi(x,t)|\,dxdt\right)^2+\varepsilon\|\varphi^o\|_{L^{p'}(\Omega)}-\int_\Omega y_d\,\varphi^o dx$$

over $L^{p'}(\Omega)$, where $\varphi(x,t)$ is the solution of the retrograde problem

$$\begin{cases}-\varphi_t-\Delta\varphi+a(x,t)\varphi=0 & \text{in } Q,\\ \varphi=0 & \text{on } \Sigma,\\ \varphi(T)=\varphi^o & \text{on } \Omega\,.\end{cases}$$

Now, by using Proposition 1, the coercitivity of the functional $J(\cdot)$ is shown. Therefore, $J(\cdot)$ attains its minimum over $L^{p'}(\Omega)$ in a unique point $\hat{\varphi}^0$. Further, if $\hat{\varphi}(x,t)$ is the solution of the corresponding retrograde problem, through the subdifferential of the functional J, it can be shown that there exists $v\in sgn(\hat{\varphi}(x,t))\chi_O$ such that the solution of

$$\begin{cases}y_t-\Delta y+a(x,t)y=|\hat{\varphi}|_{L^1(0)}v\chi_O & \text{in } Q,\\ y=0 & \text{on } \Sigma,\\ y(0) & \text{on } \Omega,\end{cases}$$

satisfies $\|y(T)-y_d\|_{L^p(\Omega)\le\varepsilon}.$ ■

Proposition 3 *(Fabre-Puel-Zuazua [1992] [1995]). Let M be the mapping*

$$\begin{aligned}M:\ L^p(\Omega)\times L^\infty(Q)&\to L^{p'}(Q)\\ (y_d,a)&\mapsto \hat{\varphi}^0\,.\end{aligned}$$

Then, if K is a compact subset of $L^p(\Omega)$ and B is a bounded subset of $L^\infty(Q)$, then $M(K\times B)$ is a bounded subset of $L^{p'}(\Omega)$. □

Proposition 4 *Let $a=a(t,x)\in L^\infty(Q)$. Then there exists a constant $C>0$ such that for every $k\in L^p(Q)$ and $\omega^0\in L^p(\Omega)$ the solution ω of*

$$\begin{cases} \omega_t - \Delta\omega + a(t,x)\,\omega = k & \text{in } Q, \\ \omega = 0 & \text{on } \Sigma, \\ \omega(0) = \omega^0 & \text{on } \Omega, \end{cases}$$

satisfies

$$(7) \qquad \| \omega \|_{L^\infty(0,T;L^p(\Omega))} \le C\left(\| \omega^0 \|_{L^p(\Omega)} + \|k\|_{L^p(Q)}\right).$$

Moreover, if $\omega^0 \equiv 0$ *and* $a(\cdot,\cdot) \equiv 0$, *then* $\omega \in X^p(0,T)$ *and there exists a constant* $C > 0$ *such that*

$$(8) \qquad \| \omega \|_{X^p(0,T)} \le C\|k\|_{L^p(Q)},$$

where

$$X^p(0,T) = L^p\left(0,T;W_o^{1,p}(\Omega)\right) \cap W^{1,p}\left(0,T;L^p(\Omega)\right),$$

and

$$\| \cdot \|_{X^p(0,T)} = \| \cdot \|_{L^p(0,T;W_o^{1,p}(\Omega))} + \| \cdot \|_{W^{1,p}(0,T;L^p(\Omega))}.$$

Proof . For (7) see Theorem 9.1, page 341 of Ladyzenskaja-Solonnikov-Uralceva [1968] and pages 226-228 of Pazy [1993]. For (8) see pages 341-342 of Ladyzenskaja-Solonnikov-Uralceva [1968]. ■

Remark 2 If $a(\cdot,\cdot) \equiv 0$, then using Proposition 4 and Gronwall's Lemma it is easy to prove that $\omega \in X^p(0,T)$ and

$$\| \omega \|_{X^p(0,T)} \le C_a \|k\|_{L^p(Q)},$$

with $C_a = o\left(1 + \| a \|_{L^\infty(Q)} \exp\left(\|a\|_{L^\infty(Q)}\right)\right)$. □

We shall need to use a fixed point theorem for multivalued operators:

Definition 1 *Let X, Y two Banach spaces and* $\Lambda: X \to \mathcal{P}(Y)$ *a multivalued function. We say that* Λ ***is upper hemicontinuous*** *at* $x_0 \in X$, *if for every* $p \in Y'$, *the function*

$$x \to \sigma(\Lambda(x), p) = \sup_{y \in \Lambda(x)} < p, y >_{Y' \times Y}$$

is upper semicontinuous at x_0. *We say that the multivalued funcion is upper hemicontinuous on a subset K of X, if it satisfies this property for every point of K.*

Theorem 2 (Kakutani's fixed point Theorem). *Let* $K \subset X$ *be a convex and compact subset and* $\Lambda : K \to K$ *an upper hemicontinuous application with convex, closed and nonempty values. Then, there exists a fixed point of* Λ. □

For a proof see, for instance, Aubin [1984] page 126.

Proof of Theorem 1. We fix $y_d \in L^p(\Omega)$, $\varepsilon > 0$ and we define

$$g(s) = \begin{cases} \dfrac{f(s) - f(s_o)}{s - s_o} & s \neq s_o, \\ \\ f'(s_o) & s = s_o, \end{cases}$$

As f satisfies (4), (5) and (6) then $g \in L^\infty(\mathbb{R}) \cap C(\mathbb{R})$. Indeed: Let $\hat{M} = \max\{M, |s_o| + 1\}$. Then, as $g \in C(\mathbb{R})$ (by construction),

$$\max\left\{|g(s)| : s \in \left[-\hat{M}, \hat{M}\right]\right\} < \infty .$$

Moreover, from (5),

$$\sup_{|s| > \hat{M}} |g(s)| \leq \sup_{|s| > \hat{M}} \left(\frac{|f(s)|}{|s - s_o|} + \frac{|f(s_o)|}{|s - s_o|} \right) \leq \sup_{|s| > \hat{M}} \left(\frac{c_1 + c_2 |s|}{|s - s_o|} \right) + |f(s_o)| < \infty$$

since $|s - s_o| \geq 1$ and $\lim_{|s| \to \infty} \dfrac{c_1 + c_2 |s|}{|s - s_o|} = c_2$.

Now, for each $z \in L^p(Q)$ and for each $b \in \beta(z)$ we can split $y = L_b(z) + Y_b(z)$, where $L = L_b(z)$ is the (unique) solution of

$$(9) \qquad \begin{cases} L_t - \Delta L + g(z)L = -f(s_o) + g(z)s_o - A(x,t)b & \text{in } Q, \\ L = 0 & \text{on } \Sigma, \\ L(0) = y_o & \text{on } \Omega, \end{cases}$$

and by using Proposition 2, for each $\varepsilon > 0$ it is possible to find two functions $\varphi(z,b) \in L^{p'}(Q)$ and $v(z,b) \in sgn\ (\varphi(z,b))\chi_O$ such that the solution $Y = Y_b(z)$ of

$$(10) \qquad \begin{cases} Y_t - \Delta Y + g(z)Y = u\chi_O & \text{in } Q, \\ \quad Y = 0 & \text{on } \Sigma, \\ \quad Y(0) = 0 & \text{on } \Omega, \end{cases}$$

with $u = \| \varphi(z,b) \|_{L^1(O)} v(z,b)$ satisfies

$$(11) \qquad \| Y(T) - (y_d - L(T)) \|_{L^p(\Omega)} \leq \varepsilon .$$

Now, by using Remark 2 and that

$$(12) \qquad X_p(0,T) \subset C([0,T]; L^p(\Omega)) \text{ with compact embedding}$$

(see Lemma 4 and Theorem 3 of Simon [1987]) we have that

(13) $\{y_d - L(T) : z \in L^p(Q),\ b \in \beta(z)\}$ is a relatively compact subset of $L^p(\Omega)$.

Further, $y = L + Y$ is solution of

$$(14) \qquad \begin{cases} y_t - \Delta y + g(z)y = -f(s_o) + g(z)s_o - Ab + u\chi_O & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_o & \text{on } \Omega \end{cases}$$

and satisfies

(15) $$\| y(T) - y_d \|_{L^p(\Omega)} \leq \varepsilon ,$$

with $u = \| \varphi(z,b) \|_{L^1(O)} v(z,b)$. As $g(\cdot)$ is bounded, from (13) and Propositions 3 and 4 we obtain that

(16) $\left\{ \| \varphi(z,b) \|_{L^1(O)} v(z,b),\ z \in L^p(Q),\ b \in \beta(z) \right\}$ is bounded in $L^\infty(Q)$

and then

(17) $$M = \sup_{\substack{z \in L^p(Q) \\ b \in \beta(z)}} \| \varphi(z,b) \|_{L^1(O)} < \infty .$$

Obviously $u = \| \varphi(z,b) \|_{L^1(O)} v(z,b)$ satisfies

(18) $$\| u \|_{L^\infty(Q)} \leq M.$$

Therefore, if we define the operator

$$\Lambda : L^p(Q) \to \mathcal{P}(L^p(Q))$$

by

$\Lambda(z) = \{\ y$ satisfies (14), (15) for some $b \in \beta(z)$ and some u satisfying (18)$\}$,

we have seen that for each $z \in L^p(Q)$, $\Lambda(z) \neq \varnothing$. In order to apply Kakutani's fixed point Theorem, we have to check that the next properties hold:

(i) There exists a compact subset U of $L^p(Q)$, such that for every $z \in L^p(Q)$, $\Lambda(z) \subset U$.

(ii) For every $z \in L^p(Q)$, $\Lambda(z)$ is a convex, compact and nonempty subset of $L^p(Q)$.

(iii) Λ is upper hemicontinuous.

The proof of those properties is as follows:

(i) From Remark 2 we know that there exists a bounded subset U of $L^p(Q)$ such that for every $z \in L^p(Q)$, $\Lambda(z) \subset U$. Now, to see that we can choose such a set U being compact, we shall prove that the set

$$\mathcal{Y} = \left\{ y \text{ satisfying (14) for some } z \in L^p(Q),\ b \in \beta(z) \text{ and } u \text{ verifying (18)} \right\}$$

is a relatively compact subset of $L^p(Q)$. If $y \in \mathcal{Y}$, there exist $z \in L^p(Q), b \in \beta(z)$ and $u \in L^\infty(Q)$ satisfying (18) such that $y = u^1 + u^2 + Y$, where Y is given by (10) and u^1, u^2 are given by

$$\begin{cases} u_t^1 - \Delta u^1 = -f(s_o) & \text{in } Q, \\ u^1 = 0 & \text{on } \Sigma, \\ u^1(0) = y_o & \text{on } \Omega, \end{cases}$$

$$\begin{cases} u_t^2 - \Delta u^2 + g(z)\left(u^1 + u^2\right) = g(z)s_o - A(x,t)b & \text{in } Q, \\ u^2 = 0 & \text{on } \Sigma, \\ u^2(0) = 0 & \text{on } \Omega\,. \end{cases}$$

Since u^1 is a fixed point in $L^p(Q), \left\{g(z)u^1,\ z \in L^p(Q)\right\}$ is a bounded subset of $L^p(Q)$. Then, from Remark 2, the solution u^2 lies on a bounded subset of $X^p(0,T)$. But, as $X^p(0,T) \subset L^p(Q)$ with compact embedding (see (12)), u^2 lies on a compact subset K_1 of $L^p(Q)$. On the other hand, by Remark 2, $Y(v)$, solution of (10), lies in a bounded subset of $X^p(0,T)$, and $Y(v) \subset K_2$, with K_2 being a compact subset of $L^p(Q)$.Therefore $\mathcal{Y} \subset u^1 + K_1 + K_2$, which is a relatively compact subset of $L^p(Q)$.This concludes the proof of (i) if we take $U = \overline{\mathcal{Y}} \subset L^p(Q)$.

(ii) We have already seen that for every $z \in L^p(Q)$, $\Lambda(z)$ is a nonempty subset of $L^p(Q)$. Further $\Lambda(z)$ is obviously convex, because $B(y_d,\varepsilon), \beta(z)$ and $\{u \in L^\infty(Q)$: satisfying (18)$\}$ are convex sets. Then, we have to see that $\Lambda(z)$ is a compact subset of $L^p(Q)$. In (i) we have proved that $\Lambda(z) \subset U$ with U compact. Let $(y^n)_n$ be a sequence of elements of $\Lambda(z)$ which converges on $L^p(Q)$ to $y \in U$. We have to prove that $y \in \Lambda(z)$. We know that there exist $b^n \in \beta(z)$ and $u^n \in L^\infty(Q)$ satisfying (18) such that

$$
(19)\quad \begin{cases} y_t^n - \Delta y^n + g(z)y^n = -f(s_o) + g(z)s_o - Ab^n + u^n\chi_O & \text{in } Q, \\ y^n = 0 & \text{on } \Sigma, \\ y^n(0) = y_o & \text{on } \Omega, \\ \| y^n(T) - y_d \|_{L^p(\Omega)} \le \varepsilon. \end{cases}
$$

Now, using that β is a bounded maximal monotone graph and that the controls u^n are uniformly bounded, we deduce that $u^n \to u$ and $b^n \to b$ in the weak topology of $L^p(Q)$. Further u satisfies (18) and since any maximal monotone graph is strongly-weakly closed (see Proposition 3.5, of Barbu [1976]) over Banach spaces whose topological duals are uniformly convex (for instance, $L^p(Q)$ with $1 < p < \infty$) we obtain that $b \in \beta(z)$. Therefore, if we pass to the limit in (19) we obtain:

$$
\begin{cases} y_t - \Delta y + g(z)y = -f(s_o) + g(z)s_o + Ab + u\chi_O & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_o & \text{on } \Omega. \end{cases}
$$

Further, by the smoothing effect of the heat equation, $y^n(T)$ converges to $y(T)$ on $L^p(\Omega)$ (using again Remark 2 and the compactness result (12)) and $\| y(T) - y_d \|_{L^p(\Omega)} \le \varepsilon$. This proves that $y \in \Lambda(z)$ and concludes the proof of (ii).

(iii) We must prove that for every $z_o \in L^p(Q)$

$$
\limsup_{z_n \to z_o} \sigma(\Lambda(z_n), k) \le \sigma(\Lambda(z_o), k), \quad \forall k \in L^{p'}(Q).
$$

We have seen in (ii) that $\Lambda(z)$ is a compact set, which implies that for every $n \in \mathbb{N}$ there exists $y^n \in \Lambda(z_n)$ such that

$$
\sigma(\Lambda(z_n), k) = \int_Q k(t,x) y^n(t,x)\,dxdt.
$$

Now by (i) $(y^n)_n \subset U$ (compact set). Then, there exists $y \in L^p(Q)$ such that (after extracting a subsequence) $y^n \to y$ on $L^p(Q)$. We shall prove that $y \in \Lambda(z_o)$. We know that there exist $b^n \in \beta(z_n)$ and $u^n \in L^\infty(Q)$ satisfying (18) such that

$$(20) \qquad \begin{cases} y_t^n - \Delta y^n + g(z_n)y^n = -f(s_o) + g(z_n) - Ab^n + u^n \chi_O & \text{in } Q, \\ y^n = 0 & \text{on } \Sigma, \\ y^n(0) = y_o & \text{on } \Omega, \\ \| y^n(T) - y_d \|_{L^p(\Omega)} \leq \varepsilon. \end{cases}$$

Then there exists $u \in L^\infty(Q)$ satisfying (18) such that $u_n \to u$ in the weak-* topology of $L^\infty(Q)$. On the other hand, using again that β is a bounded strongly-weakly closed graph and the smoothing effect of the heat equation, we deduce that y satisfies (14) and (15) with $z = z_o$ for some $u \in L^\infty(Q)$ satisfying (18) and some $b \in \beta(z_o)$, which implies that $y \in \Lambda(z_o)$. Then, for every $k \in L^{p'}(Q)$,

$$\sigma(\Lambda(z_n), k) = \int_Q k(t,x) y^n(t,x) dxdt \to \int_Q k(t,x) y(t,x) dxdt \leq$$
$$\leq \sup_{\bar{y} \in \Lambda(z_o)} \int_Q k(t,x) \bar{y}(t,x) dxdt = \sigma(\Lambda(z_o), k),$$

which proves that Λ is upper hemicontinuous and concludes the proof of (iii). Finally, if we restric Λ to $K = conv(U)$ (the convex enveloppe of U), which is a compact set in $L^p(\Omega)$, Λ satisfies the assumptions of the Kakutani's fixed point Theorem. Then, Λ has a fixed point $y \in K$. Further, by construction, there exists a control $u \in L^\infty(Q)$ satisfying (18) such that

$$(21) \qquad \begin{cases} y_t - \Delta y + f(y) + A(x,t)\beta(y) \ni u \chi_O & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_o & \text{on } \Omega, \\ \| y(T) - y_d \|_{L^p(\Omega)} \leq \varepsilon. \end{cases}$$

Therefore, y is the solution that we were looking for. ■

We can improve the boundedness condition on β

Corollary 2 Let $\overline{\beta}:\mathbb{R} \to \mathcal{P}(\mathbb{R})$ *be a maximal monotone graph such that there exist two positive constants* c_1 *and* c_2 *such that*

$$|b| \leq c_1 + c_2 |r| \quad \forall b \in \overline{\beta}(r).$$

Assume also that the set of points where $\overline{\beta}$ *is multivalued is, at most, a set of the form* $\{x_i : i \in \mathbb{Z}\}$,

$$\ldots < x_{-2} < x_{-1} < x_o < x_1 < x_2 < \ldots,$$

such that

$$\sum_{i \in \mathbb{Z}} \mu\left(\overline{\beta}(x_i)\right) < \infty$$

if μ *is the Lebesgue's measure on* $\mathbb{R}$. *Suppose, finally that* $\overline{\beta}$ *is differentiable at some point* x_o. *Then, the problem*

$$\begin{cases} y_t - \Delta y + \overline{\beta}(y) \ni u\chi_O & \text{in } Q, \\ y(x,t) = 0 & \text{on } \Sigma, \\ y(x,0) = y_o(x) & \text{on } \Omega, \end{cases}$$

has the approximate controllability property with state space $X = L^p(\Omega)$ $(1 < p < \infty)$ *and control space* $\mathcal{U} = L^\infty(Q)$.

Proof of Corollary 1. If we call $\beta(\cdot)$ to the maximal monotone graph such that $\beta(x_o) = \overline{\beta}(x_o)$, $\beta(\cdot)$ is constant over each interval (x_i, x_{i+1}) $(i \in \mathbb{Z})$ and $\mu(\beta(x_i)) = \mu(\overline{\beta}(x_i))$ $(i \in \mathbb{Z})$, then, β is a bounded maximal monotone operator and $f = \overline{\beta} - \beta$ is a nondecreasing function satisfying the conditions of Theorem 1. The proof concludes by applying this theorem with $f + \beta = \overline{\beta}$. ■

Remark 3 It seems important to point out that Theorem 1 is established for any open set O of Q. The case in which the controllability set O is reduced to a single point $\{(t_o,x_o)\}$ of Q, or a segment $(0,T)\times\{x_o\}$, for some $x_o\in\Omega$, needs a different approach. Notice, that, for instance, any control u with such support is not an element of the dual of the energy space $\left(\text{in particular } u\notin L^2(0,T;H^{-1}(\Omega))\right)$ and so the associated state $y(t,\cdot:u)$ can not be found, in general, in the natural energy space $L^2(0,T;H_o^1(\Omega))$. A very special controllability result when $O=(0,T)\times\{x_o\}$ $(N=1,\ f$ linear, $A\equiv 0,\dots)$ was the object of the works Lions [1994] and Glowinski-Lions [1995]. □

3. Negative results for the superlinear case $(r>1)$.

In this case, the result about approximate controllability is, in general, negative. For instance, if $O=\omega\times(0,T)$ with ω an regular open subset of Ω and $A(\cdot,\cdot)\equiv 0$, then any solution can be uniformly estimated (independently of the controls) over $\overline{\Omega}\setminus\overline{\omega}$. A first example is due to A. Bamberger (see Henry [1978]): Given $\Omega=(0,1), r>1$, and $v\in\mathcal{U}=L^2(0,T)$, we consider the problem:

$$\begin{cases} y_t-y_{xx}+|y|^{r-1}y=0 & \text{in } Q,\\ y_x(t,0)=v(t);\ y(t,1)=0 & \text{on } \Sigma,\\ y(0)=0 & \text{on } \Omega.\end{cases}$$

Then, if $\Omega_\varepsilon=(\varepsilon,1)$ $(0<\varepsilon<1)$, we have that

$$\int_{\Omega_\varepsilon}|y(T,x)|^2dx\le C_\varepsilon \text{ (independently of } v). \quad □$$

A different method was developped in Díaz [1990] for the global boundary control case. Here we adapt his proof to problem $(\mathcal{P})$ with $A(\cdot,\cdot)\equiv 0$.

Theorem 3 *Asume $p>1$. Let $u\in\mathcal{U}=L^2(\omega\times(0,T))$ arbitrary and let $y(x,t:u)$ be the corresponding solution of the problem $(\mathcal{P})$ with $A(\cdot,\cdot)\equiv 0$ and $f(s)=|s|^{p-1}s$. Then*

$$|y(x,t:u)| \le C(r,n)\left(\frac{1}{d(x)^{\theta}} + \frac{1}{t^{\frac{\theta}{2}}}\right) \qquad a.e.\,(x,t) \in (\Omega \setminus \overline{\omega}) \times (0,T),$$

with

$$\theta = \frac{2}{p-1}, \quad and \quad d(x) = dist\,(x, \partial\,\omega) \qquad \square$$

Proof. It suffices to prove that

$$y(x,t:u) \le C(r,n)\left(\frac{1}{d(x)^{\theta}} + \frac{1}{t^{\frac{\theta}{2}}}\right) \quad a.e.\ (x,t) \in (\Omega) \setminus \overline{\omega}) \times (0,T),$$

(the other inequality is analogous). We define the function

$$Y(x,t) = C(r,n)\left(\frac{1}{d(x)^{\theta}} + \frac{1}{t^{\frac{\theta}{2}}}\right).$$

Now, for every $x_o \in \Omega \setminus \overline{\omega}$, $t_o \in (0,T)$ and $k = \dfrac{d^2(x_o)}{t_o}$, we consider the function

$$U(t,x) = \frac{C}{\left(kt - r^2\right)^{\theta}}\left(= C\left(kt - (x - x_o)^2\right)^{-\theta}\right),$$

over the set

$$S = \left\{(t,x) \in Q \setminus (\overline{\omega} \times (0,T)) : |\,x - x_o|^2 < kt, 0 < t \le t_o\right\},$$

with $r = |\,x - x_o|$ and C a constant that we shall choose later. Following the work Kamin-Peletier-Vázquez [1989] we shall show that for C large enough,

$U \geq u$ in S. First of all, we point out that $U = \infty$ over the parabolic boundary of S. Further, if we denote (by simplicity) $\psi = kt - r^2$, we obtain that

$$\begin{aligned}
L(U) &\equiv U_t - \Delta U + U^p \\
&= -kC\theta\psi^{-(\theta+1)} - \operatorname{div}\left[2\theta C\psi^{-(\theta+1)}\left(x_i - x_{o,i}\right)\right] + C^p\psi^{-\theta p} \\
&= -kC\theta\psi^{-(\theta+1)} - \sum_{i=1}^{n}\left[4\theta C(\theta+1)\psi^{-(\theta+2)}\left(x_i - x_{o,i}\right)^2\right] - \sum_{i=1}^{n}\left[2\theta C\psi^{-(\theta+1)}\right] \\
&\quad + C^p\psi^{-\theta p} \\
&= -kC\theta\psi^{-(\theta+1)} - 4\theta C(\theta+1)r^2\psi^{-(\theta+2)} - 2n\theta C\psi^{-(\theta+1)} + C^p\psi^{-\theta p}
\end{aligned}$$

Therefore, if we choose C such that

$$\begin{cases} \dfrac{1}{3}C^{p-1} \geq 4\theta(\theta+1)r^2 \\ \dfrac{1}{3}C^{p-1} \geq 2n\theta\psi \\ \dfrac{1}{3}C^{p-1} \geq k\theta\psi, \end{cases} \tag{22}$$

we obtain that $L(U) \geq 0$. Now, as $r^2 + \psi \leq d^2(x_o) = kt_o$, (22) is satisfied if

$$C = c(p,n)\left[d(x_o)^\theta + k^{\frac{1}{p-1}}d(x_o)^{\frac{2}{p-1}}\right].$$

Then, by applying the maximum principle to u and U in S, we obtain

$$\begin{aligned}
u(t_o,x_o) \leq U(t_o,x_o) &= \frac{C}{(kt_o)^\theta} = c(p,n)\frac{d(x_o)^\theta + k^{\frac{1}{p-1}}d(x_o)^\theta}{(kt_o)^\theta} = \\
&= c(p,n)\left[\frac{1}{d(x_o)^\theta} + \frac{k^\theta t_o^{\frac{1}{p-1}}}{(kt_o)^\theta}\right] = c(p,n)\left[\frac{1}{d(x_o)^\theta} + \frac{1}{t_o^{\frac{\theta}{2}}}\right].
\end{aligned}$$

■

Corollary 3 *If* $r > 1$, *the problem* $(\mathcal{P})$ *with* $A(\cdot,\cdot) \equiv 0$ *does not verify the approximate controllability properyty.* □

Remark 4 It seems possible to extend the results of this paper to other nonlinear problems of more complex structure than the semilinear one. This is the case, for instance, of the porous medium equation

$$y_t - \Delta y^m = u\chi_O$$

and the $(m+1)$-Laplacian equation

$$y_t - \Delta_{(m+1)} y = u\chi_O$$

So, if $m > 1$ we have already obtained negatives answers. For $0 < m < 1$ we conjecture that the answer is positive. □

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