# *On superbarrelled spaces. Closed graph theorems*

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## Abstract

J. Arias de Reyna [1] has proved that the space $l_0^\infty(\Omega, \Sigma)$ is not totally barrelled, except for trivial cases. Nevertheless, such space has a high degree of barrelledness, as has been proved by M. López Pellicer [4]. In this paper the spaces with López Pellicer's property (that we call super-barrelled), are studied, in connection with the barrelledness classes introduced by the author in [5] and [8], we prove that a space is super-barrelled if and only if is is barrelled of class $\alpha$, for every $\alpha$.

**Definition 1**. A locally convex space (l.c.s. in short) $E$ is called *superbarrelled* if for every increasing web $W = \{E_{n_1 \dots n_p} : p, n_1, \dots, n_p \in \mathbb{N}\}$ of vector subspaces of $E$, there exists a sequence $(n_k)_{k \in \mathbb{N}}$ so that $E_{n_1 \dots n_k}$ is barrelled and dense in $E$.

$\mathbb{W}$ is an increasing web if every sequence $(E_{n_1 \dots n_{k-1} n})_{n \in \mathbb{N}}$ is not decreasing and

$$E = \bigcup_{n=1}^{\infty} E_n$$

and

$$E_{n_1 \dots n_k} = \bigcup_{n=1}^{\infty} E_{n_1 \dots n_k n}$$

for all $k \in \mathbb{N}$

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If $\Sigma$ is a $\sigma$-algebra on a set $\Omega$, following Valdivia [9] we denote by $l_o^\infty(\Omega,\Sigma)$ (or $l_o^\infty(\Sigma)$) the real or complex space spanned by the characteristic functions $\chi_A$, with $A\in\Sigma$, endowed with the sup norm $\|x\|=\sup\{|x(\omega)|:\omega\in\Omega\}$. Theorem 1 of [4] asserts that $l_o^\infty(\Omega,\Sigma)$ is superbarrelled.

**Proposition 2**. *If E is a superbarrelled space and* $(E_n)$ *is a non decreasing sequence of vector subspaces whose union is E, then there exists a subsequence* $(E_{n_k})$ *consisting of superbarrelled spaces.*

**Proof**. Suppose the proposition false. Then there is a non decreasing sequence $(E_n)$ of subspaces of *E* covering *E*, none of them superbarrelled. Hence, for every $n_1\in\mathbf{N}$ there exists an increasing web $W_{n_1}=\{E_{n_1\ldots n_p}:p,n_2,\ldots,n_p\in\mathbf{N}\}$ of subspaces of $E_n$ such that for every sequence $(n_k)$ there is as $E_{n_1\ldots n_k}$ which is not barrelled or is not dense in *E*. If we consider the web $W=\{E_{n_1\ldots n_p}:p,n_1,\ldots,n_p\in\mathbf{N}\}$, we then get a contradiction.

**Definition 3**. (By Transfinite induction). Every barrelled space is said to be of class 0. If $\alpha$ is a transfinite ordinal number of the first class (i.e., with predecesor), the barrelled space *E* is said to be of class $\alpha$ if, for every non decreasing sequence $(E_n)$ of subspaces covering *E*, there is an *n* so that $E_n$ is a barrelled space of class $\alpha-1$, dense in *E*. If $\alpha$ is a transfinite ordinal number of the second class, *E* is said to be barrelled of class $\alpha$ if E is barrelled of every class $\alpha'<\alpha$.

**Proposition 4**. Every superbarrelled space is of class $\alpha$, for all $\alpha$.

**Proof.** Let us suppose there is a superbarrelled space $E'$ which is not of class $\alpha'$. Then there is a transfinite number $\alpha$ and a superbarrelled space *E* of class $\alpha$, but not of class $\alpha+1$, such that every superbarrelled space is of class $\geq\alpha$. Let $(E_n)$ be a non decreasing sequence of subspaces covering *E*. Then, by Proposition 2, there exists a subsequence $(E_{n_k})$ of superbarrelled subspaces, hence of class $\geq\alpha$, covering *E*. But then, according to Definition 3, *E* would be a barrelled space of class $\geq\alpha+1$.

**Proposition 5.** *A barrelled space E is superbarrelled if and only if it is of class* $\omega_1$, *where* $\omega_1$ *is the first uncountable ordinal.*

**Proof**. If *E* is a superbarrelled space, by Proposition 4, *E* is of class $\omega_1$. Conversely, let us suppose that *E* is a barrelled space of class $\omega_1$ and let $(E_n)$ be a non decreasing sequence of subspaces covering *E*. If no $E_n$ is of class

$\omega_1$, there would be and $\alpha < \omega_1$ such that no $E_n$ is of class $\alpha$ and so *E* would not be of class $\alpha + 1 < \omega_1$. Thus, there are infinite $E_n$ of class $\omega_I$.

Let now $W = \{E_{n_1 \dots n_p} : p, n_1, \dots, n_p \in \mathbb{N}\}$ be an increasing web of subspaces of *E*. By induction, using the preceding argument, we get a sequence $(n_k)$ such that every $E_{n_1 \dots n_k}$ is a barrelled space of class $\omega_1$, dense in *E*. Thus, *E* is superbarrelled.

**Corollary 6**. *Every barrelled space of class* $\omega_1$ *belongs to any other class* $\alpha$.

**Proof**. It follows immediately from Propositions 4 and 5.

The following notions are related with those considered above:

**Definition 7**. Given a l.c.s. *E*, a $\Gamma_r(E)$- space is a l.c.s. *F* such that every linear mapping $T: E \to F$ with closed graph, is continuous. $\Gamma_r(E)$ will denote also the class of all $\Gamma_r(E)$- spaces. A $\Gamma_r^{(o)}$ (or simply $\Gamma_r$) space is a l.c.s. F belonging to $\Gamma_r(E)$, for every barrelled space *E*. If $\alpha$ is a transfinite ordinal of the first class, a $\Gamma_r^{(\alpha)}$-space is a space F which is the union of a non decreasing sequence $(F_n)$ of $\Gamma_r^{(\alpha-1)}$- subspaces. If $\alpha$ is a transfinite number of the second class, a $\Gamma_r^{(\alpha)}$- space is a $\Gamma_r^{(\alpha')}$- space, for some $\alpha' < \alpha$, depending on the space:

$$\Gamma_r^{(\alpha)} = \cup \left\{ \Gamma_r^{(\alpha')} : \alpha' < \alpha \right\}$$

Clearly, $\Gamma_r^{(\alpha)} = \Gamma_r^{(\omega_1)}$, for every $\alpha \geq \omega_1$.

As an application of these notions, we have:

**Theorem 8**. [5, **Theorem** 11]. *Let E be a barrelled space of class* $\alpha$ *and F be a* $\Gamma_r^{(\alpha)}$*-space. If* $T: E \to F$ *is a linear map with closed graph, then T is continuous and there exists a subspace* $F_o$ *of F which is a* $\Gamma_r$*-space and contains the range* $T(E)$ *of T.*

**Theorem 9**. [7, **Theorem** 1]. *Let m:* $\Sigma \to E$ *(l.c.s.) be a mapping such that* $\Sigma$ *is a* $\sigma$*-algebra of subsets of a set* $\Omega$, *and* $E \in \Gamma_r(l_o^\infty(\Sigma))$. *If* $x^* \circ m$ *is bounded and finitely additive for every* $x^*$ *beloging to a total subset* $\Gamma$ *of the dual* $E^*$ *of E, then m is a bounded vector measure.*

**Theorem 10**. [7, **Theorem** 2]. *Let* $\Sigma$ *be a* $\sigma$*-algebra of subsets of* $\Omega$. *If E (l.c.s.) is not a* $\Gamma_r(l_o^\infty(\Sigma))$*-space, there exists a nonbounded, finitely additive* *vector measure* $m:\Sigma \to E$ *and a total subset* $\Gamma$ *of* $E^*$ *such that for every* $x^* \in \Gamma, x^* \circ m$ *is a bounded maesure.*

Let us notice that from Theorem 8 and [4, Theorem 2] it follows that $l_o^\infty(\Sigma)$ is of class $\omega_1$ and that $\Gamma_r^{(\omega_1)} \subset \Gamma_r(l_o^\infty(\Sigma))$, for every $\sigma$-algebra $\Sigma$ of subsets of a set $\Omega$.

With our terminology, Valdivia [9, Theorem 1] has proved that $l_o^\infty(\Sigma)$ is of class 1, the author [5, Theorem 1] has proved that $l_o^\infty(\Sigma)$ is of class 2, and Ferrando and López Pellicer [2, Theorem 1] have proved that $l_o^\infty(\Sigma)$ is of class $n$ for every $n \in N$ (see also [3 Theorem 1]).

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