Numerical analysis of semilinear parabolic problems with blow-up solutions.

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Abstract

In this paper we analyze the discretization in time of semidiscretized parabolic initial-boundary-value problems whose solutions blow up in finite time. We focus on collocation using piecewise linear functions and discuss the choice of the time step sequence (as a function of the collocation parameter). In addition, we present corresponding error bounds and convergence results. Numerical examples (including problems with delay arguments) illustrate the analysis

1. Introduction

In this paper we consider semilinear parabolic problems of the type

$$\text{(P)} \qquad \begin{cases} u_t(x,t) = \Delta u(x,t) + f(u) \text{ in } D\times(0,T), \\ u(x,t) = 0 \text{ on } \partial D\times(0,T), \\ u(x,0) = u_o(x) \geq 0 \text{ in } D, \end{cases}$$

where $D \subset \mathbb{R}^N$ is a bounded domain with smooth boundary ∂D, and $f:\mathbb{R}^+ \to \mathbb{R}^+$ satisfies the following conditions:

(F-1) f is locally Lipschitz continuous and non-decreasing;
(F-2) $lim_{s\to\infty} f(s)/s = \infty$;

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(F-3) $\int^{\infty} \frac{ds}{f(s)} < \infty.$

The initial function u_o is assumed to be sufficiently regular on D to guarantee the existence of a local (classical) solution. It is well known that for large u_o no solution can exist globally in time: for such initial functions u_o the solution $u(x,t)$ of (P) blows up in finite time; i.e. there exists a $T_b < \infty$ such that $u(x,t)$ exists for all (x,t) with $x \in D$, $t < T_b$, but

$$\lim_{t \to T_b^-} \sup_D u(x,t) = \infty$$

(see [11, 16] and the references mentioned in these papers).

The purpose of this paper is to analyze the blow-up properties of a class of simple discretization schemes (based on collocation in time) for the given problem (P). This analysis can easily be extended to problems where the Laplace operator is replaced by a more general elliptic operator of second order with time-independent coefficients. Our work generalizes results obtained in [17, 4, 5] in several ways; for example, we do not assume singlepoint blow-up (cf. [11, 5, 6]), and instead of employing the explicit Euler method and/or a discretization of the right-hand side of (P) which is linear in its implicit part our method is fully implicit. We note in passing that the existence and non-existence of solutions of certain elementary difference schemes for a related blow-up problem based on

$$u_t(x,t) = (u^{\sigma+1}(x,t))_{xx} + u^p,$$

with $\sigma > 0$ and $p > 1$, has been analyzed in [12, 13].

We shall give an error estimate for the difference between $u(x,t)$ and the approximate solution. This estimate will imply that the blow-up time $\tilde{T}_b$ of the discretized problem converges to T_b. However, we have not yet been able to derive a realistic error bound for $T_b - \tilde{T}_b$. Compare also [2, 5, 8, 9], as well as [19] (blow-up in ODEs), for similar analyses and difficulties. Although some results exist ([partial 17, 18]), many problems remain open.

2. Setting of the approximating problems

Consider in $\mathbb{R}^N$ a rectangular grid with mesh length $h>0$, parallel to the axes of the Cartesian coordinate system, and denote by e_k the unit vector in the direction of the x_k axis. Let $\{P_i\}$ be the set of grid points and extend $u(x,t)$ to the whole space $\mathbb{R}^N$ by setting $u(x,t)=0$ for $x\notin D$. Assume that $P_1,\ldots,P_r$ denote the grid points contained in the domain *D*. At such a grid point P_i we shall replace the derivative $u_{x_k x_k}$ by

$$D_k^2 u := h^{-2}\cdot\left(u(P_i+he_k,t)-2u(P_i,t)+u(P_i-he_k,t)\right)$$

The spatially discretized version of (*P*) is then given by

$$(P)_U \qquad \begin{cases} \dot{U}(P_i,t)=\sum_{k=1}^{N} D_k^2\, U(P_i,t)+f\left(U(P_i,t)\right)=:G_i(U), \\ U(P_i,t)=0 \text{ if } P_i\notin D, \\ U(P_i,0)=u_o(P_i)(i=1,\ldots,r) \end{cases}$$

In the following, let *U* (*t*) denote the vector whose components are given by $U(P_i,t)(i=1,\ldots,r)$.

We now approximate $U(t)$ by piecewise linear functions $V(t)$ in $C^o(0,T)$. Consider a variable grid for $[0,T]:0=t_o<t_1<\ldots<t_M=T$, with $\tau_m:=t_{m+1}-t_m$, and let $c\in[0,1]$ be a given collocation parameter. We assume that the approximating linear spline function $\mathbf{V}(t):=\left(V(P_1,t),\ldots,V(P_r,t)\right)^T$ has its knots at $t_1,\ldots,t_{M-1}$, and that in $[t_m,t_{m+1}]$ it is determined by

$$V(P_i,t)=V(P_i,t_m)+\frac{t-t_m}{\tau_m}\left(V(P_i,t_{m+1})-V(P_i,t_m)\right), \tag{2.1}$$

and the collocation equation

$$\dot{V}(P_i,t_m+c\tau_m)=\sum_{k=1}^{r} D_k^2\, V(P_i,t_m+c\tau_m)+f\left(V(P_i,t_m+c\tau_m)\right) \tag{2.2}$$

$(m=0,\ldots,M-1)$. For $P_i\in D$, the collocation equation (2.2), together with (2.1), leads to the system.

$$(P)_V \begin{cases} V(P_i,t_{m+1}) = V(P_i,t_m) \\ \qquad + \tau_m \sum_{k=1}^{r} D_k^2 \left(V(P_i,t_m) + c\left(V(P_i,t_{m+1}) - V(P_i,t_m) \right) \right) \\ \qquad + \tau_m \cdot f\left(V(P_i,t_m) + c\left(V(P_i,t_{m+1}) - V(P_i,t_m) \right) \right) \\ V(P_i,t_k) = 0 \text{ if } P_i \notin D (k = 1,\dots,M) \\ V(P_i,0) = u_o(P_i) \text{ for } i = 1,\dots,r \end{cases}$$

Equation $(P)_V$ and (2.1) together represent a *continuous one-stage implicit Runge-Kutta method* for the time-discretization of $(P)_U$

Three specific finite-difference methods known in the literature (forward Euler, a variant of Crank-Nicolson, and backward Euler) are particular cases of $(P)_V$ corresponding to collocation employing $c = 0$, $c = 1/2$, and $c = 1$, respectively.

3. Discussion of the approximating problems

Let us first focus on Problem $(P)_U$.

Lemma 3.1. *For any initial function $u_o \geq 0$, $(P)_U$ possesses a nonnegative local solution U which ceases to exist when blow-up occurs.*

Proof: Let U_ε be the solution of $(P)_U$ with initial data $u_o + \varepsilon (\varepsilon > 0)$ and boundary data ε . With this particular choice, the classical theory for ordinary differential equations implies that there exists a unique solution for small t. This solution is positive. Moreover, U_e achieves its minimum either at $t = 0$ or at the boundary $\partial D \times (0, t_o')$, where t_o' is the maximal time for which U_e exists. Suppose the contrary and let $(P', t') \in D \times (0, t_o')$ be the point where the minimum is attained. Then

$$\dot{U}_e(P',t') \leq 0, \quad \sum_{(k)} D_k^2 U_e(P',t') \geq 0, \quad \text{and} \quad f\left(U_e(P',t')\right) \geq 0.$$

If $f\left(U_e(P',t')\right) > 0$, $(P)_U$ yields a contradiction immediately. Otherwise we deduce that $\sum_{(k)} D_k^2 U_e(P',t') = 0$ and thus $U_e(P' \pm h e_j, t') = 0$ for all

$j = 1,\dots,N$. Repetition of this argument leads to a contradiction, and hence we have $U_e(P',t) \geq \varepsilon$. Since U_e depends continuously on ε we obtain

$$U := \lim_{e\to o} U_e \geq 0 \ \ for \ \ t \in (0,t')$$

This establishes the assertion.

Remark: Lemma 3.1 remains true if $f(s)$ is is replaced by $f(s) - \beta s$, $\beta > 0$.

By (F-2) there exists a function $M_o(\varepsilon) > 0$ such that

$$\frac{2sN}{h^2 f(s)} \leq \varepsilon \ \text{ for } \ s \geq M_o(\varepsilon).$$

Theorem 3.2 *Assume that (F-1) - (F-3) hold. Suppose that*
$\max_{(i)} U(P_i,t') = M_o(\varepsilon)$ *for some* $\varepsilon < 1$ and $t' > 0$,

and let

$$T'(\varepsilon) := sup\left\{t > 0 : \max_{(i)} U(P_i,t) < M_o(\varepsilon)\right\} < \infty$$

Then blow-up occurs at some finite time T_U which can be estimated by

$$\int_{M_o(e)}^{\infty} \frac{ds}{f(s)} \leq T_U - T'(\varepsilon) \leq \int_{M_o(e)}^{\infty} \frac{ds}{(1-\varepsilon) f(s)}$$

Proof: Set $\chi(t) := max_{(i)} U(P_i,t)$. It is a Lipschitz function satisfying

$$f(\chi) \geq \dot{\chi} \geq -\frac{2N}{h^2}\chi + f(\chi). \tag{3.1}$$

Observe that $\dot{\chi} \geq 0$ if $\chi \geq M_o(\varepsilon)$, $\varepsilon < 1$. Consequently, if

$$-\frac{2N}{h^2}\chi(t) + f(\chi(t)) \geq (1-\varepsilon) f(\chi(t))$$

holds for some $t_o \geq T'(\varepsilon)$, then this inequality is also true for all $t \geq t_o$. The proof follows now immediately from (3.1). □

Remark: The above theorem holds for all functions f which are positive and Lipschitz continuous on $\mathbb{R}^+$

Consider now problem $(P)_V$. Put

$$\delta_m(P_i) := V(P_i, t_{m+1}) - V(P_i, t_m),$$

and

$$z_m(P_i) := V(P_i, t_m) + c\delta_m(P_i).$$

We shall first address the question regarding the existence of a solution to $(P)_V$. In order to do so let us distinguish between two cases.

Case (i): $c = 0$

This corresponds to the *explicit Euler method.* Given $V(P_i, 0)$, we can compute successively $V(P_i, t_1), V(P_i, t_2), \ldots$

Case (ii): $c \neq 0$.

The equations for z_m are of the form

$$z_m(P_i) = c\tau_m \sum_{(k)} D_k^2 z_m(P_i) + c\tau_m f(z_m(P_i)) + V(P_i, t_m) \tag{3.2}$$

Write

$$|V(\cdot, t_m)|_\infty = \max_{(i)} |V(P_i, t_m)|$$

and correspondingly

$$|z_m|_\infty = \max_{(i)} |z_m(P_i)|.$$

Given $V(P_i, t_m)$ for all $P_i \in D$, (3.2) represents a nonlinear system of algebraic equations for the unknowns $z_m(P_i)$, $i = 1, \ldots, r$ (recall that r denotes the number of grid points lying in D; by assumption on D, r is finite). Existence will be proved by means of a fixed-point argument. For this purpose define the map $H: \mathbb{R}^r \to \mathbb{R}^r$ whose ith component is given by

$$H_i(w) := c\tau_m \sum_{(k)} D_k^2 w(P_i) + c\tau_m f(w(P_i)) + V(P_i, t_m),$$

with $w := (w(P_1), \ldots, w(P_r))^T$. The vector $z_m := (z_m(P_1), \ldots, z_m(P_r))^T$ is then a fixed point of this map. It is readily seen that for τ_m and $\rho := |V(\cdot, t_m)|_\infty$ sufficiently large, (3.2) does not have a solution. Since we are especially interested in large solutions (near blow-up), we must allow τ_m to be small.

Define, for given $\alpha > 1$, the ball

$$B(\alpha\rho) := \{w \in \mathbb{R}^r : |w| < \alpha\rho\}.$$

For $w \in B(\alpha\rho)$ we have

$$|H(w)|_\infty \le c\tau_m\left(\frac{4N}{h^2}\alpha\rho + f(\alpha\rho)\right)+\rho \tag{3.3}$$

This estimate leads to the following

Theorem 3.3 *If*

$$\tau_m < \frac{\rho(\alpha-1)}{c\left(4N\alpha\rho/h^2 + f(\alpha\rho)\right)},$$

then $(P)_V$ *has a solution in* $B(\alpha\rho)$.

Proof: The above assertion is a direct consequence of Brouwer's fixed-point theorem. □

In general, however, the solution of $(P)_V$ is not unique. We have certainly uniqueness if H is a contraction mapping.

Theorem 3.4 *Let* $L(M)$ *denote the Lipschitz constant of f in [0,M]. If*

$$\tau_m < min\left\{\frac{(\alpha-1)\rho}{c\left(4N\alpha\rho/h^2 + f(\alpha\rho)\right)}, \frac{1}{c\left(4N/h^2 + L(\alpha\rho)\right)}\right\},$$

then $H: B(\alpha\rho) \to B(\alpha\rho)$ *is a contraction.*

Proof: Observe first that

$$|H(w)-H(v)|_\infty \le c\tau_m\left(4N/h^2 + \frac{|f(w)-f(v)|_\infty}{|w-v|_\infty}\right)|w-v|_\infty .$$

If $w, v \in B(\alpha\rho)$, then by (F-1) we have $|f(w)-f(v)|_\infty \le L(\alpha\rho)\cdot|w-v|_\infty$. The assertion is now obvious. □

Remark: The above result remains true for large $\alpha\rho$ if $f(s)$ is replaced by $f(s)-\beta s$, $\beta > 0$.

Illustration:

For $f(u) = u^p - \beta u$ $(p > 1, \beta \ge 0)$ the inequality of Theorem 3.4 may be written as

$$\tau_m < min\left\{\frac{\alpha-1}{\alpha}\cdot\frac{1}{c\left(4N/h^2+(\alpha\rho)^{p-1}\right)-\beta},\frac{1}{c\left(4N/h^2+p(\alpha\rho)^{p-1}-\beta\right)}\right\},$$

or, for brevity, as

$$\tau_m < min\{B_1, B_2\},$$

with obvious meaning of B_1, B_2. We have $B_1 = B_2$ if α is a root of

$$\alpha^p - \frac{p}{p-1}\alpha^{p-1} + \frac{\beta-4N/h^2}{(p-1)\rho^{p-1}} = 0.$$

If, for efficient numerical computation, we want τ_m to be as large as possible, then we must choose α to be the smallest positive root. Let this root be denoted by α^*. (Note that α^* depends on m, $\alpha^* = \alpha_m^*$, since we have $\rho = \left| V(\cdot, t_m)\right|_\infty = \rho_m$). If $p = 2$, then

(3.4) $$\alpha^* = 1+\sqrt{1+(4N/h^2-\beta)/\rho}.$$

Remarks:

(1) Under the above conditions, the fixed point z_m depends continuously on the collocation parameter c.

(2) It follows immediately from $(P)_V$ and (F-2) that, if $max_{(i)} z_m(P_i) =: z_m(P_s)$ is sufficiently large, then

$$V(P_s, t_{m+1}) - V(P_s, t_m) \geq 0$$

(3) If $\rho_m \to \infty$, as $m \to \infty$, then (F-2) implies that $\tau_m \to 0$.

(4) If $|V(t_m)|_\infty = \rho_m \to \infty$, as $m \to \infty$, and if $\sum_{(m)} \rho_m / f(\alpha\rho_m) < \infty$, then $\sum_{m=1}^{\infty} \tau_m < \infty$. Consequently, $(P)_V$ has finite blow-up time. Other criteria which guarantee that the approximate blow-up time is finite can be found in [17] (for $N=1$, $f(u)=u^2$, and $c=0$); [18] (finite element method of lumped type on D and explicit Euler ($c = 0$)); [4] ($N = 1$), method implicit in linear part only, with $f(u) = u^p$).

4 Error estimates

We first derive an error estimate for $e(t) := U(t) - V(t)$ where $U(t)$ and $V(t)$ are, respectively, the solutions of problems $(P)_U$ and $(P)_V$ (i.e. $e(t)$ is the collocation error corresponding to the collocation solution $V(t)$ to $(P)_U$).

Lemma 4.1 *Assume* $(F-1)$, *and suppose that*

$$0 \le |U(t)|, |V(t)| \le M \quad \text{for} \quad t \in (0, t'] .$$

Set

$$m := \min\{k : t_k \ge t\} \quad \text{and} \quad \tau := \max_{l \le m}\{\tau_l\}$$

Then

$$|e(t)| \le B\tau \quad \text{for} \quad t \le t',$$

for some finite constant $B = B(M, t')$

Proof: By (2.1) we have

$$V(t_m + s\tau_m) = V_m + s(V_{m+1} - V_m), \quad V_k := V(t_k), \quad s \in [0,1] \tag{4.1}$$

Using Taylor's formula we may write

$$U(P_i, t_m + s\tau_m) = U(P_i, t_m) + s\tau_m \dot{U}(P_i, t_m) + R_m(P_i, s), \tag{4.2}$$

where

$$R_m(P_i, s) = s\tau_m\left[\dot{U}(P_i, t_m^{'}) - \dot{U}(P_i, t_m)\right]$$

for some $t_m^{'} = t_m^{'}(P_i) \in [t_m, t_m + s\tau_m]$. Thus, in view of $(P)_U$,

$$R_m(P_i, s) = s\tau_m\Big[h^{-2}\sum_{k=1}^{N} D_k^2\big(U(P_i, t_m^{'}) - U(P_i, t_m)\big) + f\big(U(P_i, t_m^{'})\big) - f\big(U(P_i, t_m)\big)\Big] \tag{4.3}$$

By (F-1), $|f(s_1) - f(s_2)| \le L(M)|s_1 - s_2|$ for all $s_1, s_2 \in [0, M]$. By applying once more Taylor's formula and observing the differential equation in Problem $(P)_U$ we conclude that

$$|R_m(P_i, s)|_\infty \le (s\tau_m)^2\{C_1 M + C_2 f(M) + C_3 L(M) M + C_4 L(M) f(M)\} \tag{4.4}$$

Here, the C_i are structural constants. If we write $\dot{U}(t) = G(U(t))\ (cf.(P)_U)$, then $e(t)$ satisfies

$$\dot{e}(t_m + c\tau_m) = G(U(t_m + c\tau_m)) - G(V(t_m + c\tau_m)).$$

Let $e_i(t) := U(P_i, t) - V(P_i, t)$ be the ith component of $e(t)$. By (4.1) and (4.2),

$$e_i(t_m + s\tau_m) = e_i(t_m) + s\tau_m \dot{U}(P_i, t_m) + R_m(P_i, s)$$
$$-s\left(V(P_i, t_{m+1}) - V(P_i, t_m)\right),$$

which we write as

(4.5) $$e_i(t_m + s\tau_m) = e_i(t_m) + s\tau_m \beta_{m-1}(P_i) + R_m(P_i, s).$$

Also,

$$\dot{e}_i(t_m + s\tau_m) = \beta_{m-1}(P_i) + \dot{R}_m(P_i, s)$$

where by (4.2)

$$\left|\dot{R}_m(P_i, s)\right| \leq s\tau_m \{C_1 M + C_2 f(M) + C_3 L(M) M + C_4 L(M) f(M)\}.$$

It follows from (4.5) that

$$\beta_{m-1}(P_i) + \dot{R}(P_i.c) = \sum_{k=1}^{N} D_k^2\, e_i(t_m + c\tau_m)$$
$$+ f\left(U(P_i, t_m) + c\tau_m\right) - f\left(V(P_i, t_m + c\tau_m)\right)$$
$$= \sum_{k=1}^{N} D_k^2 \left[e_i\left(t_m + c\tau_m \beta_{m-1}(P_i) + R_m(P_i, c)\right]\right.$$
$$+ \frac{f\left(U(P_i, t_m + c\tau_m) - f\left(V(P_i, t_m + c\tau_m)\right)\right)}{U(P_i, t_m + c\tau_m) - V(P_i, t_m + c\tau_m)}$$
$$\times\left[e_i(t_m) + c\tau_m \beta_{m-1}(P_i) + R_m(P_i, c)\right]$$
$$=: G'\left(\tilde{U}_m\right)\left[e_i(t_m) + c\tau_m \beta_{m-1}(P_i)\right] + \tau_m^2 \tilde{R}(P_i, c)$$

Hence,

(4.6) $$\left(1 - c\tau_m G'\left(\tilde{U}\right)\right)\beta_{m-1}(P_i) = G'\left(\tilde{U}_m\right) e_i(t_m) + r_m,$$

where

$$|r_m| \le \tau_m^2 K(M).$$

Observe that upon setting $s = 1$ in the above expressions for $e_i(t_n + s\tau_m)$ we find

$$e_i(0) = 0, \quad e_i(t_i) = \tau_o \beta_{-1}(P_i) + \tau_o^2 \tilde{R}_o(P_i, 1),$$

and

$$e_i(t_m) = \sum_{l=o}^{m-1} \tau_l \beta_{l-1}(P_i) + \sum_{l=o}^{m-1} \tau_l^2 \tilde{R}_l(P_i, \xi_l), \quad n \ge 1$$

Inserting this last excpression in (4.6) we obtain

$$\left(1 - c\tau_m G'\left(\tilde{U}_m\right)\right)\beta_{m-1}(P_i) = G'\left(\tilde{U}_m\right)\sum_{l=o}^{m-1} \tau_l \beta_{l-1}(P_i)$$

$$+G'\left(\tilde{U}_m\right)\sum_{l=o}^{m-1} \tau_l \tilde{R}_l(P_i, \xi_l) + r_m^2 \hat{R}_m(P_i, c).$$

Set $\beta_{m-1} := \left(\beta_{m-1}(P_1), \ldots, \beta_{m-1}(P_r)\right)^T$. Then

$$\left|G'\left(\tilde{U}_m\right)\beta_{m-1}(P_i)\right| \le k(M)|\beta_{m-1}|_\infty.$$

If $c\tau_m k(M) < 1$, then we have the estimate

$$|\beta_{m-1}| \le C_o \tau \sum_{t=o}^{m-1} |\beta_{l-1}| + C_1 \tau$$

The discrete version of Gronwall's Lemma yields

$$|\beta_{m-1}| \le C_l \tau \cdot \exp(C_o m\tau) =: B_o \tau$$

Hence,

$$|e(t_m)| \le B_o \tau \sum_{l=o}^{m-1} \tau_l + C_2 \tau = \tau(B_o t_m + C_2) =: B_1 \tau$$

and consequently,

$$|e(t_m + s\tau_m)| \le |e(t_m)| + \tau|\beta_{m-1}| + C_3 \tau^2 \le B\tau,$$

where $B = B(M, t')$. This completes the proof. □

Remark: If c = 1/2 then the collocation method corresponds to the one stage Runge-Kutta-Gauss method, and the resulting order is two:

$$\left|e(t_m)\right| \le B\tau^2.$$

While Runge-Kutta-Gauss methods with $s > 1$ stages exhibit order reduction (from $p = 2s$ to $p = s$) when applied to stiff systems or semilinear systems with stiff linear part (compare [3], this order reduction does not arise for $s = 1$ ([7,14]).

Let us now study the connection between $(P)_U$ and (P). We have

$$u(P_i + he_k, t) = u(P_i, t) + u_{x_k}(P_i, t)h + u_{x_k x_k}(P_i, t)h^2 / 2 + R_k^+$$

and

$$u(P_i - he_k, t) = u(P_i, t) - u_{x_k}(P_i, t)h + u_{x_k x_k}(P_i, t)h^2 / 2 + R_k^-,$$

and hence

$$D_k^2 u(P_i, t) = u_{x_k x_k}(P_i, t) + R_k / h^2$$

If $|u(x,t)| \le M$ in $\overline{D} \times [0, T']$, then according to the Schauder estimates for parabolic equations ([10]) we have $R_k = o(h^2)$. Therefore, $u(P_i, t), i = 1, \ldots, r$, solves a system of the form

$$(P') \qquad \dot{u}(P_i, t) = \sum_{k=1}^{N} D_k^2 u(P_i, t) + f(u(P_i, t)) + \omega$$

where $\omega \to 0$ as $h \to 0$. The order of $\omega(h)$ depends on the regularity of f.

In order to compare the solutions of $(P)'$ to those of $(P)_U$ we shall discuss some maximum principles for $(P)_U$. These are discrete versions of the well-known maximum principles for the continuous case (P). In the following we write $W \ge U$ for two elements in $\mathbb{R}^r$ if the inequality holds for each of the components.

Lemma 4.2 Let and W solve, respectively, $\dot{U} = G(U)$ and $\dot{W} \ge G(W)$ in $(0, T')$. If $W(0) \ge U(0)$, then $W(t) \ge U(t)$ for all $t \in (0, T')$.

Proof: Set

$$M := \max\left\{\sup_{(0,T')} |U|_\infty, \sup_{(0,T')} |W|_\infty\right\}.$$

Since f is Lipschitz in $(0, M)$, there exists a real number $L > 0$ such that $f(s) - Ls$ is stricly decreasing in $(0, M)$. Observe that $\hat{U} := \exp(-Lt)U$ and $\hat{W} := \exp(-Lt)W$ satisfy

$$d\hat{U}/dt = \exp(-Lt)\dot{U} - L\hat{U} = \exp(-Lt)G(U) - L\hat{U}$$

and

$$d\hat{W}/dt = \exp(-Lt)\dot{W} - L\hat{W} \geq \exp(-Lt)G(W) - L\hat{W},$$

or, equivalently,

(4.7)
$$\frac{d\hat{U}(P_i,t)}{dt} = \sum_{(k)} D_k^2\hat{U}(P_i,t) + \exp(-Lt)\left(f(U(P_i,t)) - LU(P_i,t)\right)$$
$$\frac{d\hat{W}(P_i,t)}{dt} \geq \sum_{(k)} D_k^2\hat{W}(P_i,t) + \exp(-Lt)\left(f(W(P_i,t)) - LW(P_i,t)\right).$$

Suppose that the assertion is false. Then $\hat{W}(P_i,t) - \hat{U}(P_i,t)$ takes in $(0,T')$ its negative minimum at some point (P_j, t_o). At this point,

$$\frac{d}{dt}\left(\hat{W} - \hat{U}\right) \leq \text{ and } \sum_{(k)} D_k^2\left(\hat{W} - \hat{U}\right)(P_i,t) \geq 0,$$

and, because of the strict monotonicity of $f(s) - Ls$, we have

$$f(W) - f(U) - L(W - U) > 0$$

This is obviously impossible. □

Let us go back to Problem $(P)'$. If $|\omega|_\infty \leq \delta$ in $(0,t')$, then $u(P_i,t)$ solves

(4.8)
$$G(u) - q \leq \dot{u} \leq G(u) + q, \quad |q_i| := \delta.$$

Lemma 4.3 *Suppose* $(F-1)$ *and* $|u|_\infty \leq M$ *for* $t \leq t'$. *Then we have*

$$\left|u(P_i,t) - U(P_i,t)\right| \leq C(M,t')\delta \quad (i = 1,\ldots,r) \quad \textit{whenever} \quad t \leq t'.$$

Proof: Consider the function $r(t) := q\exp(L(M)t) - q/L(M)$, where $L(M) > 1$ is the Lipschitz constant of f in the interval $[0, M]$. This function is positive and satisfies $\dot{r} = Lr + q$. For $W = u + r(t)$ we obtain

$$\dot W(P_i,t)=\dot u(P_i,t)+Lr+q$$
$$\geq \sum_{(k)} D_k^2 W(P_i,t)+f\big(W(P_i,t)-r\big)+Lr$$
$$\geq G_i\big(W(P_i,t)\big)$$

(cf. Problem $(P)_U$). By Lemma 4.1, $W\geq U$. Similary, we show that $Z\leq U$, where $Z:=u-r(t)$. This completes the proof. □

If we now combine the results of Lemma 4.1 and Lemma 4.3 we obtain

Theorem 4.4. Assume $(F-1)$. Let $M>0$ be any given real number and suppose that $u(x,t)$, $U(t)$ and $V(t)$ satisfy

$$|u(\cdot,t)|_\infty,\ |U(t)|_\infty\ |V(t)|_\infty<M \text{ for } t\in\big(0,t'(M)\big).$$

Then

$$|u(P_i,t)-V(P_i,t)|\leq B(M,t')\tau+C(M,t')h,$$

where $\tau:=\max_{(k)}\tau_k$.

Remarks: If $T_b<\infty$ is the blow-up time for (P) and $T_b(h,\tau)$ is the blow-up time for the corresponding discretized problem $(P)_V$, then

$$\lim_{h,\tau\to o} T_b(h,\tau)=T_b\ .$$

Note that in general $t'(M)$ is unknown since it depends on $u(x,t)$ and $U(t)$ for which only the approximation $V(t)$ is given. An estimate can be obtained in the following way. Observe that the solution of

$$\dot z=f(z),\ z(0)=\max_D u_o(x)=:m$$

is an upper solution for $(P)_U$. Hence,

$$u(x,t)\leq z(t)\leq M \text{ if } t<\int_m^M\frac{ds}{f(s)}=:t''.$$

Similary, the function $W(t)$ with $W_i(t)=z(t)$ satisfies

$$\dot W(t)=G(W(t)) \text{ in } (0,t''),\ W(0)\geq U(0).$$

By Lemma 4.2, $U(t)\leq W(t)$ in $(0,t'')$. Consequently if $|V(t)|_\infty\leq M$ in $(0,\tilde t)$, then $t'\geq\min(t'',\tilde t)$,.

5. Further remarks: extensions

The analysis of the preceding sections can be readily extended to semilinear parabolic problems where the reaction term $f(u)$ contains a *delay argument*, or where $f(u)$ is replaced by a *memory term.*

Consider first the problem

$$(PD)\begin{cases} u_t(x,t) = \Delta u(x,t) + f_1(u(x,t)) - f_2(u(x,t-r)) \text{ in } D\times(0,T), \\ u(x,t) = 0 \text{ on } \partial D\times(0,T), \\ u(x,t) = A\phi(x,t) \geq 0 \text{ on } D\times[-r,0] \end{cases}.$$

Here, the reaction term $f(u)$ of (P) is replaced by the difference of $f_1(u)$ and a delayed reaction term $f_2(u(\cdot,t-r))$, where $r>0$ is a given constant delay. The initial function ϕ is assumed to be sufficiently regular on its domain.

The (analytical and approximate) blow-up results obtained in [15] are similar to those for (P) and its discretizations. We refer to this thesis for details. However, since [15] does not contain any numerical examples, we illustrate the blow-up behavior of (PD) and its discretization corresponding to $(P)_V$ by means of an example (Example 6.2 below).

In [1] the blow-up results for (P) were generalized to

$$(PM)\begin{cases} u_t(x,t) = \Delta u(x,t) + \int_o^t a(t-s) f(u(x,s))ds \text{ in } D\times(0,T), \\ u(x,t) = 0 \text{ on } \partial D\times(0,T), \\ u(x,0) = u_o(x) \geq 0 \text{ in } D \end{cases}$$

Here, the *memory kernel* $a(t)$ is assumed to be strictly positive and nonincreasing.

It is clear from the foregoing discussion that our analysis will carry over to (PM), except that we have to deal with an additional discretization step, namely the approximatioin by suitable quadrature formulas of the memory term. Details of this analysis will be given elswehere.

6. Numerical examples

Example 6.1: We consider problem (P) with $N=1$, $D=(0,1)$, and

$f(u) = u^p - \beta u$ (whit $p > 1$, $\beta \geq 0$). Let the initial function u_o be given by

$$(6.1) \quad u_o(x) = \begin{cases} A \sin(\pi x / (2x_a)) & \text{if } 0 \leq x \leq x_a \\ A \cos(\pi(x - x_a) / (2(1 - x_a))) & \text{if } x_a \leq x \leq 1 \end{cases}$$

Here, $x_a \in (0,1)$ is given; in the numerical examples it will be chosen as $x_a = 0.2$. The above initial function leads to single-point blow-up at $x_b = 0.5$ for any $x_a \in (0,1)$.

The (uniform) spatial grid in $(P)_V$ is determined by $h = 1/M$, and the sequence $\{\tau_m\}$ of time steps was selected according to Theorem 3.4 (with < replaced by =); the value of the parameter $\alpha(= \alpha_m)$ is as in (3.4). We computed approximations corresponding to the values $c = 1/2$ and $c = 1$ (backward Euler) of the collocation parameter; as can be seen from the result in Theorem 3.4, using $c = 1$ requires twice the number of time steps needed for $c = 1/2$.

In Tables 1-4 we list a sample of computed blow-up times $\tilde{T}_b$ correspondings to the values A = 14, $p = 2$ and $\beta = 0$, $\beta = 1$, respectively; the "blow-up thresholds" M_i were chosen as $M_o = 4 \cdot 10^3$ and $M_1 = 10^4$. The number of time steps needed to reach the blow-up threshold M_i is denoted by $\mu(M_i)$

M	$\tilde{T}_b(M_o)$	$\tilde{T}_b(M_1)$	$\mu(M_o)$	$\mu(M_1)$
80	0.19731	0.19745	5415	5424
160	0.19751	0.19766	20936	20961

Table 1: *Example 6.1*: A = 14, *p* = 2, β = 0, *c* = 1

M	$\tilde{T}_b(M_o)$	$\tilde{T}_b(M_1)$	$\mu(M_o)$	$\mu(M_1)$
80	0.19751	0.19766	2711	2716
160	0.19756	0.19772	10471	10484

Table 2: *Example 6.1*: A = 14, *p* = 2, β = 0, *c* = 0.5

M	$\tilde{T}_b(M_o)$	$\tilde{T}_b(M_1)$	$\mu(M_o)$	$\mu(M_1)$
80	0.27447	0.27462	7491	7500
160	0.27479	0.27494	29048	29074

Table 3: *Example 6.1*: $A = 14, p = 2, \beta = 1, c = 1$

M	$\tilde{T}_b(M_o)$	$\tilde{T}_b(M_1)$	$\mu(M_o)$	$\mu(M_1)$
80	0.27468	0.27484	3749	3754
160	0.27485	0.27501	14528	14541

Table 4: *Example 6.1*: $A = 14, p = 2, \beta = 1, c = 0.5$

Example 6.2: For $N = 2$, the numerical computations we are aware of were carried out for the case where D is a disk and the solution u is *radially symmetric* (compare [5,8,9]). In this example we consider (P) with $N = 2$, $D = (0,1) \times (0,1)$, and $f(u) = u^p - \beta u (p > 1, \beta \geq 0)$. The initial function is

$$u_o(x) = A u_o^{(1)}(x_1) u_o^{(2)}(x_2)$$

where

$$u_o^{(i)}(x_i) := \begin{cases} \sin(\pi x_i / (2a_i)) & \text{if} \quad 0 \leq x_i \leq a_i \\ \cos(\pi(x_i - a_i)/(2(1-a_i))) & \text{if} \quad a_i \leq x_i \leq 1 \end{cases}$$

It has a peak of height A at $x_a = (a_1, a_2)$; the a_i are given parameters in (0,1).

We are using a uniform (rectangular) spatial grid with $h = 1/M$, and the time step sequence $\{\tau_m\}$ will be chosen as described in Theorem 3.4, with (optimal) value of $\alpha (= \alpha_m)$ given by

$$\alpha^* = 1 + \sqrt{1 + (8/h^2 - \beta)/\rho}$$

(recall (3.4)). A selection of numerical results is given in Tables 5-8.

M	$\tilde{T}_b(M_o)$	$\tilde{T}_b(M_1)$	$\mu(M_o)$	$\mu(M_1)$
20	4.6296 D-02	4.6346 D-02	111	112
40	4.6537 D-02	4.6592 D-02	363	365
50	4.6566 D-02	4.6662 D-02	545	547

Table 5: *Example 6.2*: $A = 40, p = 2, \beta = 0, x_a = (0.5, 0.5), c = 0.5$

M	$\tilde{T}_b(M_o)$	$\tilde{T}_b(M_1)$	$\mu(M_o)$	$\mu(M_1)$
20	6.4017 D-02	6.4069 D-02	147	148
40	6.4374 D-02	6.4429 D-02	493	495
50	6.4422 D-02	6.4477 D-02	743	745

Table 6: *Example 6.2*: $A = 40, p = 2, \beta = 0, x_a = (0.1,0.1), c = 0.5$

M	$\tilde{T}_b(M_o)$	$\tilde{T}_b(M_1)$	$\mu(M_o)$	$\mu(M_1)$
20	1.3128-D-03	1.3128 D-03	346	350
40	1.3150 D-03	1.3151 D-03	1180	1185
50	1.3154 D-03	1.3154 D-03	1788	1794

Table 7: *Example 6.2*: $A = 20; p = 3, \beta = 1, x_a = (0.5,0.5), c = 0.5$

M	$\tilde{T}_b(M_o)$	$\tilde{T}_b(M_1)$	$\mu(M_o)$	$\mu(M_1)$
20	1.3674 D-03	1.3674 D-03	353	357
40	1.3519 D-03	1.3519 D-03	1205	1210
50	1.4507 D-03	1.3507 D-03	1823	1830

Table 8: *Example 6.3*: $A = 20, p = 3, \beta = 1, x_a = (0.1,0.1)\ c = 0.5$

Example 6.3: We consider the delay problem (PD) with $N = 1$, $D = (0,1)$, $f_1(u) = u^p$ and $f_2(u) = u$. Let the initial function $\phi\ (x,t)$ be given by $\phi(x,t) = u_o(x)\phi_o(t)\ (x \in (0,1), t \in [-r,0])$, where $u_o(x)$ is as in (6.1) and

$$\phi_o(t) = \exp(\gamma t), \quad \gamma \geq 0 \quad (t \in [-r,0])$$

As in Example 6.1, this choice of the initial function leads to single-point blow-up at $x = x_b = 0.5$ (cf. [15]).

The spatial grid in $(P)_V$ is again determined by $h = 1/M$, and the sequence $\{\tau_m\}$ of time steps was selected according to Theorem 3.4 (with < replaced by =); the value of the parameter $\alpha(=\alpha_m)$ is as in (3.4) (with $\beta = 0$). The computed approximations correspond to the values c = 1/2 and c = 1 (backward Euler) of the collocation parameter.

In Tables 9 and 10 we list a sample of computed blow-up times $\tilde{T}_b$ corresponding to the values p = 2, A = 14, r = 0.5, and γ = 0, 10, respectively; as before, the "blow-up thresholds" M_i were chosen as $M_o = 4 \cdot 10^3$ and $M_1 = 10^4$.

M	$\tilde{T}_b(M_o)$	$\tilde{T}_b(M_1)$	$\mu(M_o)$	$\mu(M_1)$
80	0.26299	0.26315	3591	3596
160	0.26315	0.26331	13912	13925

Table 9: *Example 6.3*: A = 14, p = 2, γ = 0, c = 0.5

M	$\tilde{T}_b(M_o)$	$\tilde{T}_b(M_1)$	$\mu(M_o)$	$\mu(M_i)$
80	0.19795	0.19811	2717	2722
160	0.19801	0.19817	10495	10508

Table 10: *Example 6.3*: A = 14, p = 2, γ = 10, c = 0.5

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