Almost continuous functions of two variables

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Recibido: 13 de Octubre de 1993

Presentado por el Académico Numerario D. Pedro Jiménez Guerra

Abstract

In this paper necessary and sufficient conditions for almost continuity of extensions and Cartesian products of almost continuous functions are studied. The results are generalizations of Lipinski Theorem of [6].

AMS Subject Classification (1991): Primary **54D30**. Secondary **54C08**.

Introduction

Let us establish some terminology to be used. $\mathbb{R}$ denotes the real line. Letters X, Y, Z and T will denote topological spaces with topologies τ_X, τ_Y, τ_Z and τ_T , respectively. We consider a function $f: T \rightarrow Z$ and its graph (i.e. a subset of $T \times Z$) to be coincident. A function $f: T \rightarrow Z$ is *almost continuous* (in the sense of Stallings) iff each open neighbourhood G of f in $T \times Z$ contains a continuous function $g: T \rightarrow Z$ [8], [4]. The class of all almost continuous functions from T into Z will be denoted by $A(T, Z)$. By $C(T, Z)$ and *Const* (T, Z) we denote the classes of all continuous functions from T into Z, and all constant functions fronm T into Z, respectively.

For a function $f: X \times Y \rightarrow Z$ and $x \in X$, we denote by f_x the x-section of f, i.e. f_x is a function from Y into Z, defined by $f_x(y) = f(x, y)$ for $y \in Y$. The symbol $rng(f)$ denotes the range of f.

For every function $f: X \rightarrow Z$, we denote by $F_{f,Y}$ the function from $X \times Y$ into Z defined by

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Both authors are supported by KBN Research Grant 1992-94, N° 2 1144 91 01.

$$F_{f,Y}(x,y)=f(x)$$

for $(x,y)\in X\times Y$. Note that $F_{f,Y}$ is continuous ($F_{f,Y}$ is constant) iff f is so. Moreover, one can easily prove the following proposition.

Proposition 1. *Let $f:X\to Z$ and let Y be any topological space. Then f is almost continuous whenever $F_{f,Y}$ is so.*

However, the fact that f is almost continuous does not imply that $F_{f,Y}$ is so, even if $X=Y=Z=\mathbb{R}$. In fact, J. Lipinski has proved recently the following theorem.

Theorem 1. *If $f:\mathbb{R}\to\mathbb{R}$ is not continuous then $F_{f,\mathbb{R}}$ is not almost continuous, either* [6].

In particular, for the function $f_0:\mathbb{R}\to\mathbb{R}$ defined by

$$f_0(x)=\begin{cases} sin(1/x) & if\ x\neq 0,\\ 0 & if\ x=0,\end{cases}$$

the extension $F_{f_0,\mathbb{R}}$ is not almost continuous (this example was given by Lipinski [5]). However, it is well-known that if $f\in A(X,Z)$ and Y is a compact space, then $F_{f,Y}\in A(X\times Y,Z)$ (cf [7, Corollary 4.2, (1)]). The foregoing suggests the problem of characterization of compactness of Y in terms of almost continuity.

We say that a space T is countably compact iff it is a Hausdorff space and

(*) for every countable descending sequence of non-empty, closed subsets of T, its intersection is non-empty. [3, p. 253].

We will write that a space T is *quasi countably compact* iff it satisfies condition (*). (Cf the definition of quasi compact spaces in [3, p. 171].)

We say that a space T is regular iff it is a T_1-space and

(*) for every $x\in T$ and every closed set $F\subset T$ with $x\notin F$, there exist disjoint open sets $U_1,U_2\subset T$, such that $x\in U_1$ and $F\subset U_2$. [3, p. 58].

We will write that a space T is *quasi regular* iff it satisfies condition (*).

Remark 1. Evidently, every (countably) compact space is quasi countably compact. Moreover, if T is countably compact, then both metrizability and the Lindelöf condition imply that T is compact. However, there exist non-compact, countably compact spaces [3, Examples 1-4, p. 257].

1. Extensions of almost continuous functions

For topological spaces X, Y, Z we define the following conditions:
(i) Y is quasi countably compact,
(ii) $F_{f,Y} \in A(X\times Y,Z)$ for each $f \in A(X,Z)$
(iii) there exists $f \in A(X,Z)/\, C(X,Z)$ such that $F_{f,Y} \in A(X\times Y,Z)$,
(iv) there exists $f\colon X \to Z$, such that $F_{f,Y} \in A(X\times Y,Z)$ and $f \notin C(X,Z)$.

Proposition 2. *Assume that X and Z are first countable spaces. Then*

$$(i) \Rightarrow (ii).$$

Proof. Let $f \in A(X,Z)$. For each $x \in X$, let $(W_{n,x})_{n\in N}$ and $(V_{n,x})_{n\in N}$ be descending bases of X at x and of Z at $f(x)$, respectively. Let $G \subset X\times Y\times Z$ be an open neighbourhood of $F_{f,Y}$. Note that for each $x \in X$ there exist open neighbourhoods W_x of x and V_x of $f(x)$, such that $W_x \times Y\times V_x \subset G$. Indeed, fix $x \in X$. Let U_n be the set of all $y \in Y$ for which there exists an open neighbourhood U_y of y such that $W_{n,x} \times U_y \times V_{n,x} \subset G$. Then $(U_n)_{n\in N}$ is an ascending sequence of open sets. Since for each $y \in Y$ there exist an open neighbourhood U_y of y and $n \in N$ such that $W_{n,x} \times U_y \times V_{n,x} \subset G$, $Y = \cup_{n\in N} U_n$. By the quasi countable compactness of $Y, Y = U_m$ for some $m \in M$, and we can set $W_x = W_{m,x}$ and $V_x = V_{m,x}$.

Define $G_1 = \cup_{x\in X} W_x \times V_x$ and $\hat{G}_1 = \cup_{x\in X} W_x \times Y\times V_x$. Note that $f \subset G_1$, so there exists a continuous function $g\colon X \to Z$ contained in G_1. Then $F_{g,Y}$ is a continuous function contained in $\hat{G}_1 \subset G$.

□

Note that, in general, condition (ii) does not imply (i). Indeed, let Z be a discrete space. It is well-known (and easy to prove) that then for any connected topological T_1-space T, $A(T,Z)=C(T,z)=Const(T,Z)$ (cf [7, Corollary 1.2]). Therefore if $X = Y = \mathbb{R}$ (with Euclidean topology) then the condition (ii) holds, while the condition (i) fails. (Observe that Z is a metric space.)

Condition (ii) does not imply condition (i) even if $A(X,Z)/C(X,Z)\neq\varnothing$.Indeed, *let* $Z=[0,1]$ and

$$\tau_Z=\{A\subset Z\colon 0\in A\}\cup\{\varnothing\}.$$

Then for every topological space T, each function $f\colon T\to Z$ is almost continuous. (If $G\subset T\times Z$ is open and $G\supset f$, then G contains a continuous function $g\equiv 0,\ g\colon T\to Z.$) Setting $X=Y=\mathbb{R}$ (with Eucliean topology) we get that conditions (ii)-(iv) hold, while condition (i) fails. However, Z is neither quasi regular, nor T_1-space.

In the proof of the following proposition we use an idea of [6].

Proposition 3. *Assume that x is a first countable, Hausdorff space, Y is a connected space and Z is a quasi regular space. Then*

$$(iv)\Rightarrow(i)$$

Proof. Assume that Y is not quasi countably compact space, and function $f\colon X\to Z$ is not continuous at $x\in X$. It follows by assumptions on X and Z that there exist a descending base of X at $x,(W_n)_{n\in N}$, a sequence $(x_n)_{n\in N}$ in X, and an open neighbourhood V of $f(x)$ such that $lim_{n\to\infty}x_n=x,\ x_n\in W_n\setminus W_{n+1}$ and $f(x_n)\notin\overline{V}(n\in N)$. Since Y is not quasi countably compact space, there exists an ascending sequence $(H_n)_{n\in N}$ of open subsets of Y such that $Y=\bigcup_{n\in N}H_n$ and $H_n\neq H_m$ for $m\neq n$. Set

$$G_1=\bigcup_{n\in N}W_n\times H_n\times V,$$

$$G_2=X\times Y\times Z\setminus\left(\{x\}\times Y\times Z\cup\bigcup_{n\in N}\{x_n\}\times Y\times\overline{V}\right).$$

$$G=G_1\cup G_2.$$

Then G is an open neighbouurhood of $F_{f,Y}$. Indeed, for each $t \in X$, either $t \neq x$ and $(t, y, F_{f,Y}(t, y)) = (t, y, f(t)) \in G_2$, or $t = x$ and $(t, y, F_{f,Y}(t, y)) \in G_1$.

Suppose that there exists a continuous function $g: X \times Y \to Z$ contained in G. Let $y \in Y$. Then $g(x, y) \in V$, and therefore there exists an $n \in N$ such that $g(t, y) \in V$ for each $t \in W_n$. In particular, $g(x_n, y) \in V$, so $g_{x_n}(y) \in V$. Since $g_{x_n}(u) \notin \overline{V}$ for $u \notin H_n$ and $rng(g_{x_n}) \subset V \cup (Z \setminus \overline{V}), rng(g_{x_n})$ is not connected. Hence $F_{f,Y} \notin A(X \times Y, Z)$.

□

Using the two above propositions we get the following theorem.

Theorem 2. *Assume that x is a first countable, Hausdorff space, Y is a connected space, Z is a first countable, quasi regular space, and moreover,* $A(X, Z) / C(X, Z) \neq \phi$. *Then condiions (i).(iv) are equivalent.*

Corollary 1. *For every connected space Y the following are equivalent:*
(a) Y is quasi countably compact,

(b) $F_{f,Y} \in A(\mathbb{R} \times Y, \mathbb{R})$ *for each* $f \in A(\mathbb{R}, \mathbb{R})$,

(c) there exists $f \in A(\mathbb{R}, \mathbb{R}) / C(\mathbb{R}, \mathbb{R})$ *such that* $F_{f,Y} \in A(\mathbb{R} \times Y, \mathbb{R})$,

(d) there is $f: \mathbb{R} \to \mathbb{R}$ *such that* $f \notin C(\mathbb{R}, \mathbb{R})$ *and* $F_{f,Y} \in A(\mathbb{R} \times Y, \mathbb{R})$,

(e) there exists a first countable space X and a function $f: X \to \mathbb{R}$, *such that* $f \notin C(X, \mathbb{R})$ *and* $F_{f,Y} \in A(X \times Y, \mathbb{R})$.

Remark 2. Note that there exist countably compact and connected topological spaces which are not compact (see [3, Example 3.10.2, p. 257]), so conditions (b)-(e) of Corollary 1 do not imply compactness of Y.

However, by Remark 1, we get

Corollary 2. *Assume that Y is connected and either it is metrizable, or it is a* T_2*-space which satisfies the Lindelöf condition. Then the statement*
(•) Y is compact,
is equivalent to conditions (b)-(e) of Corollary 1.

Now we drop the asumption that Y is connected. First we formulate a general problem.

Query 1. *Characterize topological spaces Y for wich the condition (ii) holds.*

Though the above problem is open, we can prove some partial results.

Proposition 4. *Assume that X is a first countable, Hausdorff space and Z is a first countable quasi regular space. Then (iv) implies the following condition:*

(i') every component of Y is quasi countably compact.

Proff. This follows easily from Proposition 3 and the fact that restrictions of almost continuous functions to closed sets are almost continuous (cf [8]).

□

On the other hand, the condition (i') does not imply (ii) even if X, Y and Z are metrizable.

Example. *There exists a subspace Y of the plane with countably many compact components such that ¬ (iv) whenever X is a first countable, Hausdorff space and Z is a first countable quasi regular space.*

Let $y=(0,0), Y_0=\{y\}, Y_n=[0,1]\times\{1/n\}$ for $n\in N$, and let $Y=\bigcup_{n=0}^{\infty} Y_n$.

Let $f:X\to Z, x,(x_n)_n,(W_n)_n$ and V be as in the proof of Proposition 3 and let $U_0=[0,1/2)\times[0,1]$. Set

$$G_1=\bigcup_{n\in N} W_{n+1}\times(Y_n\cup\bigcup_{k>n} Y_k\cap U_0\cup Y_0)\times V,$$

$$G_2=X\times Y\times Z\setminus\left(\{x\}\times Y\times Z\cup\bigcup_{n\in N}\{x_n\}\times Y\times\overline{V}\right),$$

$$G=G_1\cup G_2.$$

Then G is an open neighbourhood of $F_{f,Y}$

Suppose that there exists a continuous function $g:X\times Y\to Z$ contained in G. Since $g(x,y)\in V$, there exists an $n>1$ such that $g(x_n,y)\in V$ and $g_{x_n}^{-1}(V)\cap Y_n\neq\varnothing$. Consider the function $g_{x_n}|Y_n$. Since $g_{x_n}(u)\notin\overline{V}$ for $u\in Y_n\setminus U_0$ and $g_{x_n}(Y_n)\subset V\cup\overline{V}$, $rng\left(g_{x_n}|Y_n\right)$ is not connected. Hence

$F_{f,Y} \notin A(X \times Y, Z)$.

□

However, it is easy to observe the following:

Proposition 5. *Let X and Z be first countable, let Y satisfy (i') and moreover assume that all components of Y are open. Then (ii) holds.*

Now we will considerer again the Lipinski example. We shall determine for which subspaces Y of the real line the function $F_{f_0,Y}$ is almost continuous.

We say that a space T is an *extensor* for a space X if for each closed subset $F \subset X$ and every $f \in C(F,T)$ there exists an $f^* \in C(X,T)$ such that $f^*|F = f$. It is well-known that every convex subset of a locally convex linear topological space is an extensor for every metrizable space [1].

Recall that $f: X \to Z$ is *peripherally continuous* if for each $x \in X$ and each pair of open sets $U \subset X$ and $V \subset Z$ such that $x \in U$ and $f(x) \in V$, there exists an open subset W of U such that $x \in W$ and $f(bd(W)) \subset V (bd(W)$ denotes the boundary of W), cf. e.g. [2].

Proposition 6. *Assume that X and Z are first countable spaces, there exists a base of Z composed of extensors for X and $f: X \to Z$ is a peripherally continuous function wich is discontinuous at most at one point. Suppose moreover that Y satisfies (i') and*

(i^+) if **U** *is an open cover of Y such that every component of Y is contained in some element of* **U**, *then there exists a pairwise disjoint open refinement of* **U**.

Then $F_{f,Y}$ is almost continuous.

Proof. *Assume that f is discontinuous at $x \in X$.*

First note that $f \in A(X,Z)$. Indeed, let $G \subset X \times Z$ be an open neighbourhood of f and let W and V be open sets such that $(x, f(x)) \in W \times V \subset G$ and V is an extensor for X. Since f is peripherally continuous at x, there exists an open neighbourhood W_0 of x such that $W_0 \subset W$ and $f(bd(W_0)) \subset V$. Since $f|bd(W_0)$ is continuous, there exists a continuous function $f^*: X \to V$ such that $f^*|bd(W_0) = f|bd(W_0)$. Hence $g = (f|(X \setminus W_0)) \cup (f^*|W_0)$ is a continuous function and $g \subset G$.

Now we shall verify that $F_{f,Y}$ is almost continuous. Let G be an open neighbourhood of $F_{f,Y}$. Observe that, by (*i'*), for every component L of Y there exist open sets $W_L \subset X, U_L \subset Y$ and $V_L \subset Z$ such that

$$\{x\} \times L \times \{f(x)\} \subset W_L \times U_L \times V_L \subset G$$

Thus the family $\boldsymbol{U} = \{U_L : L \textit{ is a component of } Y\}$ is an open of Y. In view of (i^+), there exists a pairwise disjoint open refinement $\boldsymbol{U}_0$ of $\boldsymbol{U}$.

Note that $F_{f,Y} = \bigcup_{U \in \boldsymbol{U}_0} F_{f,U}$. For each $U \in \boldsymbol{U}_0$ there exist open neighbourhoods W_U of x and V_U of $f(x)$ such that $W_U \times U \times V_U \subset G$ and $f(bd(W_U)) \subset V_U$.Fix $U \in \boldsymbol{U}_0$. It Follows by the first part of the proof that there exists a continuous function $g_U : X \to Z$ such that $g_U|(X \setminus W_U) = f|(X \setminus W_U)$ and $g_U | W_U \subset W_U \times V_U$. Then $F_{g_U,U}$ is a continuous function contained in G. Since elements of $\boldsymbol{U}_0$ are pairwise disjoint and open, $g = \bigcup_{u \in u_0} g_U$ is a continuous function contained in G.

□

Observe that the condition (i^+) is satisfied for subspaces of $\mathbb{R}$ (obvious) and for countably paracompact strongly zero-dimensional normal spaces (cf Dowker's Theorem, [3 5.2.3]). Thus Proposition 6 yields the following consequences.

Corollary 3. *Let Y be a subspace of* $\mathbb{R}$. *Then* $F_{f_0,Y} \in \boldsymbol{A}\ (\mathbb{R} \times Y, \mathbb{R})$ *iff all componentess of Y are compact.*

Corollary 4. *If Y is a subspace of* $\mathbb{R}$ *then the condition (i') and (c)-(e) are equivalent.*

Query 2. *Does the implication* $(i') \Rightarrow (b)$ *hold whenever* $X = Z = \mathbb{R}$ *and* $Y \subset \mathbb{R}$?

2. Cartesian products of almost continuous and continuous functions

Given functions $f_1: X_1 \to Y_1$ and $f_2: X_2 \to Y_2$ we denote by $f_1 \times f_2$ the *Cartesian product* of f_1 and f_2 , i.e. the function from $X_1 \times X_2$ into $Y_1 \times Y_2$ defined by

$$(f_1 \times f_2)(x_1, x_2) = (f_1(x_1), f_2(x_2)).$$

Observe that $\pi_1 \circ (f_1 \times f_2) = F_{f_1, X_2}$, where $\pi_1 : Y_1 \times Y_2 \to Y_1$ denotes the projection. This fact yields that the Cartesian product of almost continuous functions and continuous funcions need not be almost continuous (see thee Lipinski example, cf [7]. However, if X_2 is compact then $f_1 \times f_2$ is almost continuous whenever $f_1 \in A(X_1, Y_1)$ and $f_2 \in C(X_2, Y_2)$ [7, Theorem 4.1].

Analogously to the previous section we considerer the following conditions:

(i) X_2 is quasi countably compact,

(ii) $f_1 \times f_2 \in A(X_1 \times X_2, Y_1 \times Y_2)$ for each $f_1 \in A(X_1, Y_1)$ and $f_2 \in C(X_2, Y_2)$,

(iii) there exists $f_1 \in A(X_1, Y_1) \backslash C(X_1, Y_1)$ such that $f_1 \times f_2 \in A(X_1 \times X_2, Y_1 \times Y_2)$ for each $f_2 \in C(X_2, Y_2)$,

(iv) there exists $f_1: X_1 \to Y_1$ such that $f_1 \times f_2 \in A(X_1 \times X_2, Y_1 \times Y_2)$ for each $f_2 \in C(X_2, Y_2)$ and $f_1 \notin C(X_1, Y_1)$,

(v) there exist $f_1: X_1 \to Y_1$ and $f_2 \in C(X_2, Y_2)$ such that $f_1 \notin C(X_1, Y_1)$ and $f_1 \times f_2 \in A(X_1 \times X_2, Y_1 \times Y_2)$,

(vi) there exists $f_1: X_1 \to Y_1$ and $f_2 \in Const\,(X_2, Y_2)$ such that $f_1 \notin C(X_1, Y_1)$ and $f_1 \times f_2 \in A(X_1 \times X_2, Y_1 \times Y_2)$

(vii) there exists $f_1: X_1 \to Y_1$ such that $F_{f_1, X_2} \in A(X_1 \times X_2, Y_1)$ and $f_1 \notin C(X_1, Y_1)$.

Similary to the proofs of Proposition 2 and [7, Theorem 4.1] we can prove the following.

Proposition 7. *Assume that X_1, Y_1 and Y_2 are first countable spaces. Then*

$$(i) \Rightarrow (ii).$$

Proposition 8. *We have*

$$(vi) \Rightarrow (vii).$$

Proof. Recall that F_{f_1,X_2} is the composition of the almost continuous function $f_1 \times f_2$ and the continuous function π_1. So the Proposition is a consequence of the fact that such compositions are almost continuous [8].

□

Corollary 5. *Assume that X_1 is a first countable, Hausdorff space, X_2 is a connected space, Y_1 and Y_2 are first countable, Y_1 is quasi regular, and moreover, $A(X_1, Y_1) \setminus C(X_1, Y_1) \neq \varnothing$. Then conditions (i)-(vii) are equivalent.*

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