Estimation and tests of the discrete probability law based on the empirical generating function. (two dimensional case)

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Abstract

A large portion of statistical literature pertains to the theory and application of non-parametric methods of inference. This work presents a new approach to some well-known statistical problems based on observation of stochastic process. The recent development of the theory of probability allows us to consider these observations as ones of random variables with values in an infinite dimensional vector space. This paper will be devoted to estimation and test by means of empirical generating function G_n. We use a similar procedure to that of CRAMER-VON MISES for various hypotheses testing problems.

0. Introduction

Here we present some general and basic notation which we have tried to keep coherently throughout the paper.

The generating function of the probability law $P = \left(p_{ij}\right)_{i,j\geq 0}$ defined on $\left(\mathbb{N}^2, \mathbb{P}(\mathbb{N}^2)\right)$ is the complex function $G(s,t)$ defined in $\left\{(s,t)\in \mathbb{C}^2 \,/\, |s|\leq 1, |t|\leq 1\right\}$ by $G(s,t)= \sum_{i,j\geq 0} p_{ij}s^i t^j$. Subsequently with language abuse, we call more generating function of P the restriction of G on $T=[0,1]\times[0,1]$.

Let $(X_1,Y_1);\ldots;(X_n,Y_n), n\geq 1$ be independent indentically distributed (iid) random variables $(r.v's)$ from a probability space $(\Omega, \breve{a}, P)$ with values

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in $\left(\mathbb{N}^2, \mathbb{P}\left(\mathbb{N}^2\right)\right)$ and let $\mathbb{P}_n = \frac{1}{n}\sum_{i=1}^{n} \delta_{(X_i, Y_i)}$ be the corresponding empirical measure. G_n its generating function. We denote by:

$*T = [0,1]\times[0,1]$

$*C = C(T)$ be the separable Banach space of all continuous functions endowed with a norm $\|f\| = \sup_T |f(s,t)|$. $B(C)$ its Borel σ -field.

$*M = M(T)$ the space of all bounded measures defined on $(T, B(T))$

$*C$ and M are paraid spaces, by the pairing functional

$< f, \mu >= \int_T f(u,v)d\mu(u,v)$

$*m_\mu(k,l) = \int_T u^k v^l d\mu(u,v); (k,l) \in \mathbb{N}^2, \mu \in M$.

In order to be more explicit and so that the subject can be accessible to anyone wishing to read it let us list some important concepts and results that are necessary to prove several results stated in laters parts. They are moreover presented in perspective with the historical development and strong interaction between Probability theory and Analysis.

A: MATHEMATICAL TOOLS
A.1: MEASURABLE VECTOR SPACE

Definition 1: Given a vector space E over $\mathbb{R}$ and a σ -field $\breve{a}$ of subsets of E, the pair $(E, \breve{a})$ will be called "Measurable vectorspace" (M.V.S) if the mappings: $(\lambda, x) \rightarrow \lambda.x$ of $\mathbb{R} \times E$ onto E and $(x,y) \rightarrow x+y$ of $E x E$ onto E are measurable when $\mathbb{R}$ is endowed with its Borel σ -field.

Example 1: If E is a separated metric vector space $(E, B(E))$ is $M.V.S.$

A.2: DUALITY AND MEASURABILITY
Weakned σ-field.

Let E and F be two spaces paraid by a bilinear form $(x,y) \rightarrow < x,y >$. F is identified with a vector subspace of the algebraic dual E^* of E.

Definition 2: We call weak σ -field on *E*, defined by the duality between *E* and *F*, and denoted $w(E,F)$, the weakened sub -σ -field of $\breve{a}$, such that every linear form on $E: x \to < x, y >,\ y \in F$, is continuous.

Remark 1: The weak σ -field is identical with the σ -field generated by algebraic base of *E*. So by subset of *F* spaning a $\sigma(F,E)$-sequentially dense subspace.

Construction of weak σ-field: $(E,F)^f$ denotes the family of all the finite codimensional σ (E,F)-closed subspaces of *E* ordinate by the inclusion $\supset$. If $G \in (E,F)^f, \pi_G$ is the canonical linear mapping of *E* onto *E/G*. A cylinder set in *E* is any subset of *E* of the form $\pi_G^{-1}(B)$ where *B* is a Borel set in *E/G*.

Theorem 1: $w(E,F)$ is the σ -algebra generated by a Boolean algebra $\alpha(E,F) = \bigcup_{G \in (E,F)^f} \pi_G^{-1}(B(E/G))$

The vector space $(E,\breve{a})^m$ of all measurable linear forms on $(E,\breve{a})$ will be called the "Dual of the M.V.S.$(E,\breve{a})$". If The canonical duality between *E* and $(E,\breve{a})^m$ is separated in $E\left(i.e < x, f >= 0,\ \forall f \in (E,\breve{a})^m \Rightarrow x = 0\right)$, then $(E,\breve{a})$ will be called "separated" (S.M.V.S.)

We have $\left(E, w\left(E,(E,\breve{a})^m\right)\right)^m = (E,\breve{a})^m$.

Theorem 2: Every Borel linear form defined on a separable Frechet space is continuous, wich my be written $(E,B(E))^m = E'$. However it is well known that $w\left(E,E'\right) = B(E)$.

We refer the reader to $([33],[4])$ for a detailed study of this theory of "measurable duality" and its applications, especially for a characterization of the dual of certain types of M.V.S.

A. 3 RANDOM VECTORS AND THEIR OBSERVATIONS.

A. 3.1 Definitions and examples

Let $(\Omega,\breve{a},P)$ be a fundamental probability space. Let *E* and *F* be two linear spaces paraid by a bilinear from <.,.>.

Definition 3: An E valued random $X:\Omega \to E$ is an $\breve{a} - w(E,F)$ measurable function.

Theorem 3: A function $X:\Omega \to E$ is $\breve{a} - w(E,F)$ measurable function if and only if $< X, y >$ defined on the same probability space is a random variable.

The previous remark 1-A.2 will be useful. Indeed it is enough to verify that for restricted subspace of measurable linear functionals.

The important example of the random variables with values in an infinite dimensional vector spaces is constitued by stochastic process with paths in some space of functions. Let $(\Omega, \breve{a}, P)$ denotes a probability space, $(X_t, t \in T)$ denotes a $\mathbb{R}$-valued stochastic process defined on $(\Omega, \breve{a}, P)$. If E is the vector space of real functions on T so that for all $\omega \in \Omega$, the sample $X_.(\omega)$ belongs to E, then we consider the application $X:\omega \to X_.(\omega)$ as an E valued random if there exists a vector space F so that E and F are paraid. That is, when F holds the set of the linear forms $\delta_t : x \to x(t)$ generating a $\sigma(F,E)$-sequentially dense vector sub space.

If T is a σ-compact metric space and $(X_t, t \in T)$ with a.s-continuous sample paths. It is easily seen that $X:\omega \to X_.(\omega)$, $\omega \in \Omega$ defines a random vector$X$taking values in the S.M.V.S. (C, B(C).) whose topological dual is identified with the space $M_C(T)$ of all regular measures with compact support on T. The canonical duality $< C, M_C(T) >$ being defined by the bilinear form $< f, \mu > = \int_T f d\mu$.

A. 3.2. CHARACTERISTIC ELEMENTS OF RANDOM VECTOR

Let $X:(\Omega, \breve{a}, P) \to (E, W(E,F), P_X)$ a random vector.

Definition 4: We call the characteristic function of X the function,

$$y \in F; \varphi_X(y) = E\left(e^{i<X,y>}\right) = \int_E e^{i<x,y>} dP_X(x) = \int_{\mathbb{R}} e^{it} dP_{<X,y>}(t).$$

Definition 5: For random vectors $(X_n; n \geq 1)$ we say that $(X_n; n \geq 1)$ converges in cylindrical probability to a random vector X, provided $\varphi_{X_n}(y) \underset{n\to\infty}{\longrightarrow} \varphi_X(y)\ \forall y \in F$.

Definition 6: Let $X:(\Omega,\breve{a},P)\rightarrow(E,w(E,F),P_X)$a random vector. X is scalarly integrable if, $\forall y\in F\ E_p(<X,y>)$exists. It will be said to be integrable if there exists $EX\in E$ (identified with a vector subspace of F^*) such that $<EX,y>=E_P(<X,y>)$for all $y\in F$.

We now announce the most important results that we need for the proof of our main results.

Theorem 4: Let E a separable Frechet space which topology determined by the increasing sequence of semi norms $(p_n,n\geq 1)$. Then X integrable if $p_n(X)$ it is for all n.

In particular, if E is a separable Banach space EX exists as soon as $\|X\|$ be an integrable real random variable.

Proof: ([1]. p 112)

Under the same preceding hypotheses. We have,

Theorem 5: a) If $E(P_n(X_1))<\infty,\forall n\geq 0$, there exists a random vector X so that $S_n=\frac{1}{n}\sum_{i=1}^{n}X_i\rightarrow X\ a.s.$

b) If $E\|X_1\|<\infty$, then $\|S_n-EX_1\|\overset{n\rightarrow\infty}{\rightarrow}0\ a.s.$

Proof: ([1] p. 114)

For more details see ([1]).

We close this section with the following example of random vector: Let $(\Omega,\breve{a},P)$ be an arbitrary complete probability space, and let T be the mapping: $\omega\rightarrow\delta_\omega$ (δ_ω being the Dirac measure en $\omega\in\Omega$) from Ω to the vector space E of bounded measures with finite support on Ω, endowed with the greatest-field ξ which yields T measurable. By $([33].p121)$ we have: for all integer n, the functional defined in $(E,\xi)^m$ by:

$$f\in\mathcal{L}_0(\Omega,\breve{a})=(E,\xi)^m,\varphi(f)=\left(\int_\Omega e^{if}dP\right)^n,$$

Where $\mathcal{L}_0(\Omega,\breve{a})$ denotes the vector space of all real measurables functions on T, is the Fourier transform of the unique probability on the separated M.V.S (E,ξ).

φ is the Fourier transform of the random vector T^n from the product space $(\Omega, \breve{a}, P)^n$ into (E, ξ) defined by $T^n(\omega_1, \dots, \omega_n) = \sum_{i=1}^{n} \delta_{\omega_i}$. The probability distribution of T^n is: the generalized multinomiale distribution $m_{(\Omega, \breve{a})}(P, n)$ with parameters P and n.

For each measurable partition $A_1, \dots, A_K$ of Ω, the distribution of the random vector $(< T^n, 1_{A_1} >, \dots, < T^n, 1_{A_k} >)$ is multinomiale with parameters $(P(A_1), \dots, P(A_K))$ and n ([33], p 122). This implies that for all $A \in \breve{a}, < T^n, 1_A >$ is a random variable whose distribution is binomiale $B(P(A), n)$ and then $E(< T^n, 1_A >) = nP(A)$ which imply that for all $f \in \mathcal{L}_o(\Omega, \breve{a}), E(< T^n, f >) = nE_P(f)$ as soon as f is P-integrable. Note that T^n is not scalarly integrable relatively to the duality between E and $(E, \xi)^m = \mathcal{L}_o(\Omega, \breve{a})$ and it's natural to ask when it is. To obtain this it's enough to consider T^n as a random vector with values in the vector space $m(\Omega, \breve{a})$ of bounded measures $w.r$ to the duality $< m(\Omega, \breve{a}); \mathcal{L}_o(\Omega, \breve{a}) >$ where $\mathcal{L}_o(\Omega, \breve{a})$ denotes the space of all real bounded measurable functions on $(\Omega, \breve{a})$. The weakened σ-field $w(m(\Omega, \breve{a}), \mathcal{L}_\infty(\Omega, \breve{a}))$ is generated by the applications $\mu \to < \mu, 1_A > = \mu(A), A \in \breve{a}$. The restriction of the functional φ on $\mathcal{L}_o(\Omega, \breve{a})$ characterizes entirely the law of T^n which is in this case scalarly integrable and integrable with $ET^n = nP \in m(\Omega, \breve{a})$.

Consider the empirical probability $\mathbb{P}_n = \frac{1}{n} T^n$. $\mathbb{P}_n$ is an unbiased estimator of the law P. Let, the random vector $\mathbb{D}_n = \sqrt{n}(\mathbb{P}_n - P)$ which values in $< m(\Omega, \breve{a}), w(m(\Omega, \breve{a}), \mathcal{L}_\infty(\Omega, \breve{a}))$. we have $\varphi_{\mathbb{D}_m}(f) \overset{n \to \infty}{\to} \exp -\frac{1}{2}(< f^2, P > - < f, P >^2)$.That is to say, the sequence $(\mathbb{D}_n)$ converges in cylindrical law to a central Gaussian cylindrical probability on $(\mathcal{L}^*_\infty(\Omega, \breve{a}), w(\mathcal{L}^*_\infty, \mathcal{L}_\infty))$ which determines a unique central Gaussian probability.

A.4. NORMAL LAW AND GAUSSIAN RANDOM VECTOR

Taking into consideration the exceptional place which the normal law occupies, and due to its numerous aplications which is witnessed by Probability theory and the mathematical statistic, it is necessary in this list of mathematical tools to devote a place to this law and some of its characteristics. We end this part by the contribution which was brought by the WISHART law to the quadratic analysis of some the Gaussian random functions.

Let E and F be two linear spaces paraid by a bilinear form $<.,>$

Definition 7: A random vector X with values in E is Gaussian if for any linear functional $y \in F, < X, y >$ is a real valued Gaussian variable.

Definition 8: A random vector X with values in a S.M.V.S. E is Gaussian if it's relatively to the canonical duality $< E, E^m >$.

A.4.1. GAUSSIAN CYLINDRICAL PROBABILITY

Theorem 6: For all $x_o \in E$ and non-negative quadratic form Q on F, there exists an unique cylindrical probability $\gamma_{x_o,Q}$ whose Fourier transform is such that for every $y \in F$

$$\varphi_{\gamma_{x_o,Q}}(y) = \exp\left\{ i < x_o, y > -\frac{1}{2} Q(y) \right\}$$

Proof: (see $[33]$ p. 78).

$\gamma_{x_o,Q}$ is called Gaussian cylindrical probability with mean x_o and variance the non-negative quadratic form Q on F.
The terms "mean" and" variance" are the language abuse borrowed from a traditional terminology of Probability.

Theorem 7: Let E_1, F_1 others paraid vector spaces and u the linear mapping from E into $E_1, \sigma(E,F) - \sigma(E_1, F_1)$ continuous. Then $u(\gamma_{x_o,Q}) = \gamma_{u(x_o),Q o^t u}$

Proof: It's easy: simple application of the Fourier transform.

Definition 8: Let m be any fixed element of a real Hilbert space $H.\mu_m = \gamma_{m,\|\cdot\|_H^2}$ denotes the canonical Gaussian probability on H, whose Fourier transform is:

$$\varphi_{\mu_m}(y) = \exp\left\{i < m, y >_H - \frac{1}{2}\|y\|_H^2\right\}, y \in H.$$

Theorem: Let $\mu = (\mu_M)_M$ be a cylindrical probability. If for any linear $y \in F$, $y(\mu)$ (see $[8]$ p. 72) is a real Gaussian probability, then μ is a Gaussian cylindrical probability.

Proof: (see $[8]$, p. 78).

A. 4.2. GAUSSIAN PROBABILITY AND GAUSSIAN VECTOR-NORMAL LAW

Definition 8. A probability P defined in a weak measurable vector space $(E, w(E,F))$ is said to be Gaussian if the associatied cylindrical probability $(P_M)_{M \in (E,F)^f}$ is Gaussian.

In a particular, a probability P defined on a S.M.V.S $(E, \breve{a})$ will be said to be Gaussian if it is on a weakned S.M.V.S $\left(E, w\left(E, (E,\breve{a})^m\right)\right)$.

A random vector X in E w.r to the duality $< E, F >$ is said to be Gaussian if his law P_X is Gaussian.

Definition 9: For $m \in E$ and a non-negative quadratic form Q on F, a random vector X in E is distributed as Normal law if his characteristic function is,

$$y \in F, \varphi_X(y) = \exp\left\{i < m, y > -\frac{1}{2}Q(y)\right\}$$

We note $X \to N_{E,F}(m, Q)$.

Example 2: Let T be a set and $(X_s, s \in T)$ a Gaussian real random function defined on $(\Omega, \breve{a}, P)$ with mean function $m = (m(t), t \in T)$ and covariance function $K(s,t) = E((X_s - m(s))(X_t - m(t))), (s,t)T xT.(X_s, s \in T)$ defines a random vector in $E = \mathbb{R}^T$ relatively to the duality between E and $F = \sum_{t \in T} \mathbb{R}_t (\mathbb{R}_t = \mathbb{R}, \forall t \in T)$ i.e in the S.V.S $\left(R^T, \underset{t \in T}{\otimes} B(\mathbb{R})\right) = (E, w(E,F))$. The mappings $\delta_t : x \to x(t), t \in T$ form a base of F and the form $Q_k\left(\sum_{i \in I} \lambda_i \delta_{t_i}\right) = \sum_{I \times I} \lambda_i \lambda_j K(t_i, t_j)$ where I is finish, defines a non negative quadratic form on F. It's abviously $X \to N_{E,F}(m, Q_K)$. Now, let T be a σ-compact metric space. Suppose that m and K are continuous and the process $(X_s, s \in T)$ with a.s-continuous sample paths. The associated random vector X is with values in a separable Frechet space $C(T)$ of all real continuous functions on T, whose topological dual is identified as the space $M_C(T)$ of all regular Borel measures with compact support on T. Moreover, the distribution of X is $N_{C(T),M_C(T)}(m, Q_k)$ where Q_k is the non-negative quadratic form on $M_C(T)$ defined by

$$Q_k(\mu) = \int_{T\times T} K(s,t)d\mu(s)d\mu(t)$$

So that, $\varphi_X(\mu) = E\left(e^{i<X,\mu>}\right) = \exp\left\{i < m, \mu > -\frac{1}{2}Q_k(\mu)\right\}, \mu \in M_C(T)$ (1)

Conversely, let P be a Gaussian probability on $(C, B(C))$ and let

$m(t) = \int_C x(t)dP(x),\ K(s,t) = \int_C < x, \delta_s >< x, \delta_t > dP(x) - m(s)m(t).$

Then, the characteristic function of P is given by (1) ([31], p. 303).

A. 4.3. SUPPORT OF GAUSSIAN PROBABILITY

This section contains the main results concerning the support of Gaussian probabilities which borings into evidence the important role played by the reproducing kernel Hilbert space (RKHS).

Theorem 9: Let S be any non empty set with $N(S) \leq N_1$ the cardinality of the continuum; and $E = \mathbb{R}^S$ be the complete Hausdorff separable locally convex space of all real functions on S endowed with the topology of pointwise convergence. Let P be a centreted inner regular Gaussian on $(E, B(E))$. Then the support of P is $\overline{H(K)}$ where $H(K)$ is the *RKHS* of the function *K*. More, $H(K) \in B(\mathbb{R}^S)$ and $P(H(K)) = 0$ if and only if $H(K)$ is infinite dimensional.

Proof: ([31]).

Theorem 10: Let P be a zero-mean Gaussian probability on $(E, B(E))$, where E is any separable Frechet space with variance the non negative quadratic form Q on the topological dual E'. Then there exists a separable Hilbert space $H(Q)$ with following properties:

i) $H(Q) \subset E$ and $\overline{H(Q)} = \text{supp}P$.

ii) The injection map j of $H(Q)$ into E is continuous and $j\left(\gamma_{o,\|\cdot\|_H^2}\right) = P$.

iii) $H(Q) \in B(E)$ and $P(H(Q)) = 0$ if $H(Q)$ is infinite dimensional

Proof: See ([31]). $H(Q)$ is the R.K.H.S where
$K(s,t) = \frac{1}{2}\left(Q\left(\delta_s + \delta_t\right) - Q\left(\delta_s\right)Q\left(\delta_t\right)\right)$.

A. 4.4. QUADRATIC ANALYSIS OF CERTAIN GAUSSIAN PROCESS

This section deals with the derivation of the characteristic function of quadratic forms of certain real Gaussian process.

Theorem (J.L. SOLER) 11: Let T be a σ-compact metric space and let $(X_t, t \in T)$ be a real Gaussian process with a.s. continuous sample paths on T such that:

a- $E(X_t) = m(t), t \in T$
b- $E(X_s X_t) - m(s)m(t) = K((s,t);(s,t)) \in TxT$;
c- the covariance function K is continuous on TxT,
d- the mean function m belongs to the R.K.H.S. $H(K)$;
then, for every regular Borel measure υ with compact support on TxT such that $\upsilon(AxB) = \upsilon(BxA)$ for all Borel subsets A,B of T, the characteristic function of the r.r.v:

$Z_\upsilon = \int_{TxT} X_s X_t \, d\upsilon(s,t)$ is given by:
$a \in \mathbb{R}, \varphi_{Z_\upsilon}(a) = \det\left(1 - 2iaA_\upsilon^k\right)^{-1/2} \exp\left\{ia < m, \left(1 - 2iaA_\upsilon^k\right)^{-1} \circ A_\upsilon^k(m) >_{H(K)_C}\right\}$
where A_υ^k is the self-adjoint nuclear operator in $H(K)$ defined by:

$f \in H(K), A_\upsilon^k(f)(\cdot) = \int_{TxT} f(s)K(t,.)d\upsilon(s,t)$;
and
$E(Z_\upsilon) = < m, A_\upsilon^K(m) >_{H(K)} + \text{Trace}\left(A_\upsilon^K\right) = \int_{TxT} (m(s)m(t) + K(s,t))\, d\upsilon(s,t)$.

As a particular case, for every regular Borel measure μ with compact support on T, the characteristic function of the r.r.v: $Y_\mu = \int_T X_t^2 d\mu(t)$ is given by:

$$a \in \mathbf{R}, \varphi_{Y_\mu}(a) = \det\left(1 - 2iaB_\mu^k\right)^{-1/2} \exp\left\{ia < m, \left(1 - 2iaB_\mu^k\right)^{-1} \circ B_\mu^k(m) >_{H(K)c}\right\}$$

where B_μ^K is the self-adjoint nuclear operator in $H(K)$ defined by:

$$f \in H(K),\ B_\mu^K(f)(.) = \int_T f(t)K(t,.)d\mu(t)$$

and $E\left(Y_\mu\right) = < m, B_\mu^K(m) >_{H(K)} + \text{Trace}\left(B_\mu^K\right) = \int_T \left(m^2(t) + K(t,t)\right)d\mu(t)$

Proof:([33])

This theorem unifies and generalizes the known results about quadratic forms of Gaussian processes related to brownian motion in the central case. Moreover, the derivation of the characteristic function of these quadratic forms for a wide class of non central Gaussian processes, constitutes a progress which allow us to look for statistical applications in two differents fields.

Finally, it looks to us that we dispose of enough tools that we could follow next part of our work.

B. EMPIRICAL GENERATING FUNCTION

The generating function may be used instead of the densities or distribution functions in problems of inferences. We shall now study some further properties of the empirical generating function and discuss its applitation in tests for goodness of fit.

Consider a random variable $(X,Y):(\Omega,\breve{a},P) \to \left(\mathbf{N}^2, \mathbf{P}(\mathbf{N}^2), P_{(X,Y)}\right)$. Let $(X_1,Y_1);\ldots;(X_n,Y_n)$ be iid random with distribution $P_{(Y,Y)}$.

$\mathbf{P}_n = \frac{1}{n}\sum_{i=1}^{n} \delta_{(X_i,Y_i)}$ be the empirical probability and G_n be the empirical generating function based on $(X_1,Y_1);\ldots;(X_n,Y_n)$.

Proposition 1: G_n is a random·vector in $\left(C,B(C)\right)w.r$ to the duality $< C,M >$ scalarly integrable and integrable such that $EG_n = G$. More $\|G_n - G\|_\infty \overset{n\to\infty}{\to} 0\ a.s$

Proof: $G_n:(\Omega,\breve{a},P) \to \left(C,B(C)\right)$

$$\omega \to G_n(\omega)((s,t) \to \frac{1}{n}\sum_{i=1}^{n} s^{X_i(\omega)} t^{Y_i(\omega)}).$$

It is enough to remark that for all $(u,v) \in T$,

$< G_n, \delta_{(u,v)} >= G_n(u,v) = \frac{1}{n}\sum_{i=1}^{n} u^{X_i} v^{Y_i}$ is a real valued random variable. The G_n is a random vector in $(C, B(C))$ scalarly integrable, since $< G_n, \mu >$ is a r.v.r with $E(< G_n, \mu >) =< G, \mu >$. As $(C, \| \ \|)$ is a separable Banach space and $\|G_n\|$ is integrable we know from (th. 5, A.3) that G_n integrable such that $EG_n = G$. We can write $G_n = \frac{1}{n}\sum_{i=1}^{n} Z_i ; Z_i = (.)^{X_i}(.)^{Y_i}$. So that $E(\|Z_1\|) < \infty$ and the hypotheses of theorem 5.A3 are fulfilled we obtain $\|G_n - G\|_\infty \overset{n\to\infty}{\to} 0$ a.s.

B.1.1. SUFFICIENCY

Définition 1: Let $(\Omega, \breve{a}, \mathbb{P})$ be a statistical space. A sub-σ-algebra $B \subset \breve{a}$ is called sufficient (for $\mathbb{P}$) if and only if for every $A \in \breve{a}$ there is some B-measurable g_A such that for every $P \in \mathbb{P}$, the conditional probability $P(A / B) = g_A$ a.s.
The essential point is that g_A does not depend on *P*.
The intersted reader will find in ([5],[32]) various expositions on this subject.

Example 1: Let $(\Omega, \breve{a}, \mathbb{P})$ be a statistical space $(\Omega^n, \breve{a}^{\otimes n}, \mathbb{P}^n = \{P^{\otimes n}, P \in \mathbb{P}\})$ its *n*-fold product statistical space, φ_n designs the sub-σ-algebra of $\breve{a}^{\otimes n}$ consisting of sets invariant under all permutations of the coordinates. Denote by $f_\sigma : (\omega_1, \ldots, \omega_n) \to (\omega_{\sigma(1)}, \ldots, \omega_{\sigma(n)})$ where $\sigma \in S_n$. f_σ is a measurable isomorphism of $(\Omega^n, \breve{a}^{\otimes n})$ and preserves each $P^{\otimes n}$. Then for any $A \in \breve{a}^{\otimes n}$ and $B \in \varphi_n$ we have

$$P^{\otimes n}(A \cap B) = \frac{1}{n!}\sum_{\sigma \in S_n} P^{\otimes n}(A \cap f_\sigma(B)) = \frac{1}{n!}\sum_{\sigma \in S_n} P^{\otimes n}(f_\sigma(A) \cap B)$$

Thus $P^{\otimes n}(A / \varphi_n) = \frac{1}{n!}\sum_{\sigma \in S_n} 1_{f_\sigma(A)}(.)$ a.s $P^{\otimes n}$ and φ_n is sufficient.

A present let $G_n : (\mathbb{N}^2, \mathbb{P}(\mathbb{N}^2), \mathbb{P}) \to (C, B(C))$

$\overline{(x,y)} = ((x_i, y_i), 1 \le i \le n) \to G_n \overline{(x,y)} : (s,t) \to \frac{1}{n} s^{x_i} t^{y_i}$

Set $\tilde{S}_n = G_n^{-1}(B(C))$. It's abvious that $\tilde{S}_n \subset \varphi_n$. Let us prove the converse inclusion. For each $A \in (P(\mathbb{N}^2))^{\otimes n}$, let $s(A) = \underset{\sigma \in S_n}{\cup} f_\sigma(A) \in \varphi_n$. Then if $B \in \varphi_n, s(B) = B$.

$$\text{Let } \mathcal{F} = \left\{ A \in (\mathbb{P}(\mathbb{N}^2))^{\otimes n} : s(A) \in \tilde{S}_n \right\}$$

We want to prove $\mathcal{F} = (\mathbb{P}(\mathbb{N}^2))^{\otimes n}$. Now $\mathcal{F}$ is a montonoe class. Also, any finite union of sets in $\mathcal{F}$ is in $\mathcal{F}$.
Given $A_1, \ldots, A_n$ in $\mathbb{P}(\mathbb{N}^2), A_1 x \ldots x A_n \in \mathcal{F}$. Since the collection C of finite unions of such sets is an algebra; the smallest monotone class including C is $(\mathbb{P}(\mathbb{N}^2))^{\otimes n}$.

CONCLUSION: G_n is a sufficient, consistent, unbiased estimator of the generating function G of the unknown law P.

B.2. EMPIRICAL GENERATING PROCESS

We now discuss the properties of the empirical generating process $E_n(s,t) = \sqrt{n}(G_n(s,t) - G(s,t)); (s,t) \in T$.

Note that:

i) $E(E_n(s,t)) = 0, \forall (s,t) \in T$

ii) $K((s,t);(u,v)) = E(E_n(s,t)E_n(u,v)) = nE(G_n(s,t)G_n(u,v)) - nG(s,t)G(u,v)$

$$= \frac{1}{n} E\left(\sum_{i=1}^{n} (su)^{X_i} (tv)^{Y_i} + \sum_{i \ne j} (s^{X_i} t^{Y_i})(u^{X_j} v^{Y_j})) - nG(s,t)G(u,v) \right)$$

$$= G(su, tv) - G(s,t)G(u,v)$$

Proposition 2: The sequence $(E_n, n \ge 1)$ converges in cylindrical law to $X \to N_{C,M}(0, Q_K)$.

Proof: Let $\mu \in M$,

$\varphi_{E_n}(\mu) = E_p(\exp i < E_n, \mu >) = E_p(\exp i\sqrt{n} < G_n, \mu >) \exp - i\sqrt{n} < G, \mu >$

Since $< G_n, \mu >= \frac{1}{n}\sum_{i=1}^{n} m_\mu(X_i, Y_i)$. Hence

$$\varphi_{E_n}(\mu) = \left[\exp - \frac{i}{\sqrt{n}} < G, \mu >. E_p\left(\exp\frac{i}{\sqrt{n}} m_\mu(X, Y)\right)\right]^n = A^n$$

Put $f = m_\mu(X, Y) \in \mathcal{L}_\infty(\Omega, \breve{a})$. We have

$< G, \mu >= E_p(f)$

and

$$A = \exp\left(-\frac{i}{\sqrt{n}} E_p(f)\right) \times E_p\left(\exp i\frac{f}{\sqrt{n}}\right)$$

$$A = \left(1 - \frac{i}{\sqrt{n}} E_p(f) + \frac{(E_p f)^2}{2n} + 0\left(\frac{1}{n}\right)\right) E_p\left(1 + i\frac{f}{\sqrt{n}} - \frac{f^2}{2n} + 0\left(\frac{1}{n}\right)\right)$$

$$A = \left(1 - \frac{1}{2n}\left(E_p(f^2) - E_p^2(f)\right) + 0\left(\frac{1}{n}\right)\right).$$

Therefore,

$\lim_{n\to+\infty} \varphi_{E_n}(\mu) = \exp - \frac{1}{2}\sigma_p^2(f)$, where $\sigma_p^2(f) = E_p(f^2) - (E_p(f))^2$

Note that $\sigma_p^2(f) = \int_{T\times T} K((s,t);(u,v))\, d\mu(s,t)\, d\mu(u,v) = Q(\mu)$

Thus $\varphi_{E_n}(\mu) \overset{n\to\infty}{\to} \exp - \frac{Q(\mu)}{2} = \varphi_X(\mu)$

That's to say $(E_n, n \geq 1)$ converges in cylindrical law to the random vector $X \to N_{C,M}(0, Q)$.

Of course for any finite collection $(s_1, t_1), \ldots, (s_k, t_k) \in T -$ taking

$$\mu = \sum_{j=1}^{k} a_j \delta_{(s_j, t_j)} -, \ \varphi^{(a_1,\ldots,a_k)}_{(E_n(s_1,t_1),\ldots,E_n(s_k,t_k))} \overset{n\to\infty}{\to} \varphi^{(a_1,\ldots,a_k)}_{(X(s_1,t_1),\ldots,X(s_k,t_k))}$$

The sequence $\{(E_n(s_1, t_1), \ldots, E_n(s_k, t_k)); n \geq 1\}$ converges in law to $(X(s_1, t_1), \ldots, X(s_k, t_k))$ but an example ([9]), p20) shows that the finite-dimensional set do not form a convergence determing class.

Let consider the random vector in $C: Z = (\cdot)^X (\cdot)^Y - G$. We have

$$E_p\left(\|Z\|_\infty^2\right) < \infty \text{ and } E_p\left(\sup_{(s,t),(u,v)\in T} \frac{|Z(s,t)-Z(u,v)|^2}{\|(s,t)-(u,v)\|^2}\right) < \infty$$

Hence by lemme 4.1 $([21]\,\text{p.}7)$ $\left(\text{n}^{-1/2}\sum_{i=1}^{n} Z_i = E_n\right)$ is weakly convergent in C and this limit is $X \to N_{C,M}(0,Q)$.

In particular,

$$\left(\int_T E_n^2(s,t)d\mu(s,t); n \geq 1\right) \xrightarrow{\mathcal{L}} \int_T X^2(s,t)d\mu(s,t)$$

$$\left(\sup_T E_n(s,t); n \geq 1\right) \xrightarrow{\mathcal{L}} \sup_T X(s,t)$$

Proposition 3: If the condition $E_p(X+Y)^2 < \infty$ is fulfilled, then the process $(X(s,t);(s,t)\in T)$ is almost surely continuous in *T*.

Proof: Indeed,

$E_p\left((X(s,t)-X(u,v))^2\right) = \left(G\left(s^2,t^2\right) - 2G(su,tv) + G(u^2,v^2)\right) -$
$(G^2(s,t) - 2G(s,t)G(u,v) + G^2(u,v)) = \text{Var}\left(s^X t^Y - u^X v^Y\right)$
$\|X(s,t)-X(u,v)\|_2^2 \leq E_p\left((s^X t^Y - u^X v^Y)^2\right)$
Put $[(s,t)-(u,v)] = \sup(|s-u|,|t-v|)$
$\|X(s,t)-X(u,v)\|_2^2 \leq \varphi^2([(s,t)-(u,v)])$, with
$\varphi : x \to ax$, $a = \|X+Y\|_2$ so that $\int^\infty \varphi\left(e^{-x^2}\right)dx < \infty$ which that a sufficient condition due to X. FERNIQUE $([20]\,p.77)$ for the sample continuity of $(X(s,t);(s,t)\in T)$; furthermore
$E_p\left(e^{\alpha\|X^2\|_\infty}\right) < \infty$ for all $\alpha > \frac{1}{2}\|K\|_\infty$

CONSEQUENCE $\left\{\frac{E_n}{e_n}; n \geq 1\right\}$ is a.s. relatively compact in C

$\left(e_n = 1, n = 1,2; e_n = \sqrt{2\log\log n},\ n \geq 3\right)$. Indeed let us consider the random vector in $C: Z = (\cdot)^X(\cdot)^Y - G$. We have $E_p\left(\|Z\|^2\right) < \infty$. Thus we are in a position to apply theorem 4.3 ([29] p.20).

B. 2.1. APPROXIMATION OF THE EMPIRICAL PROCESS $E_n(s,t);(s,t) \in T$

This section is mainly devoted to Brownian Bridge and Kiefer process approximation of the empirical process $(E_n(s,t);(s,t) \in)$. We begin it with a well-known results for approximation of the empirical process indexed by functions and applications.

The reading of survey articles about this written by M.CSÓRGO, S. CSÓRGO, P. PRÉVÉSZ, and D. M. MASON ([15],[17],[19]) can be highly recommended. This problem is studied for arbitrary dimensions and arbitrary continuous distributions functions.We feel there is a chance that the method could be generalized for an arbitrary distribution function. First some notations and definitions.

D.1. Wiener Process (W.P.): A separable Gaussian Process (G.P)
$W(x) = \{W(x_1,\ldots,x_d); 0 \leq x_i < \infty,\ i = 1,\ldots,d\}$ with $EW(x) = 0$ and

$$R(x_1,x_2) = EW(x_1)W(x_2) = \prod_{i=1}^{d}\left(x_{1i} \wedge x_{2i}\right), x_l = \left(x_{li}\right)_{1 \leq i \leq d}\ l = 1,2$$

D.2. Brownian Bridge (B.B):
$B(x) = W(x) - x_1 \ldots x_d W(1,\ldots,1)$ where $W(\cdot)$ is a W.P.

D.3 Kiefer Process (K.P.):
$K(x,y) = K(x_1,\ldots,x_d;y) = W(x_1,\ldots,x_d;y) - x_1 \ldots x_d W(1,\ldots,1;y)$;
Where $W(x_1,\ldots,x_d;y)$ is a W.P. of $(d+1)$- dimensions.

Let $U_{1,n} \leq \ldots \leq U_{n,n}$ denote the order statistics of the first n of independent uniform $-(0,1)(U(0,1))$ random variables $U_1 \ldots U_n$ with the corresponding uniform empirical distribution D_n defined to be right continuous and uniform empirical quantile funtion $Q_n(s) = U_{k,n}, \frac{k-1}{n} \leq s \leq \frac{k}{n}(k = 1,\ldots,n)$ where $Q_n(0) = U_{1,n}$. We define the uniform quantite process $\tau_n(s) = \sqrt{n}\left(s - Q_n(s)\right)$. and the uniform empirical process $\theta_n(s) = \sqrt{n}(D_n(s) - s), 0 \leq s \leq 1$

Theorem: With an appropriate sequence of Brownian Bridges $\{B_n(s); 0 \le s \le 1\}$,on an appropriately constructed probability space $(\Omega', \breve{a}', P')$ we have

$$\Pr\left(\sup_{0 \le s \le \frac{d}{n}} |\tau_n(s) - B_n(s)| \ge n^{-1/2}(a \log d + x) \right) \le be^{-cx}$$

$$\Pr\left(\sup_{1-\frac{d}{n} \le s \le 1} |\tau_n(s) - B_n(s)| \ge n^{1/2}(a \log d + x) \right) \le be^{-cx}$$

whenever $n_0 \le d \le n$, $0 \le x \le d^{1/2}$, where n_o , a,b and c are suitably chosen positive constants.

Proof: [15]

(*): Let $\mathcal{L}$ denotes any class of functions $\mathbf{l}$ defined on (0,1) such that: each $\mathbf{l}$ can be written as $\mathbf{l} = \mathbf{l}_1 - \mathbf{l}_2$, where $\mathbf{l}_1$and $\mathbf{l}_2$ are nondecreasing left continuous functions defined on (0,1). Let L be a positive nonincreasing function defined on $\left(0, \frac{1}{2}\right]$ slowly varying near zero and define

$$N(\delta) = \sup_{\mathbf{l} \in \mathcal{L}} \sup_{0 \le s \le \delta} \left[\left(|\mathbf{l}_1(s)| + |\mathbf{l}_2(s)| + |\mathbf{l}_1(1-s)| |\mathbf{l}_2(1-s)| \right) \right] s^{1/2} / L(s)$$

Theorem: If $\lim_{\delta \downarrow 0} N(\delta) = 0$, then on $(\Omega', \breve{a}', P')$

$$\sup_{\mathbf{l} \in L} \left| \int_0^1 \mathbf{l}(s) d\theta_n(s) - \int_{\frac{1}{n}}^{1-\frac{1}{n}} \mathbf{l}(s) dB_n(s) \right| / L\left(\frac{1}{n}\right) = o_{P(1)}.$$

Proof: ([15])

At Present let F be a right continuous distribution function

Theorem: Let $A = \{g_t(\cdot); t \in [a,b]^d\}$ be a function class. Assum that A satisfying (*) with $L(\cdot) = 1$. Assume that the function $d_A^2(s,t) = \int_{d_A}^{+\infty} (g_s(x) -$

$-g_t(x))^2 dF(x)$ is continuous in $[a,b]^d x[a,b]^d$ and let $N_{d_A}\left([a,b]^d, \in\right)$be the minimum number of d_A-balls with centres in $[a,b]^d$ and radius at most $\in \rangle 0$ that cover $[a,b]^d$. If in addition the metric entropy condition.

$$J\left([a,b]^d, d_A\right) = \int_0^{\hat{d}_A} \left(\log N_{d_A}\left([a,b]^d, \in\right)\right)^{1/2} d\in < \infty .$$

is also satisfied where $\hat{d}_A = \sup\left\{d_A(s,t); s,t \in [a,b]^d\right\}$, then, on the probability space $(\Omega', \breve{a}', P')$, as $n \to \infty$

$$\sup_{t\in[a,b]^d} \left| \int_{-\infty}^{+\infty} g_t(x) d\theta_n(F(x)) - \int_{-\infty}^{+\infty} g_t(x) dB_n(F(x)) \right| = o_p(1).$$

Proof: $([17]\text{p}.17)$ or $([15])$

Remark: a well known sufficient condition of the metric entropy condition is

$$\int_0^{\delta} \left(\varphi_A(h) / \left(h \log h^{-1}\right)\right)^{1/2} dh < \infty \text{ for some } \delta > 0$$

where $\varphi_A(h) = \sup_{\substack{s,t\in[a,b]^d \\ \|s-t\|_d \le h}} d_A$ and $\|\cdot\|_d$ stands for the maximum norm in $\mathbb{R}^d$.

In our case, consider $A = \left\{g_{(s,t)}(x,y) = s^x t^y; (x,y) \in \mathbb{N}^2, (s,t) \in T\right\}$

Since $d((s,t),(u,v)) \le \|(s,t)-(u,v)\| \left(E\left((X+Y)^2\right)\right)^{1/2}$ and $\varphi_A(\varepsilon) \le k\varepsilon$

It follows that $\displaystyle\int_0^{\delta} \frac{\varphi_A(\varepsilon)}{\sqrt{\varepsilon \log \varepsilon^{-1}}} d\varepsilon < \infty$ for some $\delta > 0$

and as a consequence of Theorem 3.2 $([19].\text{p}1462)$ we obtain on the probability space $(\Omega', \breve{a}', P')$

$$\sup_{(s,t)\in T} \left| \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g_{(s,t)}(x,y) d\theta_n(F(x,y)) - \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g_{(s,t)}(x,y) dB_n(F(x,y)) \right| =$$

$= o_p(1)$ as $n \to +\infty$

Moreover,

$$\sup_{(s,t)\in T}\left|(1-s)(1-t)E_n(s,t)-(1-s)(1-t)\sum_{i,j\geq 0} s^i t^j B_n\left(F(i,j)\right)\right| = o_p\ (1)$$

as $n \to +\infty$

Since $B_n(v) = W(v) - vW(1)$, we have

$$(1-s)(1-t)\sum_{i,j\geq 0} s^i t^j F(i,j) = G(s,t)$$

and $(1-s)(1-t)\sum_{i,j\geq 0} s^i t^j B_n\left(F(i,j)\right) \overset{d}{=} \sum_{i,j\geq 0} s^i t^j N\left(o, \sigma_{ij}\right)$ where

$$\sigma_{ij} = \frac{1}{i!\,j!}\frac{\partial^{i+j}G(s,t)}{\partial s^i \partial t^j}(0,0)$$

Thus, we obtain the following statement

Proposition 4: $\| aE_n - b \|_\infty = o_p\,(1)$ as $n \to \infty$ where

$$a(s,t) = (1-s)(1-t) \text{ and } b(s,t) = \sum_{i,j\geq 0} s^i t^j N\left(0, \sigma_{ij}\right) - G(s,t)W\ (1)$$

The results described here follows from KOMLÓS. J, MAJOR. P, TUSNÁDY.G's results $([24])$ proved in the case when F is the distribution function of the uniform distribution on $(0,1)$. When F is continuous the generalization is trivial since $F(X)$ is uniformly ditributed on $(o,1)$. It was observed in a conversation with P. RÉVÉSZ that the extension is also quite straightforward even F is entirely arbitraire $([18])$. We hope that it could be generalized to more than one dimension.

Proposition 5: There exists a sequence of $B.B(B_n)$ and $K.PK(x;y)$ such that $\| E_n - Z_n \|_\infty = O(n^{-1/2}\log n) and \| E_n - n^{-1/2}L_n \| = O(n^{-1/2}\log^2 n)$

where

$$Z_n(s,t) = (1-s)(1-t)\sum_{i,j\geq 0} s^i t^j B_n\left(F(i,j)\right); L_n(s,t) = (1-s)(1-t)\sum_{i,j\geq 0} s^i t^j K_n\left(F(i,j);n\right)$$

Proof: Let $\alpha_n = \sqrt{n}(F_n - F)$. We have $E_n(s,t) = (1-s)(1-t)\sum_{i,j\geq 0} s^i t^j \alpha_n(i,j)$

It's enough to remark,

$$|E_n(s,t)-Z_n(s,t)|\le\sup_{i,j}\left|\alpha_n(i,j)-B_n(F(i,j))\right|,\ \forall(s,t)\in T$$

In the same way with $n^{-1/2}L_n$,

We have $\| E_n - Z_n \|_\infty = O(n^{-1/2}\log n)$ and $\| E_n - n^{-1/2}L_n \|_\infty = O(n^{-1/2}\log^2 n)$

More,

$\Pr(\| E_n - Z_n \|_\infty > a_n(Z)) \le A_2\,\varepsilon^{-A_3 Z}$ where

$a_n(Z)=n^{-1/2}((A_1\log n)+Z)$ for all n and $Z. A_1, A_2, A_3$ are positive constants.

B.2.2. Inequelity of Kahane-Khintchine with standard Orlicz norm.

This section will be devoted to the study of standar Orlicz norm-the luxembourg norm-as well as the inequality of Kahane-Khintchine for empirical process E_n

I would like to express my gratitude to Goran Peskir who sent me preprints of his work and to which I was faithful.

Proposition 6: Let $\sigma^2(s,t)=G(s^2,t^2)-G^2(s,t)$. If the process $(E_n(s,t);$ $(s,t)\in T)$ is pregaussian: that is $L_{E_n(s,t)}(a)\le\exp\frac{1}{2}\sigma^2(s,t)a^2,\ \forall a\in\mathbb{R}$. Then for every $C>\sqrt{2}\,\sigma(s,t)$ the sequence $\left\{\exp\frac{E_n^2(s,t)}{C}, n\ge 1\right\}$ is uniformly integrable. (L is the Laplace transform).

Proof: Let $C>\sqrt{2}\sigma(s,t)$, it is enougt to show that for some $p\rangle 1$ we have

$$s(p)=\sup_{n\ge 1}E\left[\exp\left(\frac{E_n(s,t)}{C}\right)^2\right]^p\langle\infty.$$

We have:

$$\sup_{n\ge 1}E\left\{\exp\left(\frac{E_n(s,t)}{C}\right)^2\right\}^p=\sup_{n\ge 1}\int_0^{+\infty}\Pr\left(\exp\left(p\left(\frac{E_n(s,t)}{C}\right)^2\right)>u\right)du=$$

$$=1+\sup_{n\ge 1}\int_1^{+\infty}\Pr\left(|E_n(s,t)|>\frac{C}{\sqrt{p}}\sqrt{\log u}\right)du$$

since by our assumption $\{E_n(s,t);(s,t)\in T\}$ is pregaussian we have

$$\sup_{n\geq 1} E\left(\left(\exp\left(\frac{E_n(s,t)}{C}\right)^2\right)^p\right) \leq 1+2\int_1^\infty \exp\left(-C^2\frac{\log u}{2p\sigma^2(s,t)}\right)du = 1+2\int_1^\infty u^{-\alpha}du,$$

$\alpha = C^2 / 2p\sigma^2(s,t)$
we see that there exists $p \in \left]1, C^2/2\sigma^2\right[$ for which $s(p)\langle\infty$.

Proposition 7: $\|E_n(s,t)\|_\psi \overset{n\to\infty}{\rightarrow} \sqrt{\frac{8}{3}}\sigma(s,t) = e(s,t)$.

Before proving proposition, we will go through some other facts. First, recalling $\|\cdot\|_\psi$ denotes the standard Orlicz norm or the Luxembourg norm on $(\Omega, \breve{a}, P)$:

$\|X\|_\psi = Inf\left\{a>0 / E\left(\psi\left(\frac{X}{a}\right)\right)\leq 1\right\}$ for all real valued random variables on $(\Omega, \breve{a}, P)$, where $\psi(x) = \psi(x) = e^{x^2} - 1\ \forall x \in \mathbb{R}$ and $\inf \varnothing = 0$
If $X \rightarrow N(0, \sigma^2(s,t)), \|X\|_\psi = e(s,t)$

Proof: Let $e_n(s,t) = \|E_n(s,t)\|_\psi$. In a first step assume that $e(s,t) < \overline{lim}\, e_n(s,t)$. Thus $e(s,t)+\varepsilon < e_{n_k}(s,t)$ for some $n_1 < n_2 < \ldots$ and some $\varepsilon > 0$. Since $e(s,t) > \sqrt{2}\sigma(s,t)$. Then by proposition 6-B2.2 the sequence $\left\{\exp\left(\frac{E_n(s,t)}{e(s,t)+\varepsilon}\right)^2, n\geq 1\right\}$ is uniformly integrable, and

$$2 \geq \int_\Omega \exp\left(\frac{X}{e(s,t)}\right)^2 dP > \int_\Omega \exp\left(\frac{X}{e(s,t)+\varepsilon}\right)^2 dP \geq \lim_{n\to\infty}\int_\Omega \exp\left(\frac{E_n(s,t)}{e(s,t)+\varepsilon}\right)^2 dP \geq$$
$$\geq \lim_{k\to+\infty}\int_\Omega \exp\left(\frac{E_{n_k}(s,t)}{e(s,t)+\varepsilon}\right)^2 dP \geq 2.$$

Thus $e(s,t) < \overline{lim}\, e_n(s,t)$ is not possible. Assume now that $e(s,t) > \underline{lim}\, e_n(s,t)$. Thus $e(s,t) - \varepsilon > e_{n_k}(s,t)$ for some $n_1 \langle n_2 \langle \ldots$ and some $\varepsilon \rangle 0$, we also have that

$$2 \geq \underline{lim}\int_\Omega \exp\left(\frac{E_n(s,t)}{e_{n_k}(s,t)}\right)^2 dP > \underline{lim}\int_\Omega \exp\left(\frac{E_{n_k}(s,t)}{e(s,t)-\varepsilon}\right)^2 dP \geq \underline{lim}\int_\Omega \exp\left(\frac{X}{e(s,t)-\varepsilon}\right)^2 dP$$

Thus $e(s,t) > \underline{lim}\ e_n(s,t)$ is not possible.
This concludes the proof.

Proposition 8: Let (X_i, Y_i), $i = 1,\dots,n$ be independent a.s. bounded pairs of random variables with generating functions $G_i, i = 1,\dots,n$. If

$D_n = \sqrt{n}\left(G_n - \overline{G}_{n,1}\right)$ where $\overline{G}_{n,1} = \dfrac{G_1+\ldots+G_n}{n}$. Then

$$\| D_n(s,t) \|_\psi \le \sqrt{\frac{32}{3n}} \left(\sum_{i=1}^{n} \left\| s^{X_i} t^{X_i} - G_i(s,t) \right\|_\infty^2 \right)^{1/2}$$

Proof: Let $((Z_1, T_1), \dots, (Z_n, T_n))$ be a random vector defined on the same probability space $(\Omega, \tilde{a}, P)$, such that $(X_1, Y_1), \dots, (X_n, Y_n), (Z_1, T_1), \dots, (Z_n, T_n)$ are independent and $(X_i, Y_i), (Z_i, T_i)$ identically distributed. Put

$\tilde{D}_n = \sqrt{n}\left(\tilde{G}_n - \overline{G}_{n,1}\right)$ where $\tilde{G}_n$ is the empirical generating function of

$\mathbb{P}_n = \dfrac{1}{n} \sum_{i=1}^{n} \delta_{(Z_i, T_i)}$, then we have:

$$\| D_n(s,t) \|_\psi \le \left\| D_n(s,t) - \tilde{D}_n(s,t) \right\|_\psi = d(n)$$

Indeed, let consider the function $f : (s,t) \to \exp\left(\dfrac{s-t}{d(n)}\right)^2$. Then $t \to f(s,t)$ is a convex, for all $s \in \mathbb{R}$.

$$E\left(f\left(D_n(s,t), E\tilde{D}_n(s,t)\right)\right) \le E\left(f\left(D_n(s,t), \tilde{D}_n(s,t)\right)\right) = E\left(\exp \frac{D_n(s,t) - \tilde{D}_n(s,t)}{d(n)}\right)^2 \le 2$$

Therefore

$$\| D_n(s,t) \|_\psi \le d(n) = \left\| D_n(s,t) - \tilde{D}_n(s,t) \right\|_\psi = \frac{1}{\sqrt{n}} \left\| \sum_{i=1}^{n} \left(s^{X_i} t^{Y_i} - s^{Z_i} t^{T_i} \right) \right\|_\psi$$

As $\left(s^{X_1} t^{Y_1} - s^{Z_1} t^{T_1}, \dots, s^{X_n} t^{Y_n} - s^{Z_n} t^{T_n}\right)$ is sign-symetric $([22].\ p.8)$. Thus by theorem 6 $([27].\ p.23)$

$$\| D_n(s,t) \|_\psi \le \sqrt{\frac{8}{3n}} \left(\sum_{i=1}^{n} \left\| s^{X_i} t^{Y_i} - s^{Z_i} t^{T_i} \right\|_\infty^2 \right)^{1/2}$$

and

$$\| D_n(s,t)\|_\psi \leq \sqrt{\frac{32}{3n}}\left(\sum_{i=1}^{n} \| s^{X_i, Y_i} - G_i(s,t)\|_\infty^2\right)^{1/2}$$

The proof, therefore complete.
In particular, if $(X_i, Y_i), i=1,\ldots,n$ be independent identically distributed pair a.s-bounded, then

$$\| E_n(s,t)\|_\psi \leq \sqrt{\frac{32}{3n}}\left(\sum_{i=1}^{n} \|s^{X_i} t^{Y_i} - G(s,t)\|_\infty^2\right)^{1/2}$$

C. GAUSSIAN RANDON FUNCTION (X=X(s,t);(s,t) ∈ T)

The main purpose of this section is to study the properties of the Gaussian random function ($X = X(s,t); (s,t) \in T$) determined by its covariance structure, more precisely by the reproducing kernel Hilbert space (RKHS). Expositions on Gaussian probability measure and the RKHS have been given in particular in the celebrated course ([26])by J. NEVEU and also by RAJPUT. B.S. ([31]).

Theorem: The stochastic process $(X = X(s,t); (s,t) \in T)$ admits a version which is measurable and separable.

Proof: On the index set T, consider the pseudo-distance d_X given by.

$$d_X((s,t);(u,v)) = \| x(s,t) - X(u,v)\|_2 = \sqrt{G(su,tv) - G(s,t)G(u,v)}$$

Since d_X is separable, we therefore have the result from a theorem of X. FERNIQUE ([22]. p.38).

C.1. THE REPRODUCING KERNEL HILBERT SPACE

At present, we're studying the Gaussian probability, which constitues the law of random vector without refering to it. Let $X:(\Omega, \breve{a}, P) \to \mathbb{R}^T, T = [0,1]\times[0,1]$ be a Gaussian random function. For all function α null unless on a finite subset of T, $\sum_{(s,t)\in T} \alpha(s,t)X(s,t)$ is a real Gaussian random variable.

$L(X,T)=\left\{\sum_T \alpha(s,t)X(s,t);\alpha \text{ vanishing outside finite subset of } T\right\}$ is a vector space.

Let $H_p(X,T)$ be the closure of $L(X,T)$ in $L^2(\Omega,\breve{a},P)$ (square P-integrable functions on $(\Omega,\breve{a},P)$). $H_p(X,T)$ is constitued by Gaussian random variables and called Gaussian space Spanned by X.

Let J be the map from $H_p(X,T)$ into $\mathbb{R}^T$ defined by $J(Z)(s,t)=E(ZX(s,t))$. Set $H(K)=J(H_p(X,T))$, then J is easily seen to be (1,1) mapping of $H_p(X,T)$ onto $H(K)$.

$H(K)$ is a Hilbert space with inner product defined by

$$< f,g >_{H(K)}=< J^{-1}(f),J^{-1}(g) >_{H_p}$$

Furthermore,

$H(K)$ consists of real functions h on T such that

$$K((s,t);.)\in H(K), \forall (s,t)\in T.$$

$$\forall h\in H(K),\ \forall (s,t)\in T < h;K((s,t),.) >_{H(K)}=h(s,t)$$

Remark 1: $H(K)$ is the closure of the subvector space spanned by $\{K((s,t),.);(s,t)\in T\}$.

Proposition 1: $H(K)\subset C(T)$

Proof: Let $h\in H(K)$. For all (s,t) and (u,v) in T. We have $|h(s,t)-h(u,v)|\le$ $\le\| h\|.\| X(s,t)-X(u,v)\|_{L^2}$.

As K is continuous, the Gaussian random function $(X=X(s,t);(s,t)\in T)$ is continuous in L^2. Then,

For $\varepsilon>0,\exists\eta>0:\forall(u,v)\in T:|s-u|v|t-v|<\eta$, we have $\| X(s,t)-X(u,v)\|_{L^2}<\frac{\varepsilon}{\|h\|_{H(K)}}$ and hence $|h(s,t)-h(u,v)|<\varepsilon$ i.e h is continuous.

Remark: a) The family $\{K((s_n,t_n);.):(s_n,t_n)\in S\}$ is total, where S is a dense subset of T.

b) For $h \in H(K), \sup_T |h| \leq C \| h \|_{H(K)}$ where $C \in \mathbb{R}$

The next proposition treat the orthogonal expansion of a Gaussian random function $(X = X(s,t); (s,t) \in T)$ and its covariance.

Proposition 2: For every complete orthonormal set $\{\xi_n ; n \in \mathbb{N}\}$ of $H_p(X,T)$, there corresponds a sequence $\{h_n ; n \in \mathbb{N}\}$ with the property that,

$$X(s,t) = \sum_{n \geq 0} h_n(s,t)\xi_n \,, K((s,t),.) = \sum_{n \geq 0} h_n(s,t)h_n \,.$$

For all $((s,t);(u,v)) \in T^2$ we have $K((s,t);(u,v)) = \sum_{n \geq 0} h_n(s,t)h_n(u,v)$; more the convergence is uniform.

Proof: $\forall (s,t) \in T, X(s,t) = \sum_{n \geq 0} a_n(s,t)\xi_n$

$$a_n(s,t) = < X(s,t); \xi_n >_{H_P} = < K((s,t);.), h_n > = h_n(s,t)$$

$$So, X(s,t) = \sum_{n \geq 0} h_n(s,t)\xi_n \quad \text{and} \quad K((s,t);.) = \sum_{n \geq 0} h_n(s,t)h_n$$

Thus, $K((s,t);(u,v)) = \sum_{n \geq 0} h_n(s,t)h_n(u,v)$

In particular

$$K((s,t);(s,t)) = \sum_{n \geq 0} h_n^2(s,t)$$

Then, the monotonuous convergence of the continuous fonctions $\sum_{n \leq p} h_n^2(s,t)$ when $\rho \uparrow \infty$ to the function $K((s,t);(s,t))$ is uniform by vertue of DINI's lemme.

This shows, that the series $\sum_{n \geq 0} h_n(s,t)h_n$ converges uniformly in *(s,t).* We write,

$$\sup_T \left\| K((s,t),.) - \sum_{n \leq p} h_n(s,t)h_n \right\| \overset{p \to \infty}{\longrightarrow} 0$$

eventually,

$$\sup_{TXT}\left| K((s,t);(u,v)) - \sum_{n\le p} h_n(s,t)h_n(u,v)\right| \le \sup_T \left\| K((s,t),.) - \sum_{n\le p} h_n(s,t)h_n \right\|$$

$$\sup_T \| K((s,t),.)\| \to 0 \text{ as } p \to 0$$

Corollary 1: Suppose that T endowed with the Borelian σ -field $\mathcal{T}_o$. Then the Gaussian randonm function $(X = X(s,t), (s,t) \in T)$ admits a measurable version.

Proof: Let $(\xi_n)_n$ be orthonormal sequence of $H_p(X,T)$, it corresponds a sequence $(h_n)_n$ in $H(K)$ (by *J*). The series $\sum_{n\ge 0} h_n(s,t)\xi_n$ converges in L^2 and almost surely. We define our version $\tilde{X}$ as follows.

$$\tilde{X}((s,t);\omega) = \limsup_{p\to\infty} \sum_{n\le p} h_n(s,t)\xi_n(\omega),\ \forall((s,t);\omega) \in Tx\Omega$$

Since, the functions $h_n; n \ge o$ are $\mathcal{T}_o$-measurable, it follows that $\tilde{X}$ is measurable which is the result of the corollary.

Proposition 3: For all $\mu \in M(T)$, there exists an unique element $\overline{X_\mu} \in H_p(X,T)$ such that

$$E_p(\overline{X_\mu} Y) = \int_T E_p(X(s,t)Y)d\mu(s,t) = \int_T J(Y)(s,t)d\mu(s,t)\ \forall Y \in H_p(X,T)$$

Proof: For proof of this proposition see J. NEVEU ([26]*p*.45).

Now we consider the mapping : $h \to \int_T h(s,t)d\mu(s,t) = \int_T < K((s,t),.);$ $h > d\mu(s,t)$ from H(K) to $\mathbb{R}$. Also there exists an unique element $h_\mu \in H(K)$ such that $< h_\mu, h >= \int_T < K((s,t);.); h > d\mu(s,t)$.

Therefore

$\forall (u,v) \in T,\ h_\mu(u,v) =< h_\mu, K((u,v);.) >_{H(K)} = \int_T K((s,t),(u,v))d\mu(s,t).$

Thus h_μ coincides with the function $\int_T K((s,t);.)d\mu(s,t) = J\left(\int_T X(s,t)d\mu(s,t)\right)$

Remark 3: a-Since $h_\mu = J(\overline{X_\mu}), H' = \{h_\mu, \mu \in M\}$ is dense in $H(K)$

b- $\forall (s,t) \in T, f_{\delta_{(s,t)}} = K((s,t);.)$.

c- $\forall \mu \in M, \forall (s,t) \in T$

$f_\mu(s,t) = \int_T K((s,t);(u,v))d\mu(u,v) = \mathrm{cov}(s^X t^Y, m_\mu(X,Y))$.

Proposition 4: Let $H(K)$ be the RKHS of the covariance K, then $H(K)$ is isomorphic to a vector subspace of $L^2(\mathbb{N}^2, p(\mathbb{N}^2), P)$.

Proof: To prove this, let σ denotes the map from H' into $L^2(\mathbb{N}^2, p(\mathbb{N}^2), P)$ defined by

$\forall f_\mu \in H' \sigma(f_\mu) = N_\mu : (i,j) \to m_\mu(i,j) - < G, \mu >$

- σ is linear
- σ preserves the norm. Indeed, for all $\mu \in M$ we have

$$\|\sigma(f_\mu)\|_2^2 = \sum_{i,j} p_{ij} N_\mu^2(i,j) = \sum_{i,j} p_{ij}(m(i,j) - < G, \mu >)^2 = \sum_{i,j} p_{ij}\, m_\mu^2(i,j) - < G, \mu >^2$$

On the other hand, $\| f_\mu \|^2 < f_\mu ; f_\mu > = \int_{TxT} K((s,t);(u,v))d\mu(s,t)d\mu(u,v)$

$So \|f_\mu\|^2 = \|\sigma(f_\mu)\|^2_{L^2(\mathbb{N}^2)}$

More precisely, if be an element of $H(K)$ and (f_n) a sequence in H' such that $f_n \to f$. Since $\|f_n - f_m\| = \|\sigma| f_n\rangle - \sigma(f_m)\|$, it follows that, the sequence $(\sigma(f_n))$ is a Cauchy sequence and it's convergent. Let $g = \lim_{n\to\infty} \sigma(f_n)$, we set $g = \sigma(f)$.

Not we prove that $\sigma(f)$ is independent of the sequence (f_n). Let (f_n) and (g_n) in H' such that $(f_n),(g_n) \to f$. Assume that $(\sigma(f_n)) \to \alpha$ and $(\sigma(g_n)) \to \beta$.

$$\|\alpha - \beta\|_{L^2(\mathbb{N}^2)} \le \|\alpha - \sigma(f_n)\|_{L^2(\mathbb{N}^2)} + \|\sigma(f_n) - \sigma(g_n)\|_{L^2(\mathbb{N}^2)} + \|\sigma(g_n) - \beta\|_{L^2(\mathbb{N}^2)} \underset{n\to\infty}{\to 0}$$

then $\alpha = \beta$.

Finally we define σ on $H(K)$ by: $\sigma(f)=\lim_{n\to\infty}\sigma(f_n)$ where $(f_n)\subset H'$ and $(f_n)\to f$. More $\|\sigma(f)\|_{L^2(N^2)}=\|f\|_{H(K)}$. Let L denotes $\sigma(H(K))$. For $(a_{ij})\subset L$ we have $\sum_{i,j} p_{ij}a_{ij}=0$ and $\sum_{i,j} p_{ij}a_{ij}^2<\infty$.

Proposition 5: The injection map j of $H(K)$ into C is continuous and $P_X=N_{C,M}(0,Q)=j\left(\gamma_{0,\|\cdot\|_{H(K)}}\right)$ where $\gamma_{0,\|\cdot\|_{H(K)}}$ is the canonical Gaussian cylindrical probability.

Proof: Let $\mu\in M, j(f_\mu)(u,v)=\int_T K((u,v);(s,t))d\mu(s,t)$

$|f_\mu(u,v)|=|<f_\mu;K((u,v),.)>|\le\|f_\mu\|\|K((u,v),.)\|$.

K is continuous on a compact TxT, hence it's bounded. We have
$|i_\mu(u,v)|\le M\|f_\mu\|\ \forall f_\mu\in H'$ and $\|j(f_\mu)\|_\infty\le M\|f_\mu\|$ which be enough to say j continuous.
Let $\gamma_{o,\|\cdot\|_{H(K)}}$ be the canonical Gaussian cylindrical probability on $H(K)$.
We have $j\left(\gamma_{o,\|\cdot\|_{H(K)}}\right)=\gamma_{o\|{}^tj(\cdot)\|_{H(K)}}$

For
$(\lambda,\mu)=\in M^2,<j(f_\mu),\lambda>=\int_T j(f_\mu)(s,t)d\lambda(s,t)=\int_{TxT}K((s,t);(u,v))d\lambda(s,t)d\mu(u,v)=$

$$=<f_\mu;f_\lambda>_{H(K)}=<f_\mu;{}^tj(\lambda)>_{H(K)}$$

Thus ${}^tj(\lambda)=f_\lambda$ and $\|{}^tj(\lambda)\|^2=\|f_\lambda\|^2=Q(\lambda)$

Finally, $j\left(\gamma_{o,\|\cdot\|_{H(K)}}\right)=\gamma_{o,\|{}^tj(\cdot)\|^2_{H(K)}}=N_{C,M}(O,Q)$

$$\begin{array}{c} <H(K),H(K)> \\ j\downarrow \qquad \uparrow {}^tj \\ \langle\ C,\ M\ \rangle \end{array}$$

D. * APPLICATION
D.1. * TEST OF SEPARATE FAMILIES DISTRIBUTIONS
D.2. * GOODNESS OF FIT TESTS

Some alternative procedures for testing hypotheses are discussed here. These procedures are based on the probability generating function.

While most of the attention has been directed toward the use of the empirical characteristic function, we suggest the use of the sample generating function to test hypotheses concerning discrete random variables.

D.1. A TEST OF SEPARATE FAMILIES OF DISTRIBUTION

In testing an hypothesis that a population has a specified distribution against the alternative that its distribution belongs to a separate family in the sens that an arbitrary simple hypothesis in H_o cannot be obtained as a limit of simple hypotheses in H_1, it is well known that the likelihood ratio does no apply. COX.D.R. ([12],[13]) proposed a test based on the difference in the maximum loglikelihoods of the sample under H_o and H_1. The test was further studied by JACKSON ([27]) and developed by ATKINSON ([3]), but was shown by PEREIRA ([30]) to be not always consistent.

Denote the random variable to be observed by (X,Y) and let H_o and H_1 be respectively the hypotheses that the probability generating function is $g_o(.;\theta)$ and $g_1(.;\alpha)$, where θ and α are unknown parameters with $\theta \in \Theta \subset \mathbb{R}^k$ and $\alpha \in \tilde{\alpha} \subset \mathbb{R}^l$. Actually, we are concerned with the problem of testing H_o:" $g = g_o(.,\theta); \theta \in \Theta \subset \mathbb{R}^k$ " against the alternative H_1:" $g = g_1(.,\alpha)$ $\alpha \in \tilde{\alpha} \subset \mathbb{R}^l$ ".

We impose the following regularity conditions C_0:

a- θ must be estimated from the data. Let $\{\hat{\theta}_n ; n\}$ be a sequence of maximum likelihood estimators.

b- $(\partial / \partial\theta_i) f_o((x,y);\theta)$ exists for each $i = 1,\ldots,k; \theta \in \Theta$ and $(x,y) \in \mathbb{N}^2$.

c- $\sum_{x,y \in \mathbb{N}} \left| \left(\frac{\partial}{\partial\theta_i} f_o((x,y),\theta) \right) \right| < \infty$ for each $i = 1,\ldots,k$ and $\theta \in \Theta$.

d- The function $\theta \to \frac{\partial^2}{\partial \theta_i \partial \theta_j} \log f_o((x,y);\theta)$ is continuous uniformly in (x,y) and $E_\theta\left((\partial^2 / \partial\theta_i \partial\theta_j)\log f_o((X,Y),\theta)\right) < \infty$. For all $1 \le i,j \le k$ and all θ.

e- The matrix $I(\theta) = E_\theta\left(\left(\frac{\partial}{\partial \theta_i}\right)\log f_o((X,Y),\theta)\left(\frac{\partial}{\partial \theta_j}\right)\right.$.

$\left.(\log f_o((X,Y),\theta))\right), 1 \le i,j \le k.$ is positive definite for all θ. The matrix $I(\theta)$ is called the FISHER information matrix.

f - The map $\theta \to (\partial / \partial\theta_i) g_o((s,t),\theta)$ is continuous uniformly in (s,t) for all $i = 1,\ldots,k$.

If we denote by $S((X,Y),\theta)$ the vector of partial derivatives $\left(\left(\frac{\partial}{\partial \theta_i}\right)\log f_o((X,Y),\theta); i = 1,\ldots,k\right)$ then $\hat{\theta}_n$ is a solution of the system of equation $\sum_{i=1}^{n} S((X_i,Y_i);\theta) = 0$.

Define the *k*-dimensional vectors $a_n(s) = \frac{1}{\sqrt{n}}\sum_{1=1}^{n} S((X_1,Y_1);\theta)$ and the *kxk* matrice

$$B_n(\theta) = \frac{1}{n}\left[\frac{\partial^2}{\partial \theta_i \partial \theta_j}\sum_{1=1}^{n} \log f_o((X_1,Y_1),\theta)\right]_{1 \le i,j \le k.}$$. According to C_o ,

$E_\theta(B_n(\theta)) = -I(\theta)$.

RESULT. 1: Let $(\hat{\theta}_n)$ be a sequence of maximum likelihood estimators. Then, $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow[n \to \infty]{\mathcal{L}} N(0, I^{-1}(\theta))$.

Proof: Let $g_n(\theta)$ designate the log-likelihood gradient $\sum_{l=1}^{n} S((X_l, Y_l);\theta)$. Then, $g_n(\hat{\theta}_n) = g_n(\theta) + nB_n(\tilde{\theta}_n).(\hat{\theta}_n - \theta)$ where $\tilde{\theta}_n$ is a point on the line segement between $\hat{\theta}_n$ and θ. Thus we can write

$-\sqrt{n}\, a_n(\theta) =. nB_n(\tilde{\theta}_n).(\hat{\theta}_n - \theta)$

$\cdot B_n(\tilde{\theta}_n) = B_n(\theta) + (B_n(\tilde{\theta}_n) - B_n(\theta))$

Since $E_\theta \left| \left(\frac{\partial^2}{\partial\theta_j \partial\theta_i} \right) \log f_o((X,Y),\theta) \right| < \infty$, $\forall \theta \in \Theta$, by the Kolmogorov theorem

$B_n(\theta) \to -I(\theta)$ a.s as $n \to \infty$.

$\cdot\cdot B_n(\hat{\theta}_n) - B_n(\theta) \to 0$ a.s as $n \to \infty$

Indeed, $\tilde{\theta}_n = \upsilon\hat{\theta}_n + (1-\upsilon)\theta$ for some $0 \le \upsilon \le 1$ and $(\hat{\theta}_n) \overset{n\to\infty}{\to} \theta$ a.s.,

$(\tilde{\theta}_n) \overset{n\to\infty}{\to} 0$ a.s.

More, the function $\theta \to \left(\frac{\partial^2}{\partial\theta_i \partial\theta_j} \right) \log f_o((x,y);\theta)$ is continuous uniformly en (x,y).

Hence $B_n(\tilde{\theta}_n) = -I(\theta) + \varepsilon_n$ with $(\varepsilon_n) \overset{n\to\infty}{\to} 0$ as. $I(\theta)$ is positive definite, this implies the nonsingularity of $B_n(\tilde{\theta}_n)$; with probability one, for sufficiently large n. Thus for sufficiently large n we can write

$$\sqrt{n}(\hat{\theta}_n - \theta) = -B_n^{-1}(\tilde{\theta}_n).a_n(\theta) \text{ a.s.}$$

Finally we obtain

$$\sqrt{n}(\hat{\theta}_n - \theta) \overset{\mathcal{L}}{\to} N(0, I^{-1}(\theta)), \text{ as } n \to \infty.$$

A present, consider the statistic $H(s,t) = \sqrt{n}(G_n(s,t) - g_o((s,t);\hat{\theta}_n))$

$H(s,t) = \sqrt{n}(G_n(s,t) - g_o((s,t);\theta)) - \sqrt{n}(g_o((s,t);\theta_n) - g_o((s,t);\theta))$.

We have $\sqrt{n}\left(G_n(s,t) - g_o((s,t);\theta)\right) \underset{n\to\infty}{\overset{\mathcal{L}}{\to}} N\left(0; g_o((s^2,t^2);\theta) - g_o^2((s,t);\theta)\right)$

Let $Dg_o((s,t);\theta) = \left((\partial / \partial\theta_i) g_o((s,t);\theta); i = 1,\ldots,k\right)$.

Since $g_o((s,t);\hat{\theta}_n) = g_o((s,t);\theta) + Dg_o((s,t);\tilde{\theta}).(\hat{\theta}_n - \theta)$ where $\tilde{\theta}_n$ is a point on the line segment connecting $\hat{\theta}_n$ and θ.
$* \ \hat{\theta}_n - \theta = o_p(1)$
It occurs that,

$\sqrt{n}\left(g_o((s,t);\hat{\theta}_n) - g_o((s,t);\theta)\right) = \sqrt{n}\, Dg_o((s,t);\theta).(\hat{\theta}_n - \theta) + \gamma_n$ with
$(\gamma_n) \to 0$ as

Therefore
$\sqrt{n}\left(g_o((s,t),\hat{\theta}_n) - g_o((s,t),\theta)\right) \underset{n\to\infty}{\overset{\mathcal{L}}{\to}} N\left(0, Dg_o((s,t),\theta) I^{-1}(\theta)' Dg_o((s,t),\theta)\right)$

Finally $H(s,t) \underset{n\to\infty}{\overset{\mathcal{L}}{\to}} N\left(0, \sigma_o^2((s,t),\theta)\right)$ with

$$\sigma_o^2((s,t),\theta) = g_o((s^2,t^2);\theta) - g_o^2((s,t);\theta) - \sum_{i,j=1}^{k} \alpha_{ij}\left(\frac{\partial}{\partial\theta_i}\right) g_o((s,t);\theta) \left(\frac{\partial}{\partial\theta_j}\right) g_o((s,t);\theta)$$

Where $I^{-1}(\theta) = (\alpha_{ij})_{1\le i,j\le k}$

As, $\theta \to \sigma_o^2((s,t);\theta)$ is continuous, we obtain the follow result

RESULT 2:

The statistic $\sqrt{n}\left(G_n(s,t) - g_o((s,t);\hat{\theta}_n)\right) / \sigma_o((s,t);\hat{\theta}_n)$ converges in law to $N(0,1)$, for all $(s,t) \in T$ so that $0 < \sigma_o^2((s,t);\theta) < \infty$

We reject the hypothesis H_o if
$|A(s,t)| > u_\alpha$ where $\Pr(|N(o,1)| > u_\alpha) = \alpha$ and $A(s,t) = \dfrac{H(s,t)}{\sigma_o((s,t);\hat{\theta}_n)}$

D.2. QUADRATIC TEST

We now discuss another applications of the empirical generating function to the problem of test the hypothesis that a random variable (X,Y) has a specified generating function G. Our first exposition is taken from J.L. SOLER ([33]) to which we refer for a more complete exposition.

After the work of ANDERSON and DARLING ([2]), several authors have used a similar procedure to that of Cramer-von Mises for various hypotheses testing problems and the bibliography in this domain is large.

Briefly, for testing some hypothesis H_o on the basis of a n-sample of a random variable they use a criterion which is based on some properties of the test under H_o as $n \to \infty$, they show that T_n converges in distribution to a r.r.v. of the form $Z = \int_0^1 X^2(t)\psi(t)dt$ where $\{X(t); t \in [0,1]\}$ is a zero-mean real Gaussian process with known covariance function and ψ is some fixed measurable positive function on $[0,1]$ verifying others conditions, as the case my be.

Actually, we are concerned with the problem of testing the hypotesis H_o:"$\theta = \theta_o$" against H_1:"$\theta \neq \theta_o$" in the Gaussian statistical space: $\left(C(T); B(C(T))\right); \left\{N_{C(T)}(\theta, Q_K); \theta \in H(K)\right\}$. Without loss of generality we can assume that $\theta_o = 0$.

Definition 1: A quadratic test of H_o:"$\theta = \theta_o$" against H_1:"$\theta \neq \theta_o$" in the previous statistical space will any test of the form $x \in C(T)$, $\phi_{\mu,l}(x) = 1_{\left\{\frac{1}{2} q_\mu(x) > l\right\}}$

where $q_\mu(x) = \int_T x^2(t)d\mu(t)$, $\mu \in M_C^+(T)$: positive cone of all regular Borel measure with compact support on T, and some positive real number **l**.

Expression of the power function

Let μ be such that $\int_{-\infty}^{+\infty} \left|\varphi_{Y_\mu}^\theta(\tau)\right| d\tau < \infty$ where $\varphi_{Y_\mu}^\theta$ is the characteristic function of the r.r.v $Y_\mu = q_\mu(X)$. Then, by a classical theorem on characteristic functions, the d.f. $F_{Y_\mu}^\theta$ of Y_μ is everywhere derivable and it is given by:

$$u \in \mathbb{R}_+ ,\ F^{\theta}_{Y_\mu}(u) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{e^{i\tau u} - \mathbf{1}}{-i\tau}\varphi^{\theta}_{Y_\mu}(\tau)d\tau$$

So that the power function of the quadratic test $\phi_{\mu,\mathbf{L}}$ is given for every $\theta \in H(K)$ by:

$$\beta_{\phi_{\mu,\mathbf{L}}}(\theta) = 1 - \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{e^{i\tau l^2} - 1}{-i\tau}\varphi^{\theta}_{Y_\mu}(\tau)d\tau$$

In particular, the significance level of the test is given by:

$$\alpha = \beta_{\phi_{\mu,\mathbf{L}}}(o) = 1 - \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{e^{i\tau l^2} - \mathbf{1}}{-i\tau}\det\left(\mathbb{1} - 2i\tau B^{K}_{\mu}\right)^{-1/2} d\tau$$

which theoretically gives $\mathbf{l} = \mathbf{l}(\alpha, \mu)$

Continuity: 1-For fixed $\mathbf{l}$ and μ, the power function $\theta \to \beta_{\varphi_{\mu,\mathbf{l}}}(\theta)$ is continuous on $H(K)$.

2 - For fixed $\mathbf{l}$ and θ, the functional $\mu \to \beta_{\phi_{\mu,\mathbf{l}}}(\theta)$ is continuous on the positive cone $M^{+}_{C}(T)$ endowed with its vague topology.

Proposition 1: For every $\alpha, 0 \le \alpha \le 1$, there exists an unbiased quadratic test of size α for testing H_0:"$\theta = 0$" against H_1:"$\theta \ne 0$".

Proof: Taking for the Dirac measure δ_{t_o} at some fixed point $t_o \in T$, the test $\phi_{t_o,\mathbf{l}}$ reduces to $\mathbb{1}_{\{|x(t_o)| \ge l\}}^{(x);x\in C(T)}$.

It is similar unbiased for testing the linear hypothesis H_0:"$\theta(t_o) = 0$" against H_1:"$\theta(t_0) \ne 0$", for, it coincides with the U.M.P.B. test.

Proposition 2: For every $\alpha, 0 \le \alpha \le 1$, ϕ_{μ} is the U.M.P.B test of the size α if and only if, for all $\theta \in H(K)$, $\mu(\{t \in T{:}\theta(t) \ne 0\}) > 0$.

Proof: See $([33]\text{p.}118)$.

Finally, quadratic criteria may be used to derive tests of equality of mean functions based on observed samples x and x' of two independent Gaussian processes X and X' with same known covariance function.

Indeed, by similar arguments to that used abov, any test of the form

$$\theta_{\mu,l}(x,x') = \mathbf{1}_{\left\{\left(\int_T (x(t)-x')(t)\right)^2 d\mu(t)^{1/2} > l\right\}} ; x, x' \in C(T)$$

will be similar for testing H_o:"$\theta = \theta'$" against H_1:"$\theta \neq \theta'$" in the product statistical space:

$\left(C(T), B(C(T)), \{N_{C(T)}(m, Q_K), m \in H(K)\}.\right) \times \left(C(T), B(C(T)),\right.$

$\left.\{N_{C(T)}(\theta', Q_K), \theta' \in H(K)\}\right)$ since $X - X' \to N_{C(T)}(\theta - \theta', Q_{2K})$.

D. 2.1. GOODNESS OF FIT TESTS FOR DISCRETE DISTRIBUTION IN THE STATISTICAL SPACE $(\mathbf{N}^2, \mathbf{P}(\mathbf{N}^2), \mathbf{P})$.

In the statistical space $(\mathbf{N}^2, \mathbf{P}(\mathbf{N}^2), \mathbf{P})^n$ corresponding to an n-sample, it is desired to test the null hypothesis H_o:"$P_{(X,Y)} = P$" against the alternative H_1:"$P_{(X,Y)} \neq P$". It's equivalent to test the hypothesis H_o:"$G_{(X,Y)} = G$" against H_1:"$G_{(X,Y)} \neq G$". We suggest the quadratic test

$$\varphi\left((X_1,Y_1);\ldots;(X_n,Y_n)\right) = \mathbf{1}_{\left\{\int_T E_n^2(s,t)\,ds\,dt > l^2\right\} = \text{critical region}}$$

If $\alpha, 0 < \alpha \leq 1$, is the size of the test φ, the value $\mathbf{l}_\alpha$ so that $\mathbf{l}_\alpha^2 = aF_{\chi_1^2}^{-1}(1-\alpha)$ is the approximate value of $\mathbf{l}$ for n sufficienthy large, where $a = \int_T \sigma^2(s,t)\,ds\,dt$, $\sigma^2(s,t) = G(s^2,t^2) - G^2(s,t)$ and $F_{\chi_1^2}$ the cumulative distribution function of the chi-square distribution with one degree of freedom.

The test statistic $T_n = \int_0^1 E_n^2(s,t)\,ds\,dt$ is expressed by $T_n = Q_{n,1} - 2Q_{n,2} + Q_{n,3}$

where $Q_{n,1} = \dfrac{1}{n}\left[\sum_{i=1}^{n} \dfrac{1}{(2X_i+1)(2Y_i+1)} + 2\sum_{i \neq j, j=1}^{n} \dfrac{1}{(X_i+X_j+1)(Y_i+Y_j+1)}\right]$

$$Q_{n,2} = \sum_{i=1}^{n} T_i \ ; \ T_i = \sum_{m,l\geq 0} P_{ml}\, T_{i,m,l} \ , \ T_{i,m,l} = \frac{1}{(X_i + m + 1)(Y_i + l + 1)}$$

$$Q_{n,2} = n \int_T G^2(s,t)\,ds\,dt$$

Now we test the independence of components. Specifically, we wish to test the null hypothesis H_o that the two variables X and Y are independents, that is $G = G_1 \times G_2$

Note that $E_n = \sqrt{n}\left(G_n - G_{1,2}\right)$ where $G_{1,2}(s,t) = G_1(s)G_2(t)$, $\forall (s,t) \in T$

The test is $\phi\left((X_1,Y_1);\ldots;(X_n,Y_n)\right) = 1_{\{T_n > l^2\}}$. T_n and l are the same as before with little modification: $Q_{n,3} = n\left(\int_0^1 G_1(s)ds\right)\left(\int_0^1 G_2(t)dt\right)$.

Remark 1: If the probability generating function is exponential type we suggest a graphical method for testing $H_o : "G_{(X,Y)} = G"$ against $H_1 : "G_{(X,Y)} \neq G"$. Indeed for all $(s,t) \in T, \left(\log G_n(s,t)\right)_n$ converges almost surely to log $G(s,t)$, and if almost surely we can fit the function $(s,t) \rightarrow \log G_n(s,t)$ to the function $(s,t) \rightarrow \log G(s,t)$ we accept the hypothesis H_o .

D.2.2. APPLICATION TO THE POISSON DISTRIBUTION

Let $\left(\mathbf{N}^2, \mathbf{P}(\mathbf{N}^2), \mathbf{P}\right)^n$ be a statistical space corresponding to a n-sample and $\mathbf{P}_o$ the family of Poisson distributions. All P in $\mathbb{P}_o$ has point probability of the form

$$P(i,j) = e^{-(\alpha+\beta)} \frac{\alpha^i \beta^j}{i!\,j!},\ i,j \in \mathbf{N}; \alpha, \beta > 0.$$

The problem now is to test the hypothesis that the true distribution belongs to a given family $\mathbf{P}_o$ from the remaining n observations.

One can use the test

$$\varphi\left((X_1,Y_1);\ldots;(X_n,Y_n)\right) = \mathbf{1}_{\{T_n > l^2\}}$$

the statistic T_n can also be expressed as $T_n = Q_{n,1} - 2Q_{n,2} + Q_{n,3}$.

$Q_{n,1}$ conserves his expression

$$Q_{n,2} = \sum_{i=1}^{n} \int_T s^{X_i} t^{Y_i} \exp\left(\alpha(s-1) + \beta(t-1)\right) ds\, dt = \sum_{i=1}^{n} I_{\alpha,X_i}\ I_{\beta,Y_2}$$

For this term, as it easy to verify the relations.

$$\begin{cases} I_{\alpha,0} = \frac{1}{\alpha}\left(1-e^{-\alpha}\right); I_{\beta,0} = \frac{1}{\beta}\left(1-e^{-\beta}\right) \\ I_{\alpha,X_i} = \frac{1}{\alpha}\left(1 - X_i I_{\alpha,X_i-1}\right); I_{\beta,Y_i} = \frac{1}{\beta}\left(1 - Y_i I_{\beta,Y_i-1}\right) \end{cases}$$

$$Q_{n,3} = n \frac{\left(1-e^{-2\alpha}\right)\left(1-e^{-2\beta}\right)}{4\alpha\,\beta}$$

ESTIMATION

In this section we consider the problem of the estimation of the parameters in the statistical space $\left(\mathbb{N}^2, \mathbf{P}\left(\mathbb{N}^2\right), \mathbf{P}_o\right)^n$. If $(X_1,Y_1);\ldots;(X_n,Y_n)$ are independent identically distributed random variables with distribution given by,

$e^{-(\alpha+\beta)} \frac{\alpha^X \beta^y}{x!y!}$; $(x,y) \in \mathbb{N}^2$; $\alpha, \beta > 0$. A complete sufficient (this follows easily from NEYMAN'S criterion for sufficiency) statistic is the sample total $T = \left(\overline{X} = \sum_{i=1}^{n} X_i, \overline{Y} = \sum_{i=1}^{n} Y_i\right)$. Therefore, as $G_n(s,t)$is an unbiased estimator of $G(s,t)$, the conditional expectation function $\tilde{G}_n(s,t)(T) = E\left(G_n(s,t)/T\right)$is the uniformly minimun variance unibiased (U.M.V.U) estimator of $G(s,t)$ (see [5]. Chap VI 2). The conditional distribution of (X,Y) given T is

$$P\left((X,Y)=(x,y)/T=(k,\mathbf{l})\right) = C_k^x \left(\frac{1}{n}\right)^x \left(1-\frac{1}{n}\right)^{k-x} C_{\mathbf{l}}^y \left(\frac{1}{n}\right)^y \left(1-\frac{1}{n}\right)^{\mathbf{l}-y} \quad \text{i.e}$$

$\mathcal{L}(X,Y)/\left(\overline{X},\overline{Y}\right) = B\left(\overline{X},\frac{1}{n}\right) \otimes B\left(\overline{Y},\frac{1}{n}\right)$. $B(.,.)$ is the Binomial distribution.

It follows that

$$\tilde{G}_n(s,t)(T) = E\left(G_n(s,t)/T\right) = E\left(s^X t^Y / T\right) = \left(1+\frac{s-1}{n}\right)^{\overline{X}} \left(1+\frac{t-1}{n}\right)^{\overline{Y}}$$

In particular the U.M.V.U. estimator of $e^{-(\alpha+\beta)}$ is $\left(1-\frac{1}{n}\right)^{\overline{X}}\left(1-\frac{1}{n}\right)^{\overline{Y}}$. In the same way $\left(1-\frac{1}{n}\right)^{\overline{X}}\left(\text{resp.}\left(1-\frac{1}{n}\right)^{\overline{Y}}\right)$ is an U.M.V.U. estimator of P_o . (resp. P_o) .

Now we trow to another estimator

$$\tilde{G}_n^{(p,q)}(s,t)(T)=E\left(p!q!C_X^p C_Y^q s^{X-p}t^{Y-q} / T\right) \quad p\le X\le\overline{X};\ q\le Y\le\overline{Y}$$

$$\tilde{G}_n^{(p,q)}(s,t)(t)=\sum_{k=p}^{\overline{x}}\sum_{l=q}^{\overline{y}} p!q!C_{\overline{x}}^k\, C_k^p\, C_{\overline{y}}^l C_l^q\, s^{k-p}t^{l-q}\left(\frac{1}{n}\right)^k\left(\frac{1}{n}\right)^l\left(1-\frac{1}{n}\right)^{\overline{x}}\left(1-\frac{1}{n}\right)^{\overline{y}-l}$$

$t=\left(\overline{x},\overline{y}\right)$

$$=\left\{I=\sum_{k=p}^{\overline{x}} p!C_{\overline{x}}^k C_k^p s^{k-p}\left(\frac{1}{n}\right)^k\left(1-\frac{1}{n}\right)^{\overline{x}-k}\right\}\left\{\sum_{l=q}^{\overline{y}} q!C_{\overline{y}}^l\, C_k^q\, s^{l-q}\left(\frac{1}{n}\right)^l\left(1-\frac{1}{n}\right)^{\overline{y}-l}\right\}$$

as $C_{\overline{x}}^k C_k^p=C_{\overline{x}}^p C_{\overline{x}-p}^{k-p}$ set $k-p=a$ where $a\in\{0,\dots,\overline{x}-p\}$. We obtain

$$I=p!C_{\overline{x}}^p\left(\frac{i}{n}\right)^p\left(1+\frac{s-1}{n}\right)^{\overline{x}-p}$$

finally

$$\tilde{G}_n^{(p,q)}(s,t)(T)=p!q!C_{\overline{X}}^p C_{\overline{Y}}^q\left(\frac{1}{n}\right)^{p+q}\left(1-\frac{1}{n}\right)^{\overline{X}-p}\left(1-\frac{1}{n}\right)^{\overline{Y}-q};\overline{X}\ge p,\overline{Y}\ge q,T=\left(\overline{X},\overline{Y}\right)$$

Remark 2: $\tilde{G}_n^{(p,q)}(s,t)(T)=\left(\partial^{p+q} / \partial s^p\partial s^q\right)\left(\tilde{G}_n(s,t)(T)\right)$

The statistic

$$\sum\left((X_1,Y_1);\ldots;(X_n,Y_n)\right)=p!q!C_{\overline{X}}^{p}C_{\overline{Y}}^{q}\left(\frac{1}{n}\right)^{p+q}\cdot\left(1-\frac{1}{n}\right)^{\overline{X}-p}\left(1-\frac{1}{n}\right)^{\overline{Y}-q}1\!\!1_{\left(\overline{X}\geq p,\overline{Y}\geq q\right)}$$

is an U.M.V.M. estimator of $e^{-(\alpha+\beta)}\alpha^{p}\beta^{q}$.

D. 2.3. COMPARAISON OF TWO DISTRIBUTIONS

Let (X,Y) and (Z,T) be two iid random variables with probability generating functions respectively G_1,G_2. Let $((X_1,Y_1);\ldots;(X_n,Y_n))$ and $((Z_1,T_1);\ldots;(Z_m,T_m))$ be two samples of (X,Y) and (Z,T). G_n^1 and G_m^2 be the corresponding empirical generating function and define a new empirical process $E_{n,m}$ by:

$E_{n,m}=\left(nm/(n+m)\right)^{1/2}\left(G_n^1-G_m^2\right)$. Suppose that $\frac{m}{n}=a$.

It's easy to verify that $E_{n,m}$ is a random vector scalarly integrable and integrable. More, under the hypothesis H_o :" $G_1=G=G_2$ " we have

$$E\left(E_{n,m}(s,t)\right)=0\quad\forall(s,t)\in T$$
$$E\left(E_{n,m}(s,t)E_{n,m}(u,v)\right)=K\left((s,t);(u,v)\right)\quad\forall(s,t),(u,v)\in T$$

Proposition: Under the hypothesis H_o , $\left(E_{n,m}\right)_{n,m}$ converges in cylindrical law to the Gaussian random vector $X\to N_{C,M}(0,Q)$.

Proof: Let $\mu\in M$,

$$\varphi_{E_{n,m}}(\mu)=E\left(e^{i<E_{n,m},\mu>}\right)=E\left(\exp\left(\frac{nm}{n+m}\right)^{1/2}i<G_n^1-G_m^2,\mu\rangle\right)$$

as, $G_n^1=E_n^1/\sqrt{n}+G$ and $G_m^2=E_m^2/\sqrt{m}+G$. We have

$$\varphi_{E_{n,m}}(\mu)\underset{n,m\to\infty}{\to}\varphi_X\left(\sqrt{\frac{a}{1+a}}\mu\right)\varphi_X\left(-\sqrt{\frac{1}{1+a}}\mu\right)=\varphi_X(\mu)$$

Thus $\left(E_{n,m}\right)_{n,m}$ converges in cylindrical law to $X\to N_{C,M}(0,Q)$.

The quadratic test for testing the hypothesis H_o :"$G_1 = G = G_2$" against H_1 :"$G_1 \neq G_2$" is

$$\varphi\left((X_1, Y_1); \ldots; (X_n, Y_n); (Z_1, T_1), \ldots, (Z_m, T_m)\right) = 1_{\left\{\int_T E_{n,m}^2(s,t)\,ds\,dt > \mathbf{l}^2\right\}}$$

Where the approximate value of **l** for n sufficiently large is
$\mathbf{l}_\alpha = aF_{\chi_1^2}^{-1}(1-\alpha)(0 < \alpha \leq 1); a = \int_T \sigma^2(s,t)\,ds\,dt$ and
$\sigma^2(s,t) = G(s^2, t^2) - G^2(s,t)$

CONCLUSION

This work, as we see forms a new way of approaching some well-Known mathematical statistic problems. However the set up object is restricted. So, we have not tackle the independence of two random variables by the empirical generating function as like as it has been dealt in the continuous case by BLUM et al ([6]) and so the approximation by Brownian Bridge and the Kiefer process ([11]).

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