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Solitary waves, solitons and related (nonlinear) waves in dissipative media

POR MANUEL G. VELARDE *

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Abstract

At the end of last century the 'solitary wave', the 'wave of translation' or *The wave*, was an exotic (and to some like Airy an impossible) object in wave theory. Lamb's treatise on *Hydrodynamics,* the 'bible' of the early part of this century did only mention it slightly although great minds as Lord Rayleigh and Boussinesq, and subsequently Korteweg and de Vries and McCowan among others, did pay due attention and tribute to the August 1834 discovery by J. Scott Russell at "Turning Point" along Union Canal near Edinburgh. The situation drastically changed with the advent of computers and when in 1965 the *soliton* concept was coined and its properties described in the study of ideal, dissipation-free nonlinear integrable systems like the Korteweg-de Vries equation and later on other equations, Toda's lattice, etc. Originally appearing as *solitary* waves, eventually established by an appropriate (local) balance between nonlinearity and dispersion, their peculiar properties of invariance upon translation with energy conservation and *elastic* (overtaking) collisions experiencing at most a phase shift in trajectories led Zabusky and Kruskal to consider them as particle-like ('perfect' particles/molecules/hard spheres). In the past few years I have been engaged in establishing, analytically, numerically and experimentally, the existence of *dissipative solitons* as a consequence of instability and thus when no energy is conserved but there is however appropriate balance between energy supply/pumping/production ussualy at long wavelengths and dissipation at shorter ones by viscosity. As an illustration I take the onset and sustainment of solitonic waves in an open shallow liquid layer heated from the air side or with appropriate surfactant adsorption or desorption (a similar situation occurs with internal waves in the sheared stably stratified amosphere in the ocean). Created and maintained following an instability threshold, these propagating localized structures/pulses, 'imperfect'/van der Waals-like molecules/dissipative solitonic waves feature *elastic* and *inelastic* head-on and oblique collisions, wall reflections with and without formation of Mach stems, bound states and chaos.

* Instituto Pluridisciplinar, Universidad Complutense. Paseo Juan XXIII, nº 1. Madrid-28040.

1. Introduction

Since the discovery of the 'solitary wave', the 'wave of translation' or *The Wave* by J. Scott Russell [1] in August 1834 at "Turning Point" along Union Canal near Edinburgh, the theory developped by Lagrange, Boussinesq [2], Lord Rayleigh [3] and others like Korteweg and de Vries [4] and McCowan [5], and later on the invention of the "soliton" concept by Zabusky and Krustal [6] following their numerical integration of the Korteweg-de Vries (KdV) equation and the breakthrough by Gardner and colleagues [7] and Lax [8] on its integrability a wealth of results and even an entire body of doctrine exists about soliton-bearing integrable equations. Very litle, however, is known concerning soliton stability and much less about solitons in viscous fluids and dissipative media in general.

Zabusky and Kruskal numerical computations or computer experiment, as well as the related pioneering work by Fermi, Pasta and Ulam [9], revealed new and unexpected results. First, they discovered that the KdV equation not only sustains solitary propagating localized structures but that large-amplitude waves tend to break into a spatial series of pulses with different amplitudes and velocities. The second discovery was that these pulses retained their identity upon collision and that the only effect of their interaction was a shift in their space-time lines, corresponding to a temporary 'acceleration'. Hence their coinage of the term *soliton* to describe a solitary, uniformly propagating (localized) disturbance/pulse which preserves its structure and velocity after an interaction with another soliton like stable 'perfect' particles/molecules/hard spheres do.

The KdV equation was the first to undergo numerical investigation unravelling the particle-like behavior of its localized solutions. Since then a variety of conservation properties have been proved and a good deal of analytical techniques for solitons is now available for the the KdV equation and other soliton-bearing, integrable equations. Thus the KdV equation was not an isolated curiosity (for reviews biassed towards Fluid Physics see, e.g., Refs [10-15]).

Completely different is the case when dissipation is taken into account as it could have been the case had Zabusky and Kruskal considered collisions in their kinetic plasma model. The presence of dissipation immediately spoils integrability and little can be done analytically, but who carẽs about integrability? [16]. Moreover, dissipation usually enters with high-order derivatives which means that considering it as a perturbation yields singular expansions that are only valid either for short time scales or on short distances. Yet we may think in terms of practical existence over (long enough) time or space inervals scaled with the inverse smallness parameter introduced by the dissipation. Recent experimental results support the utility of such an approximation [23-25,32].

For an energy conserving system like the KdV equation the soliton is a consequence of a (local) balance between (inertial) nonlinearity and dispersion [17]. In the past few years I have been engaged in establishing the existence of solitons as a consequence of instability and thus when no energy is conserved but there is however balance between energy supply usually at long wavelengths and dissipation at shorter ones by viscosity. Such possibility has been analytically and numerically established for dissipation modified KdV and Boussinesq equations as well as experimentally observed [18-33].

Section 2 deals with the energy balance for dissipation-modified (KdV and Boussinesq) equations. Section 3 recalls some of the solitonic signatures and the predictions of a three-dimensional description. Section 4 describes some recent numerical findings concerning *inelastic* and *elastic* collisions with the KdV and Boussinesq equations when dissipation is present or not. Section 5 shows how 'solitons' correspond to periodic or aperiodic motions of nonlinear oscillators. Section 6 describes how solitary waves and periodic wave trains (series of solitonic crests) appear in a dissipation-modified KdV equation. Section 7 summarizes some recent experimental results and their relation to theoretical predictions. Finally, in Section 8 we provide some conclusions and comments.

2. Instability, the onset of propagating localized structures and the energy balance.

Solitons in integrable systems appear as a consequence of initial conditions. When energy is not conserved but rather there is for instance a steady energy balance the possibility exists of exciting, to a large extent irrespective of initial conditions, traveling localized dissipative structures/solitary waves or nonlinear periodic wave trains as a result of instability. Indeed, such was the theoretical prediction made in Refs. [18, 19] where a KdV-Burgers equation was derived for various heat and viscous boundary conditions (b.c) in Marangoni-Bénard convection [34-38].The coefficient of the Burgers term., a second derivative in space containing the (kinematic) viscosity and the energy input could be set to zero, positive or negative with appopriate tuning of the external constraint, i.e. by the Marangoni effect [39] through heating a Bénard liquid layer from the air side or with suitable surfactant adsorption or desorption processes. Past the instability threshold the KdV-B equation needs to be augmented with saturation terms in order to account for the evolution of the solitonic structure [sec 27 for a discussion on asymptotics]. The theoretical prediction of an instability threshold leading to long wavelength solitonic excitations is well supported by experiment.[23-25, 32].

For instance for the Bénard geometry [36] the 2D evolution of a long wavelength surface wave or periodic wave train disturbance is described by the dissipation modified KdV equation [20, 27, 33].

$$h_t + b_1 h h_x + b_2 h_{xxx} + b_3 h_{xx} + b_4 h_{xxxx} + b_5 (h h_x) + b_6 h = 0 \qquad (1)$$

where $h(x,t)$ accounts for surface deformation and the $b_i (i = 1-6)$ are coefficients which incorporate all parameters of the problem (in particular Prandtl, Bond, Galileo, capillary, Biot-like bottom friction and Marangoni numbers) whose explicit expressions are not needed here. A similar result has been found for internal waves in shallow atmospheres [40, 41]. Eq. (1) can be considered either as a generalization of the KdV equation or a generalization of the Kuramoto-Sivashinsky equation [42-45].

At variance with the KdV equation, here dissipation yields wave speed selection and thus in the supercritical state all excited solitary waves or crests in a wave train have equal phase velocity and consequently equal amplitude. Their actual values are determined by the experimental value given to the external constraint measured by a dimensionsless Marangoni number [20].

The energy balance follows immediately from Eq. (1). It suffices to multiply it by h and integrate over the entire available space. For an infinite support we have

$$E = \frac{1}{2} \int_{-\infty}^{+\infty} h^2 dx \qquad (2)$$

wich not being conserved leads to the following balance equation*

$$dE / dt = b_3 \int_{-\infty}^{+\infty} h_x^2 \, dx - b_4 \int_{-\infty}^{+\infty} h_{xx}^2 \, dx + b_5 \int_{-\infty}^{+\infty} h h_x^2 \, dx - b_6 \int_{-\infty}^{+\infty} h^2 dx \qquad (3)$$

where the first term in the r.h.s. describes the energy input at rather long wavelengths, the second term accounts for energy dissipation at shorter ones, the third one produces nonlinear feedback to the long wavelength energy input,

* A more complete description demands consideration in the energy balance of the dissipation at the upper and lower boundary layers hence leading to a theory better suited for comparison with experiment. However, for simplicity, I shall not dwell on this matther here.

positive or negative according to the sing of b_5 or h (positive for elevation and negative otherwise) and the fourth accounts for dissipation by friction at the bottom (for stress-free b.c. $b_6 = 0$).

For the 3D geometry the nonlinear evolution of surface waves in Bénard layers heated from above is a generalization of a Boussinesq system [2,10,28]. In compact form we have [27].

$$h_t = \nabla_2 \cdot [(1+h)\boldsymbol{u}] + \frac{M}{6}\nabla_2[(1+h)h] + B\Delta_2(\nabla_2 \cdot \boldsymbol{u}) + D\Delta_2^2 h \quad (4.a)$$

$$\begin{aligned} \boldsymbol{u}_t / P = & -(G+M)\nabla_2 h + A\nabla_2(\nabla_2 \cdot \boldsymbol{u}) + \Delta_2 \boldsymbol{u} + C\nabla_2(\Delta_2 h) - (\boldsymbol{u}\cdot\nabla_2)\boldsymbol{u} / P \\ & + \frac{M}{2}\nabla_2 h^2 + L\nabla_2\Delta_2(\nabla_2 \cdot \boldsymbol{u}) + E(\nabla_2 h \cdot \nabla_2)\boldsymbol{u} + F(\nabla_2 \boldsymbol{u}) \cdot \nabla h \\ & + I(\nabla_2 \cdot \boldsymbol{u})\nabla_2 h + Jh\nabla_2(\nabla_2 \boldsymbol{u}) \end{aligned} \quad (4.b)$$

where $h(x,y;t)$, as before, denotes surface deformation and $\boldsymbol{u}(x,y;t)$ accounts for 2D velocity disturbances. ∇_2 and Δ_2 are (x,y)-2D gradient and laplacian, respectively. The coefficients A-L have similar content as the b_i in Eq. (1). In particular M is the Marangoni number, P is the Prandtl number and G is the Galileo number [27]. From the system (4) follows the linear (dispersion-less) wave equation, the standard and dissipation modified KdV and (quasi 3D) Kadomtsev-Petviashvili equations [15,22,27]. Moreover although again we only have waves with a single wave speed and correspondingly with a single wave amplitude dictated by the (experimental) value of the constraint we can account for solitary wave or wave crest solitonic collisions at arbitrary angles which includes *head-on* collisions and with a suitable mathematical trick *overtaking* collisions and wall reflections at arbitrary angles.

3. Predictions of a three-dimensional theory, and some solitonic signatures.

The system (4) describes the onset and sustainment of interfacial solitary waves moving in arbitrary directions as a consequence of longwave oscillatory Marangoni instability [20,27,32,33]. The form of these solitary waves, near the instability threshold, resembles the form of ideal KdV solitons but with *asymmetric* shape. In contrast to ideal KdV solitons, which can have arbitrary amplitudes, and corresponding widths and wave velocities, the amplitudes of all Marangoni-driven "solitons" here described are strictly equal and related to the level of the thermal constraint, i.e. to the value of the (actual, supercritical and experimentally controllable) Marangoni number.

The (oblique) collision between solitary waves propagating in different directions with finite angle between the normal vectors to the wave fronts leads to a phase shift in position of the "solitons" that depends on the incident collision angle, a result that extends to driven dissipative systems the earlier theory for viscous-free, ideal systems [11]. The phase shift can change sign and indeed a *critical* collision angle (approximately $\pi / 2$) is predicted at which there is no phase shift thus separating regions of *negative* and *positive* phase shifts for angles acute enough and obtuse enough, respectively [27].

For head-on collisions the predictions [27] from the 2D reduction of the Boussinesq system (4) is that to a first approximation for humps/positive waves the interaction leaves the waves or wave crests practically unchanged with a mere *positive* phase shift, opposite to the *negative* phase shift found in overtaking collisions with the KdV equation.

Other aspects of the collision dynamics have also been investigated. Elastic and inelastic events as well as the formation of bound states and chaotic behavior have been predicted [30,31]. Indeed the dissipative terms in Eqs. (1) and (4) not only lead to wave velocity selection but also produce 'imperfect' solitons with wavy heads hence behaving like 'molecules' with infinitely long range, exponentially weak Kac-like potentials typical for the van der Waals gas [46].

The theoretical study of (strong) resonant collisions and wall reflections at arbitrary angles is still underway. There are some experimental results [25] that point to similarities with oblique angle and head-on collisions as indeed a reflection at a wall could very well be thought as a collision of an incoming soliton with its *mirror* image.

4. Inelastic collisions of solitons with or without dissipation.

When dissipation and Marangoni stresses and hence instability are disregarded in Eq. (4) the system reduces to an equation that in the context of Boussinesq paradigm [28] we consider valid for waves propagating in boh directions* . We have either

$$h_{tt} = h_{xx} - \alpha (h^2)_{xx} + \beta \, h_{xxxx} \tag{5}$$

* Strictly speaking Boussinesq equation (5) or (6) is only valid for one-side propagating waves. See, however, a discussion of the universal features of Boussinesq paradigm in Ref. [28].

(with α and β are some given parameters, whose values are not needed here; the latter one incorporates surface tension or in dimensionsless form the Bond number [20, 27] or

$$h_{tt} = h_{xx} - \alpha (h^2)_{xx} + \beta\, h_{xxtt} \tag{6}$$

where the last term shows mixed derivatives introduced to endow well possednes to Eq. (5) for all values, of β. Eqs. (5) and (6) are called in the literature the "good" (for β semidefinite negative) or Proper Boussinesq Equation (PBE) and the Regularized Long Wave Equation (RLWE), respectively [28, 47].

Solutions of Eqs. (5) and (6) are respectively

$$h = -(3/2)\left[(c^2-1)/\alpha\right] sech^2 \left\{[(x-ct)/2]\sqrt{}\left[(c^2-1)/\beta\right]\right\} \tag{7}$$

and

$$h = -(3/2)\left[(c^2-1)/\alpha\right] sech^2 \left\{\left[(x-ct)/2c\right]\sqrt{}\left[(c^2-1)/\beta\right]\right\} \tag{8}$$

where c denotes the (linear) wave velocity [10,15]. Note that as earlier stated the mathematically correct Boussinesq equation corresponds to β semidefinite negative and hence necessarily *subsonic* solitary negative/depression capillary waves. The hump/positive wave of Russell [1] is better described by solution (8) which is *supersonic* as expected.

For both the PBE and the RLWE mass and (pseudo) energy can be defined which are conserved in evolution. Recent numerical exploration of collisions of those solutions show that *subsonic* negative depression *seches* of the PBE are subject to positive phase shift while perfectly retaining their shapes after they remerge from collision. *Supersonic* positive humps of the RLWE undergo elastic or inelastic collisions depending on their wave velocities. When very slightly supersonic they behave elastically while inelasticiy (due to the non semidefinite positive nature of the pseudo-energy) and negative phase shift upon collision show up if they have 'high' (supersonic) velocities [28].

Again within the Boussinesq paradigm [28] similar computer experiments have been conducted with solutions of Eq. (1) for its two extreme cases [29]: (i) a KdV equation slightly perturbed (on a scale of a smallness parameter ε) by the last four terms in Eq. (1) (due to dissipation and the Marangoni stresses) and (ii) a dissipation dominated KdV equation when all terms are of order unity and Eq. (1) is essentially a modification of a parabolic (fourth-order derivative) Kuramoto-Sivashinsky equation [26,42].

In case (i) although the *sech* solution of the KdV equation is not a solution of Eq. (1) the mismatch for small ε is small and its impact is only felt at time distances of order $1/\varepsilon$. An important item is that if we start with an arbitrary *sech* below the instability threshold the i.c. decays to zero while if we place ourselves above threshold the i.c. grows until the *sech* corresponding to the actual value of the constraint and its appropriate velocity are attained. We have (practically) stable permanent waves in the scale $1/\varepsilon$. Yet however small ε might be, the dissipative terms controlled by ε in Eq. (1) select a single wave velocity and consequently a single value for the wave amplitude (for a solitary wave or for the crests of a periodic wave train) in the otherwise available one-parameter family of solutions of the KdV equation. Suitably introduced overtaking collisions are esentially elastic with phase shift like in the KdV problem. Formation of bound states and chaotic behavior is also numerically observed in this case [29] but this result demands further discussion beyond the scope of this paper.

In the second case (ii) *seches* and (undular) *seches* are not solutions of Eq. (1). Rather (undular) kinks/bores/hydraulic jumps are the appropriate solutions very much like the solutions of a KdV-Burges equation [48]. Then collisions are completely inelastic. Further details about the numerics and results can be found in Refs. [26, 28, 29].

5. A dynamical systems approach. The underlying nonlinear dissipative oscillator.

Search of steady solutions, propagating waves of permanent form of the standard KdV equation reduce the problem to the finding of fixed points of an underlying dynamical system. Indeed by a Galilean transformation to the moving frame, integration over one space variable and suitable reconsideration of the space coordinate as time leads to the frictionless (nonlinear) Helmholtz oscillator with asymetric cubic potential [49]. The third derivative dispersive term reduces to a second time derivative. The fixed point in phase space is a saddle and the humps/positive *seches* are simply the result of the stable/unstable character of the saddle. A similar situation occurs with Boussinesq equations (PBE and RLWE). If dissipation is added and consideration is given to the KdV-Burgers equation then we arrive at a Helmholtz oscillator with damping [49]. The Burgers second space derivative brings the damping term. Instability can also be incorporated into the picture and a dissipative hydrodynamic oscillator is obtained. This approach has already proved useful for a description of linear (harmonic) waves as well as nonlinear ones like undular bores in driven systems [48,50-56].

The dissipation modified KdV equation (1) or its reduction to the KS equation can also be described in terms of dynamical systems with however a higher order phase space due to the presence of fourth derivatives [26,30,31]. The KS case and Eq. (1) without the Marangoni nonlinearity ($b_5 = 0$) have been extensively studied in the literature (a rather complete list of papers appears in Ref. [30,31]). Let us now sketch the approach to the full Eq. (1). In the moving frame let us consider the new space coordinate $\xi = x + c_0 t$, with c_0 the phase velocity of the expected wave traveling for convenience right to left. Upon integration of (1) from $-\infty$ to the traveling $\xi-$ coordinate we obtain the following three nonlinear ordinary differential equations:

$$\dot{h} = y \tag{9.a}$$

$$\dot{y} = z \tag{9.b}$$

$$\gamma \dot{z} = -\beta z - \alpha h y - \nu y - F(h) \tag{9.c}$$

where the 'dot' denotes differentiation with respect to ξ .

$F(h) \equiv ch + h^2 / 2, c = c_0 / b_1, \nu = b_3 / b_1, \beta = b_2 / b_1, \gamma = b_4 / b_1$ and $\alpha = b_5 / b_1$. With respect to the KdV equation here the fourth-order derivative in (1) leads to a third order "time"-derivative in the compact form of the dynamical system (4) (see also [26]).

The steady solutions or fixed points of (4) are $O_1 \equiv (-2c, 0, 0)$ and $O_2 \equiv (0,0,0)$ in the space $G{:}\{h, y, z\}$. O_1 is *saddle-focus* when $\gamma > \gamma_s$ with $\lambda_3 > 0, \mathrm{Re}\,\lambda_i < 0, \mathrm{Im}\,\lambda_i \neq 0, (i = 1,2)$ whereas it is *saddle* when $\gamma > \gamma_s$ with $\lambda_3 > 0$, $\mathrm{Re}\,\lambda_i < 0$, $\mathrm{Im}\,\lambda_i = 0\ (i = 1,2)$. We have used

$$\gamma_s = \left\{ -(\nu - 2\alpha c)[9\beta c + 2(\nu - 2\alpha c)] + 2\left[(\nu^2 - 2\alpha c)^2 + 3\beta c\right]^{3/2} \right\} / 27c^2 \tag{10}$$

The λ_i denote the roots of the characteristic equation for O_1, i.e. the solutions of

$$\lambda_3 + \left(\frac{\beta}{\gamma}\right)\lambda^2 + \left(\frac{\nu + \alpha h_{o_1}}{\gamma}\right)\lambda + \frac{c + h_{o_1}}{\gamma} = 0 \tag{11}$$

Let the stable and unstable manifolds around O_1 be called W_{loc}^s and W_{loc}^u, respectively. Then W^u is a one-dimensional curve which crosses O_1 and consists of two unstable separatrices W_1^u and W_2^u located on either side of W^u.

The second fixed point O_2 is asymptotically stable when $\gamma < b\nu / c$ and characteristic roots $\kappa_3 < 0, \operatorname{Re}\kappa_i < 0 (i=1,2)$, and unstable when $\gamma > b\nu / c$ and $\kappa_3 < 0, \operatorname{Re}\kappa_i > 0 (i=1,2)$.

Changes in the qualitative structure of the neighborhood of O_2 are determined by the sign of

$$l = \left[\frac{\pi q^2 (p^2 + q)}{8p\sqrt{q}(p^2 + 4q)\Delta_0^2 \gamma^2} \right] \left[-(p^2 + 8q) + \alpha p (p^2 + 10q) - 2\alpha^2 p^2 q \right] \quad (12)$$

with $\Delta_0 = (q)^{1/2}(q^2 + rp), p = \beta / \gamma, q = \nu / \gamma$ and $r = c / \gamma . l$ may be either positive or negative on the $\gamma = \beta / c\nu$-surface. Then if the $\gamma = \beta / c\nu$-surface is intersected we have an Andronov-Hopf bifurcation, i.e., a stable limit cycle grows from O_2 when $l < 0$; otherwise, with $l > 0$ we have a *saddle*. As a consequence of this behavior periodic traveling waves are the expected solutions of Eq. (1).

To assess the stability of the stationary solutions one considers suitable Lyapunov funcions [30,31]. It can be shown that the separatrix surface W^s of O_1 splits the phase space G in two regions Q^+ and Q^- such that all trajectories of the system (4) including the separatrix W_1^u tend to O_2 as $\xi \to +\infty$ in the region Q^+ while all trajectories in the region Q^-, including the separatrix W_2^u go to infinity as $\xi \to +\infty$. It can also be shown that all trajectories of (4), with the exception of those on the separatrix surface W^s and two 1D separatrices extending to O_2, go o infinity as $\xi \to +\infty$.

Predictions are that a single soliton whose velocity is specified by the parameters of the model may propagate at $\gamma \leq \alpha\beta$ for all ν. The situation is dractically diferent if $\gamma > \alpha\beta$. When $\nu < \nu_1$, where ν_1 is some well defined value, a single velocity and a simple soliton correspond to each ν. For $\nu_0 < \nu < \nu_1$ the soliton has an oscillating head as expected in view of the (linear) dissipative term in Eq. (1). When $\nu_1 < \nu < \nu_2$, where ν_2 is another well defined value, each ν corresponds to a countable set of values of c which specify the velocity of propagation of 'soliton' trains corresponding to multiloop homoclinic trajectories. Such 'solitons' may be treated as *bound states* of model (1). An arbitrary number of "elementary" states may take part in the formation of a wave train with a series of solitons for there may exist trajectories with an arbitrary number of loops. Hence, depending on the initial conditions (i.c.), a bound state of a certain form propagatins with a *definite* velocity is realized for $\nu_1 < \nu < \nu_2$. In this respect, model (1) is highly sensitive to i.c. because there exists at each $\nu \in (\nu_1, \nu_2)$ a countable set of 'soliton' trains. In other words, we can speak about *chaos* of 'soliton' bound states for $\nu \in (\nu_1, \nu_2)$. When $\nu_1 > \nu_2$ there are no bound states and solitons of simple form may propagate. Further details about the qualitative mathematical reasoning and technique as well as a other results can be found in Ref. [30,31].

6. Wave trains with solitonic crests (series of solitons) in dissipative media.

Eq. (1) yields a dispersion relation

$$\lambda = i b_2 k^3 + b_3 k^2 - b_4 k^4 - b_6 \tag{13}$$

where here $\lambda = \mathrm{Re}\,\lambda + i\omega$. $\mathrm{Re}\,\lambda$ is the 'time constant' that determines instability and k denotes the Fourier wavenumber of a disturbance. Thus the neutral curve ($\mathrm{Re}\,\lambda = 0$) possesses the minimum off the zero wave number axis which is only attained when $b_6 = 0$ (it corresponds to stress-free b.c. at the bottom of the layer). The (first) excitable Fourier mode is $k_c = b_6 / b_4$, hence the larger b_6 the more the excitation rather than a solitary wave is a wave train with crest-to-crest distance/periodicity proporcional to the inverse of k_c. We cannot consider b_6 too large to avoid violation of the long wavelength/shallow layer assumption used in the derivation of Eq. (1).

Then the natural solutions to seek for Eq. (1) are periodic *cnoidal* waves (recall that the *sech* is a limit value of such wave train). Solutions of this kind have been obtaind [33] by approximate means when the dissipative terms in Eq. (1) are of order ε relative to the original KdV terms of order uniy. The crests of *cnoidal* wave trains possess solitonic signatures and a train is thus a series of periodically spaced 'solitons'.

7. Some experimental results.

Motivated by the theoretical insight so far obtained and sketched in the preceding sections, experiments have been performed in thin liquid layers with the surface open to air wich was either heated or full of surfactant vapor [23-25,32]. Let us recall results in the latter case, for instance, with liquid layer thickness of 18mm and lateral dimensions of 49mm×49mm. Above the liquid layer, there is a space of 70mm thickness full of hexane vapor, which is adsorbed at the surface and transferred into benzene liquid. Hexane is a surface active substance for a hexane-benzene (vapor-liquid) surface, such that $d\sigma / dC < 0$, i.e. the interfacial tension, σ, decreases with the increase of its concentration, C. As soon as the hexane vapor is adsorbed and transferred i.e. absorbed into the benzene liquid layer, surface oscillations and wave motions due to the Marangoni efrect appear on the liquid surface. At the initial stage, due to very steep concentration gradient near the surface and thus high Marangoni number, complex and irregular waves embodied in turbulent convection appear; often wave trains can be recognized. Then, approximately 40 seconds after initiation, the convective waves behave more regulary as the concentration gradient becomes moderate and the system shows two pairs of waves. Each wave pair is approximately parallel to one pair of side-walls in the vessel. The waves in a pair move in the same direction or in opposite directions. When they propagate in opposite directions, they interact somewhere in the vessel, and quite often in the center part of the vessel, then separate and approach the wall. Both of them are reflected back by the walls and move toward each other again. The damping effect due to the interaction with walls is generally negligible so that these waves appear very stable; some of them can still survive after 8 collisions with the wall and 8 collisions with another wave or even more. After some time, as the driving force fades away, there is only one pair of waves, and later on a single wave. Finally, no wave can be observed ant the liquid layer becomes quiescent as the Marangoni stresses become undercritical. Similar behavior occurs with cylindrical or annular cylindrical containers where a pair of counterrotating periodic wave trains have been observed and its collision dynamics studied [32].

To a qualitative level the apparent shape of the wave, most surely a hump, as well as fluid motions within the wave are clearly *asymmetrical.* The wave has a short head and a long tail. In the short head part, there is relatively large surface deformation and strong fluid motion which is mainly a tangential flow with the same direction as the wave. In the long tail part, surface deformation and fluid motion are weak and become weaker and weaker as we go away from the head. The penetration length of the Marangoni flow is small.

Indeed it is linked to the viscous penetration thus illustrating the major role played by dissipation in the liquid layer.

The interaction between two solitary waves can be classified into five different (experimental and theoretical) cases according to the angle measured just before the "collision" [24]:

(a) head-on collision: two solitary waves travel in opposite directions, collide in the central part of the vessel, and separate after collision.

(b) "acute" oblique interaction: two waves travel in different directions, obliquely crossing each other and interacting with an angle smaller than a "critical" angle to be defined below (see also Sect. 3 above).

(c) zero phase shift collision which defines a "critical" angle for an oblique interaction.

(d) "obtuse" oblique interaction. When the crossing angle is larger than the "critical" angle the phase shift is of opposite sign to that in the case (b). Notice that the use of the words "acute" and "obtuse" does not depend on its relation with the $\pi/2$ angle. Rather these concepts are relative to the "critical" angle defined in case (c), although in some of our experiments this critical angle was quite close to $\pi/2$ (as predicted by theory [27]).

(e) overtaking collision: two solitary waves travelling in the same direction but with different celerities and different initial positions along the same path are such that the fast wave overtakes the slow one. This is the case extensively studied since the pioneering work of Zabusky and Kruskal [6].In our experiment there is no chance to observe this case. Indeed the balance between the driving force (Marangoni stresses) and dissipation uniformly exists in the horizontal extent of the vessel and the velocities of all the (solitary) waves are the same at a time [23,24,32].

Head-on collisions have been observed and relevant details measured. For instance the phase shift varies fron -7mm to -4m with average value -5.1 mm. As the crossing angle increases to a non-zero but small value, we have the case of "acute" oblique interaction. The phase shift in this case is negative. Moreover, its absolute value decreases as the crossing angle increases. When the crossing angle reaches a certain value -the "critical" angle- the phase shift is zero. Further increasing the crossing angle leads to the "obtuse" oblique interaction region and positive phase shifts. Notice that the magnitude of both positive and negative phase shifts are apparently of the same order, O(1). For wider and wider supercritical oblique interaction angles, the appearance of the positive phase shift is instantaneous, or at least very fast. As soon as two wave fronts interact, they jump to the new positions, and a third wave -the Mach stem- is formed. This third wave is very stable; it survives after interaction with another wave.

In a head-on collision soon after the two heads separate, but with the tails of the two solitary waves still interacting, the positions of wave fronts have been recorded. It appears that the velocity of each wave is significantly lowered. Normally the minimum velocity measured is only half of that before interaction. The velocity increases from this minimum until it reaches the original value before interaction, then it moves away wih the same velocity as if nothing happened, very much as "ideal" solitons do. The outgoing trajectory of a wave after interaction is parallel to the incoming trajectory before the interaction. Although no measure has been possible yet the velocity during a short time interval when two heads of tavelling solitary waves collide head-on, an estimate comes from its knowledge before collision and right after the collision. There seems to be a very rapid decrease of velocity in this short time interval. Thus the head-on collision process seems to proceed in two stages: the initial stage of short duration and rapid decrease of velocity; and the final stage lasting a very long time with smooth increase of the velocity. Such deceleration-acceleration process during a head-on collision justifies the phase lag (or *negative* phase shift) after interaction in the cases under consideration. This process of deceleration-acceleration during the head-on interaction of two solitary waves can also be seen in "acute" oblique interaction processes. The similarity between a head-on interaction and an "acute" oblique interaction already gives us the hint that there must be some connection between these two cases. In fact, the celerities before and after the crossing point in the "acute" oblique interaction correspond to those before collision and after initial stage of collision in head-on interaction, respectively (the head-on collision can be considered as a special case of the "acute" oblique interaction with vanishing crossing angle). If we use α and β to denote the half of the crossing (bisector) angle before and after the crossing point, according to the standard deflection law, the relation between the the velocities before and after the crossing point is $v_{outgoing} / v_{incoming} = sin\beta / sin\alpha$. Thus the velocity is obtained from the shape of the wave front near the crossing point. It appears drastically lowered after the crossing point. Then, it gradually increases until it recovers its initial value before interaction. The similarity between these two classes of interactions makes the trajectory of head-on collisions in *x-t* space quite like those of wave fronts in *x-y* space in "acute" oblique interactions. It appears that for "acute" oblique interactions, this ratio is always less than unity, which corresponds to the deceleration in the early stage of interaction. This ratio increases with increasing crossing angles. When the crossing angle reaches the early defined "critical" value, the velocity ratio is equal to unity.

Evidence has been found of reflections at walls which according to the value of the incident angle lead to a reflected wave with or without Mach stem

[25]. When a solitary wave obliquely approaches a vertical wall, the result depends on the incident angle α_i . On the one hand, when the incident angle is large enough $(\pi_2 \geq \alpha_i \geq \pi/4)$, the reflecting wave is always observed, with a reflection angle, α_r , generally larger than the incident one. Considering the solitary wave 'colliding' with its *mirror* image wave, one expects from the observations made for two acute obliquely interacting solitary waves that the crossing angle before interaction, 2α, would be larger than the crossing angle after collision, $2\beta, i.e., 2\alpha > 2\beta (\alpha > \beta)$. However systematically $\alpha_i < \alpha_r$ $(\alpha = \pi/2 - \alpha_i \; and \, \beta = \pi/2 - \alpha_r)$. On the other hand, when the incident angle is small enough, say $\alpha_i < \pi/4$, the Mach stem appears. The Mach stem initiates at the leading edge of the wall where the solitary wave is interacting with. Its length increases from zero to finally reach a finite constant value, which is about several millimeters long. It appears that the transition time (or distance) from zero stem length at the leading edge to this constant value, depends on the incident angle. Generally, the larger the incident angle is, the faster the stem length becomes constant.

If one relates the oblique reflection process with the oblique collison of two solitary waves with obtuse crossing angle, one would expect the reflection angle to be the same as the incident. However, in the experiments one always observes a reflection angle larger than the incident one. The image of the reflected wave becomes weaker and weaker as the incident angle decreases. Finally, when the incident angle is smaller than about 25°, the reflected wave becomes unobservable. The interaction pattern of a solitary wave with a wall consists of only the incident wave and a Mach stem [25].

From a plot of the ratio of wave speeds after and before reflection as function of the incident angle $(v_{wave\ velocity\ after\ reflection} / v_{wave\ velocity\ before\ reflection} = sin\beta / sin\ \alpha)$ wave speeds are obtained by measuring angles. From the data obtained it appears that after reflection, the velocity goes down, even in the case of resonant interaction (Mach stem). One can also see that the ratio of wave velocities (reflected and incident, respectively) first increases as the incident angle decreases, then proceeds to a maximum of 0.7 at about 40°, to finally decrease. This could be explained as follows: if for a shallow water solitary wave, the wave speed is related to the wave amplitude or strength then the 'weaker' the wave is, the slower the wave speed becomes. As the incident angle becomes smaller, the reflected wave becomes weaker and thus slower. In order to catch up and to have a stationary reflection pattern, the reflection angle has to increase. Hence the ratio goes down at small incident angles (or large α). Since the strength of the reflected wave tends to zero below a certain critically small incident angle α_i (or large $\alpha = \pi/2 - \alpha_i$), thus leading to zero velocity for the reflected wave. Unfortunately, the accurate determination of this angle is very difficult in the experiments due to difficulties in visualizing weak waves. Extrapolation of the available data gives 25° as an estimate of this angle. A plot of β versus α shows always below the diagonal ($\beta = \alpha$ is the regular reflection law for waves) which means that the angle of reflection of dissipative solitary waves is always larger than the incident one [25].

8. Conclusion and outlook.

Analytical, numerical and experimental evidence has been provided to support the 'reality' of *solitons* and *solitonic* wave trains in non-integrable, *dissipative* systems. Companion to a (local) balance between nonlinearity and dispersion sustaining permanent solitary waves and periodic wave trains in dissipation-free systems I have introduced the input/pumping/production-dissipation energy balance allowing past an instability threshold for the onset and eventual supercritical sustainment of similar solitary waves and wave trains. I have illustrated the kind of soliton signatures, bound states and chaotic behavior exhibited by solutions of non integrable equations that incorporate dissipative elements as a (weak) perturbation of originally integrable, dissipation-free equations. I have indicated how such properties have been experimentally observed in the laboratory. Certainly neither I have provided a complete body of doctrine nor it was my pretension for in nonlinear nonequilibrium systems little thermodynamic land has been explored [57]. A new continent is in front of us. Hopefully, however, we have provided enough evidence to attract interest among researchers.

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