Relations between Daniell integral analogues

POR HANS GÜNZLER*

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Summary

Recently introduced integral extensions $\overline{B}$ and L are compared with Daniell's L^1. Always $\overline{B} \subset L^1 +$ nulfunctions of $\overline{B}$; an analogue for L however is not true, also the conjecture $L = \overline{B} +$ nulfunctions of L is shown to be false.
Finally several sufficient conditions for this decomposition of L are given.

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Introducción

Recently abstract spaces of integrable functions $\overline{B}$ and (more general) L have been introduced in [3], [5], which are constructed similar to the Daniell L^1 and which coincide with L^1 in the classical case and also with Bourbaki's L^τ, but for which, contrary to the L^1 and L^τ cases, no continuity conditions on the starting elementary integral $I|B$ are needed.

Here we obtain first $\overline{B} \subset L^1 + \{\overline{B} - nulfunctions\}$, and analogue to [4] concerning an abstract Riemann integral. The corresponding conjecture for the Schäfke localisation L of $\overline{B}, L \subset L_1 + \{L - nulfunctions\}$ is refuted by a counterexample. This gives even codim of $L_1 \cap L + \overline{B} + \{L - nulfunctions\}$ in L is infinite, so another natural conjecture, $L = \overline{B} + \{L - nulfunctions\}$, is also false in general. Nevertheless, we give several sufficient conditions for

* Mathematisches Seminar. Christian-Albrechts-Universität Zu Kiel.

$L=\overline{B}+\{L-\mathit{nulfunctions}\}$, which subsume practically all known applications and examples up to now.

1. Assumptions and notations.

In the following X is an arbitrary set $\neq \phi$, and we assume always, with pointwise =, +, $\leq$, etc. everywhere on X (see [9], (3)).

(1) B function vector lattice $\subset \mathbf{R}^X$, $I:B\to\mathbf{R}$ linear, $I(f)\geq 0$ if $0\leq f\in B$. In the next two sections we also use Daniell's condition.

(2) (1) and $I(h_n)\to 0$ if $0\leq h_{n+1}\leq h_n\in B$ and $h_n\to 0$ pointwise on X ($I|B$ $\sigma-\mathit{continuous}$, see Floret [7] p. 43).

We extend the usual + in $\overline{\mathbf{R}}$ to $\overline{\mathbf{R}}\times\overline{\mathbf{R}}$ by $r-r:=r+(-r):=0$ if $r=\pm\infty$; though + is not associative one has ($\wedge = \mathit{min}$).

(3) $|(a+b)-(c+d)|\leq|a-c|+|b-d|, |a\wedge t-b\wedge t|\leq|a-b|$

for $a,b,c,d\in\overline{\mathbf{R}}$, $0\langle t\in\overline{\mathbf{R}}$ ([1], [7]). $+M:=\{k\in M: k\geq 0\}$ if $M\subset\overline{\mathbf{R}}^X$.

Using only (1), Bobillo and Carrillo [3] introduced $B^+:=\{g\in\overline{\mathbf{R}}^X$ to each $x\in X$ exist $h_n\in B$, $h_n\leq g$, $h_n(x)\to g(x)$ $\}$, $I^+(k):=\sup\{I(h): B\ni h\leq k\}$, $B_+:=\{g\in B^+: I^+(g+l)=I^+(g)+I^+(l)\, \mathit{for\, all}\; l\in B^+\}$,

$\overline{I}(k):=\inf\{I^+(g): k\leq g\in B_+\}$ for $k\in\overline{\mathbf{R}}^X$, and $\overline{B}:=\{f\in\overline{\mathbf{R}}^X: -\overline{I}(f)=\overline{I}(f)\in\mathbf{R}\}$.

$\overline{B}$ is the closure of B in $\overline{\mathbf{R}}^X$ with respect to the "integral metric" $\overline{I}|[0,\infty]^X$, $\overline{B}$ is closed with respect to +, a., , , I I, $\overline{I}$ extends I I B and is additive, **R**-homogeneous and monotone on $\overline{B}$;

$\overline{B}_n:=\{f\in\overline{\mathbf{R}}^X:\overline{I}(|f|)=0\}\subset\overline{B}$ ($\overline{B}-or\overline{I}-\mathit{nulfunctions}$; see [1] or [6]).

With (2) also the space $L^1:=L^1(I|B)$ of Daniell I-integrable $f:X\to\overline{\mathbf{R}}$ and the Daniell integral $I_D:L^1\to\mathbf{R}$ are well defined; $L^1_n:=\{f\in L^1: I_D(|f|)=0\}$ ([1], [7]).

2. The Bobillo-Carrillo integral.

Lemma 1. *To* $g \in B_{(+)} := \{g \in B_+ : I^+(g) \langle \infty\}$ *there exist* $q \in L^1 \cap B_{(+)}, p \in +\overline{B}_n$ *with* $g = q + p, q \leq g, I_D(q) = I^+(q) = \overline{I}(q) = I^+(g) = \overline{I}(g)$.
If $g \geq 0, q \geq 0$ *is possible*.

Proof: There exist $h_n \in B$ with $h_n \leq h_{n+1} \leq g, I(h_n) \to I^+(g) = \overline{I}(g) \in \mathbf{R}$. Then q: pointwise $\lim h_n \leq g$, $q \in L^1, I_D(q) = I^+(q)$ by L^1-theory. $h_n \leq q \leq g$ imply $\overline{I}(|q - h_n|) \leq \overline{I}(|g - h_n|) = I^+(g - h_n) \to 0, q \in \overline{B}$. Since $q \in B^+, q \in B_{(+)}$ by a result of [4], p. 261, (a). With p:g-q the rest follows since $p,q \neq -\infty$.

Lemma 2

If $a,b \in L^1 \cap B_{(+)}, c,d \in +\overline{B}_n, a + c \leq b + d, I^+(a) = I_D(a), I^+(b) = I_D(b)$, *there is* $p \in +\overline{B}_n$ *with* $a + c = a \wedge b + p, a \wedge b \in L^1 \cap B_{(+)}, I_D(a) = I_D(a \wedge b) =$
$= I^+(a \wedge b)$.

Proof: L^1 and $B_{(+)}$ are $\wedge$-closed ([3] p. 248, 2)). If $p := (a + c) - (a \wedge b)$, a simple discussion $(a,b \neq -\infty)$ gives $0 \leq p \leq c + d, a + c = (a \wedge b) + p$, so $p \in +\overline{B}_n$.

$$I_D(a \wedge b) \leq I_D(a) = I^+(a) = \overline{I}(a) = \overline{I}(a + c) = \overline{I}(a \wedge b + p) = \overline{I}(a \wedge b) = I^+(a \wedge b) \leq I_D(a \wedge b)$$

Lemma 3. *If* $f \in L^1, g \in \overline{B}, f \leq g$, *then* $I_D(f) \leq \overline{I}(g)$.

Proof: By the definition of $\overline{B}$ we can assume $g \in B_{(+)}$. By definition of L^1, for every $\varepsilon > 0$ there is $k \in B^+ \cap L^1$ with $-k \leq f$ and $I_D(f) - \varepsilon \langle -I_D(k) = -I^+(k)$; then

$$0 \leq f + k \leq g + k,\ 0 \leq I^+(g + k) = I^+(g) + I^+(k), I_D(f) - \varepsilon \leq I^+(g) = \overline{I}(g).$$

Theorem. *If (2) holds,* $\overline{B} = L^1 \cap \overline{B} \cap \mathbf{R}^X + \overline{B}_n$, *i.e. to each* $f \in \overline{B}$ *exist* $p \in \overline{B}_n$ *and an* $\mathbf{R}$*-valued* $q \in L^1 \cap \overline{B}$ *with* $f = q + q. I_D(g) = \overline{I}(g)$ *for any* $g \in L^1 \cap \overline{B}$.

From the example 3 of [9], $\overline{B} \subset L^1$ is false even for probability spaces (X,Ω,μ).

Proof: First for $0 \le f \in \overline{B}$: There exist $k_n, l_n \in B_{(+)}$ with $0 \le -k_n \le -k_{n+1} \le f \le$ $\le l_{n+1} \le l_n$ and $I^+(l_n) \to \overline{I}(f), -I^+(k_n) \to \overline{I}(f)$. From Lemma 1 and 2 there exist $a_n, b_n \in L^1 \cap B_{(+)}, c_n, d_n \in +\overline{B}_n$ with $k_n = a_n + c_n, l_n = b_n + d_n, a_{n+1} \le a_n \le$ $k_n \le 0, 0 \le b_{n+1} \le b_n, I^+(k_n) = I^+(a_n) = I_D(a_n), I^+(l_n) = I^+(b_n) = I_D(b_n)$. If a: $= \lim a_n, b := \lim b_n$, then a, $b \in L^1$ with $-I_D(a) = \overline{I}(f) = I_D(b)$ from the Monotone Convergence Theorem for L^1 (e.g. [1] p. 450). With $u := (f+a)_a, v := (b-f)_+$ we hare

(4) $a \le 0 \le f \le (-a) + u, 0 \le b \le f + v, I_D(-a) = \overline{I}(f) = I_D(b)$.

$u, v \in +\overline{B}_n : b - f \le b_n - f \le (b_n + d_n) - f, 0 \le v \le l_n - f \in +\overline{B}$,

$\overline{I}(l_n - f) == \overline{I}(l_n) - \overline{I}(f) \to 0$, so $\overline{I}(|v|) = 0$; similary $u \in +\overline{B}_n$.

$(a+b)_+ \in \overline{B}_n \cap L_n^1 : a + b \le a + (f+v) \le a + (((-a)+u)+v) \le u + v$ with (3), thus $0 \le (a+b)_+ \in \overline{B}_n$; Lemma 3 gives $I_D((a+b)_+) = 0$. $I_D(a+b) = 0$ yields then $(a+b)_- \in L_n^1, a+b \in L_n^1$.

If now $b_e := 0$ where $|b| = \infty, := b$ else, also $0 \le b_e \in L^1, I_D(b_e) = I_D(b)$, $b - b_e \in L_n^1$ ($|b - b_e| \le |b - h|, L^1 =$ suitable B^q by [1], p. 448: Stone's axiom is not needed): whit $|b_e + a| \le |a+b| + |b - b_e|$ one gets $b_e + a \in L_n^1$.

Define now $r := f - (b_e + u)$ where $f \succ b_e + u, := 0$ else, $g := b_e + r, p := f - g$; then (5) $0 \le r \le f, p \le f, 0 \le g, f = g + p, g \in L^1 \cap \overline{B}, p \in \overline{B}_n$:

(Here $g \le f$ resp. $0 \le p$ is in general not possible, e.g. in ex. 3 of [9].)

$r \in L_n^1$, $g \in L^1$: Where $r \succ 0, r = f - (b_e + u) \le ((-a)+u) - (b_e + u) \le |a + b_e|$ by (3), thus $0 \le r \le |a + b_e|$ or $r \in L_n^1$; then $g \in L^1$.

$|p| \le u + v$, so $p \in \overline{B}_n$: If $f \le b_e$, there $|p| = b_e - f \le v$; if $b_e \prec f \le b_e + u, |p| =$ $= f - b_e \le u$; if $b_e + u \prec f, |p| = |f - (b_e + (f - (b_e + u)))|$; since $u \ne \infty$, the cases $f = \infty, f \ne \infty$ yield there $|p| \le u. f = g + p$, since $f \prec \infty$ implies $r \prec \infty, g \prec \infty$.

Since $g = f - p$ except where $f = p = \infty, g \in \overline{B}$ by theorem 5.2. of [2]. This gives (5).

For general $f \in \overline{B}$ one can write $f = f_e + f_u$ with $f_e \in \overline{B} \cap R^X$ and $f_u \in \overline{B}_n$ as above (see [6], Cor. II). $f_e = f_{e+} - f_{e-}$ with $f_{e\pm} \in +\mathbf{R}^X \cap \overline{B}$, (5) and $g = b_e + r$ gives $g_i \in +L^1 \cap \overline{B} \cap \overline{\mathbf{R}}^X, p_i \in \overline{B}_n \cap \overline{\mathbf{R}}^X$ with $f_e = (g_1 + p_1) - (g_2 + p_2) = g + q$ $, g := g_1 - g_2 \in \overline{B} \cap L^1 \mathbf{R}^X, f = (g + q) + f_u = g + (q + f_u), q + f_u \in \overline{B}_n$. One even has $f(x) = p(x)$ where $|f(x)| = \infty$.

$I_D = \overline{I}$ on $L^1 \cap \overline{B}$ follows from Lemma 3 for $\pm$.

3. An extension of the Bobillo-Carrillo integral.

The integral $\overline{I}|\overline{B}$ has been extended to $J|L$ in [5] with Schäfke's [11] local integral norm $\overline{I}_B : \overline{I}_B(k) := \sup\{\overline{I}(k \wedge h) : h \in +B\}, L := L(I|B) :=$ $= \overline{I}_B$ $-$closure of B in $\overline{R}^X (= R(B, I)$ in [5], $J :=$ unique $\overline{I}_B$ $-$continuous extension of $I|B$ to L. $J = \overline{I}_B$ on L, $J : L \to \mathbf{R}$ is "linear" and monotone, $\overline{B} \subset L$ with $J = \overline{I}$ on $\overline{B}$, the convergence theorems of [9] for $\overline{B}$ extend to $J|L$, in even better form. Looking at this and the definition of L it is natural to conjecture that an analogue to the Theorem of section 2 should be true for L, especially since this is true in all the examples in the literature (see section 4).

In general however L is bigger than such an analogue would allow; this is shown by the following.

Example. There is a set X, a ring Ω of subsets of X and a σ-additive $\mu : \Omega \to [0, \infty)$ *sucht that with* $B = B_\Omega :=$ *real-valued step functions over* Ω *and* $I = I_\mu := \int \ldots d\mu$ (see [9] after (17)) *one has*

(6) $L \not\subset L_1 + L_n$ *and* $L^1 \not\subset L + L_{1,n}$.

Here $L := L(I_\mu | B_\Omega), L_n := \{k \in \overline{R}^X : \overline{I}_B(|k|) = 0\}, \subset L, L^1 = L^1(I_\mu | B_\Omega) =$ usual $L^1(\mu | \Omega, \overline{\mathbf{R}}), L_1 :=$localized $L^1 = L^1 + L_{1,n}$ (see [9], section 6),

$L_{1,n} = \left\{k \in \overline{R}^X : k = 0\mu - a.e.\ on\ each\ A \in \Omega\right\}$:

$X := I \times I$ with $I := [0,1] \subset R; \Omega :=$ ring containing all $\{s\} \times E, \{s\} \times (I - E), F \times \{t\}, (I - F) \times \{t\}$ with $0 \langle s \leq 1$, E finite $\subset I, t \in I, 0 \notin F$ finite $\subset I$;

$\mu(\{s\} \times I) := 1, \mu(I \times \{t\}) := t^2, \mu(\{s\} \times E) := 0 =: \mu(F \times \{t\})$ defines a σ-additive $\mu : \Omega \to I$. Therefore $I_D \big| L^1, L_1, L_1, \overline{I} \big| \overline{B}$ and $J | L$ are well defined, (2) holds.

If $f := 1T =$ characteristic function of $T := \left\{(0, \frac{1}{n}) : n \in \mathbf{N}\right\}$, then $f \in L$, but $f \notin L_1 + L_n$:

Since $f_n := 1\{(0, 1/m) : 1 \leq m \leq n\} \to f\ (\overline{I}, B)$ (see (15) below) and $\overline{I}_B(|f_r - f_n|) \langle \langle \sum_n^{\infty} m^{-2}$ if $n \langle r$, for $f \in L$ only $1\{(0,t)\} \in L$ has to be proved by Theorem 1 of [5]. But $1 := 1((0,1] \times \{t\}) \in B^+$ and $I^+(k+1) = I^+(k) = I^+(k) + I^+(1)$ for any $k \in B^+$ by definition of $B^+ I^+, \Omega$, so $l \in B_{(+)} \subset \overline{B}$, $1\{(0,t)\} =$

$= 1(I \times \{t\}) - l \in B_{(-)} \subset \overline{B}$.

If $f = g + p$ with $g \in L_1\ p \in L_n$, one can show first that $p(0,t) = 0$ for $0 \langle t \in I$, so $g(0, \frac{1}{n}) = 1$ for $n \in N$. If $q := g$ on $A := Ix\left\{\frac{1}{n}\right\}, :0$ else, then $q \in L^1$, there are $h_m \in B$ with $I_D(|h_m - q|) \to 0, h_m = 0$ outside A and $h_m \to q$ except on a countable $M \subset A$ with $(0, 1/n) \notin M\ (L^1 = L^1(\mu \big| \Omega, \overline{R}))$. Therefore there exists a countable $P \subset I$ with $0 \notin P$ such that $g\left(s, \frac{1}{n}\right) 1$ for $s \in I - P$ and $n \in N$.

This gives a $s_o \in (0,1]$ with $p\left(s_o, \frac{1}{n} = -1\right)$ for $n \in N$.

Now if $r:=|p|$ on $C:=\{s_o\}\times I,:=0$ else, then $r\in B^+\cap L_n, C\overline{B}_{(+)}$ by Theorem 9 of [5], $r\in B_{(+)}, I^+(r)=\overline{I}(r)=0$. If now $k\left(s_o,\frac{1}{n}\right):=0, k:=1$ else in $C,:=0$ outside C, then $k\in B^+, I^+(k)=0$; this gives $1\le I^+(k+r)=I^+(k)+I^+(r)$, a contradiction.

The second part of (6) follows with $f=1\left(U_1^\infty(I\times\left\{\frac{1}{n}\right\})\right)\in L^1,\notin L+L_{1,n}$, along similar lines, we omit the details.
Furthermore one can even show that the codimension of $L_1\cap L+L_n$ in L is infinite in this example. See also (11) below.
For measure spaces the situation is different, his will be treated in Corollary IV below.

4. Relations between the preceding integrals.

Proposition 1. *If $I|B$ is σ – continuous (2), then* $\overline{B}+L_n=\left(L^1\cap\overline{B}\right)+L_n$.

This follows from Theorem of section 2, $\overline{B}+L_n=((\overline{B}\cap L^1\cap \mathbf{R}^X)+\overline{B}_n)+L_n=$ $=(\overline{B}\cap L^1\cap\mathbf{R}^X)+(\overline{B}_n+L_n)$ (though + is not associative), $\subset\overline{B}\cap L^1+L_n$ (see (18)).

Corollary I. *If* $I|B$ *is* σ*-continuous,* (8)⇔ (9)⇒(10)⇔(10'), *where*

(8) $L=\overline{B}+L_n$ (8') $L\subset\overline{B}+(L_n+L_{1,n})$ (see (18))

(9) $L=(L^1\cap\overline{B})+L_n$

(10) $L=(L^1\cap L)+L_n$ (10') $L\subset L_1+L_n$.

Proof: (8)⇔(9) by Prop. 1. If $f=g+(p+q)$ with $g\in\overline{B}, p\in L_n, q\in L_{1,n}$, the $\pm$ -closedness of L by [5], p. 81 gives $p+q\in L, q\in L\cap L_1$; then $|q|\wedge h\in\overline{B}\cap L_n^1$ if $h\in +B$ by [5], (1.- p. 82) , so $q\in L_n$ with $I_D=\overline{I}$ on $\overline{L}^1\cap B$ of the Theorem above, (8')⇒ (8). If $0\le f\in L\cap L_1$, there are $h_n\in +B$ with

$J(|f-h_n|)\to 0, f_n:=V_1^n(f\wedge h_m)\to:g$ pointwise $\le f, f_n\in\overline{B}\cap L^1$,

$\overline{I}_B(|g-f_n|)\le J(f-f_n)=J(f\wedge f-f\wedge h_n)\le J(|f-h_n|)\to 0$, so $g\in L\cap L^1$,

$f - g \in L_n$; this implies (10')⇒(10).

Corollary II. *In general (8) is false, even for* $I_\mu | B_\Omega$ *with* σ *– additive* $\mu|\Omega$.
Proof: (9) is false by (16) for the example in section 3, so also (8) by Cor. II; explicitly the $1T$ of this example $\in L, \notin \overline{B} + L_n$.
A closer look at this example even yields there

(11) $L_n = \overline{B}_n, R_1(\mu, \overline{\mathbf{R}}) \underset{\neq}{\subset} \overline{B} \underset{\neq}{\subset} L$, codim of $\overline{B} + L_n + L \cap L_1$ in L is infinite.

Proposition 2. *If B satisfies Stone's axiom* $(h \wedge 1 \in B$ *if* $h \in +B)$ *and*

$I\left(h \wedge \frac{1}{n}\right) \to 0$, $I(h - h \wedge n) \to \infty$, $h \in +B$, *then the following four conditions are equivalent:*

(8) $L = \overline{B} + L_n$ (8") $L = (\overline{B} \wedge \mathbf{R}^X) + L_n$

(12) $0 \le f$ *bounded* $\in L \Rightarrow f \in \overline{B} + L_n$

(13) $M \subset X, 1M \in L \Rightarrow 1M \in \overline{B} + L_n$.

(12)⇒ (8) is an extension of Theorem 3 of [9], for this $I\left(h \wedge \frac{1}{n}\right) \to 0$ is not needed.

(13)⇒ (12) uses the countability of the "spectrum" of a $f \in +L$ (see [2], Lemme 1, for the $\overline{B}$ case)and the closedness of $\overline{B}$ with respect to uniform convergence.

$M = U_1^\infty M_n$ with $1M_n \in \overline{B}$ and $J(1M - 1M_n) \to 0$ suffice in (13). We omit the somewhat lengthy details.

Proposition 3. *For arbitrary* $I|B$ *with* (1) *one has* $L = \overline{B} + L_n$, *if o n e of the following four conditions is true:*

(14) $I|B$ σ *– continuous,* $L^1 \subset \overline{B} + (L_n + L_{1,n})$ (see (18)).

(15) *B satisfies Stone's axiom,* $I(h - h \wedge n) \to 0$ if $h \in +B$, *there exits a indexed set* $(b_s)_{s \in S}$ *with* $b_s \in +B_{(+)}$ *such that* $\sum_{s \in e} b_s \le 1$ *for each finite*

$e \subset S$ and $\sum_e b_s \to 1X(\bar{I}, B)$ with respect to the net of finite $e \subset S$ (i.e.

$\bar{I}\left(\left|1X - \sum_e b_s\right| \wedge h\right) \to 0$ for each $h \in +B$)

(16) *B satisfies Stone's axiom, $I(h - h \wedge n) \to 0$ if $h \in +B, I^+(1X) < \infty$*

(17) *All $1\{x\} \in B^+, x \in X$.*

(16) implies even $L = \bar{B}$; (17) $\Rightarrow \bar{B} = R_1 = L \Rightarrow \bar{B} \subset R_1 \Leftrightarrow L = R_1 \Rightarrow$

$\Rightarrow L = \bar{B} + R_{1,n}$.

Most known examples are subsumed by Proposition 3:

Corollary III. *If $I|B$ is τ-continuous = Bourbaki's continuity condition, then $L^1 \subset \bar{B} = L^\tau = L^1 + \bar{B}_n \subset L = \bar{B} + L_n, L_1 \subset L$.*

Special case: $B = C_o(X, \mathbf{R})$, X locally compact, I arbitrary linear ≥ 0; if X is σ-compact (e.g. open or closed $\subset \mathbf{R}^n$), then $L = \bar{B} = L^1 = L_1$, see Cor. IV.

Proof: By [3], p. 247, $\bar{B} = L^\tau$ (see also [9], (33)); since always $L^1 \subset L^\tau, B = L^1 + \bar{B}_n$ by section 2 and (14) holds.-

Corollary IV. *If Ω is a ∂-ring and $\mu: \Omega \to [0, \infty)$ is ∂-additive, then $L_1 = R_1 \subset \bar{B} + L_n = L \subset L^1 + L_n$.*

Proof, with B = step functions $B_\Omega, I = I_\mu$ as before (6) for $\bar{B}, L, R_1 = R_1(\mu, \bar{\mathbf{R}})$ of [8], $= R_1(B_\Omega, I_\mu)$ of [10]: $L_1 = R_1$ by [8], p. 265.

$R_1 \subset \bar{B} + R_{1,n}$ by [4]. $R_{1,n} \subset L_n$ by [5], p. 82, so $L^1 \subset L_1 \subset \bar{B} + L_n$, (14) holds.-

For further inclusion of this type, see (58) of [6].

Corollary V. *If B satisfies $h \wedge 1 \in B, I(h - h \wedge n) \to 0, I\left(h \wedge \frac{1}{n}\right) \to 0$ if $h \in +B$, and B is I-separable* (i.e. there exists a at most countable $M \subset B$ such that to each $h \in B$ and $\varepsilon \rangle 0$ there is $k \in M$ with $I(|h - k|) \langle \varepsilon)$, *then* $L = \bar{B} + L_n$.

Proof: If $M = \{q_n : n \in N\}, p_n := 1 \wedge (V_1^n q_m), \varepsilon \rangle 0, h \in +B$, there is m with

$I\left(h\wedge\frac{1}{m}\right)\langle\varepsilon$, then n with $I\left(\left|m(h\wedge\frac{1}{m})-q_n\right|\right)\langle\varepsilon; h\wedge|1X_1-p_n|\le h\wedge|1X-q_n|\le$

$\le h\wedge\left|1X-m(h\wedge\frac{1}{m}\right|+\left|m(h\wedge\frac{1}{m})-c_n\right|\le h\wedge\frac{1}{m}+\left|m(h\wedge\frac{1}{m})-q_n\right|$, i.e.

$p_n\to 1X(\bar{I},B); S:=\mathbf{N}, b_n:=p_n-p_{n-1}, p_o:=0$ gives (15).-

Special cases: B finite dimensional, $B=C_o(\mathbf{R}^n,\mathbf{R}), B=B_\Omega$ with at most countable Ω ; or

Corollary VI. *If X open* $\subset\mathbf{R}^n,\Omega$ = *semiring of Lebesgue measurable sets with finite measure* $\subset X, B$ = *step functions* $B_\Omega, I=\int..d\mu_L^n$, then $L=\bar{B}+L_n$;
if Ω = {*all intervals* {a,b)$\subset X$ } *or* = {*all L-measurable sets with finite measure* $\subset X$ }, *then even* L^1 = *usual* $L^1(X,\bar{\mathbf{R}})=L_1=\bar{B}=L$.

Proof: Ω is 'μ-separable', so B_Ω is I-separable, Cor. V gives the first statement. (38) of [9] gives the first three '=' in the last statement.

If $p\in L_n, h\in +B$, then $|p|\wedge h\in\bar{B}_n=L_n^1$ by [5], 1.- p. 82; this implies $p\in L_{1,n}=L_n^1\subset L^1$ or $L_n\subset L^1=\bar{B}$.-

Corollary VII. *If, besides* $I(h-h\wedge n)\to 0$ *on +,B,* $1X\in B$ = *Stonean, or* $X\in\Omega$, *or* $\mu:\Omega\to[0,\infty)$ *is bounded on the ring* Ω, *then* $L=\bar{B}$.
Special case: X in Corollary VI has finite Lebesgue measure.
Proof: Here (16) is true.-

Corollary VIII. *(2) and any of the asumptions in Cor. III - VII or Proposition 3 imply* $L=L^1\cap\bar{B}+L_n$.

Proof: Use the Theorem of section 2 and $\left(L^1\cap\bar{B}+\bar{B}_n\right)+L_n=L^1\cap\bar{B}+L_n$.-

Proof of Proposition 3, *case (14)*: If $f\in +L$ there are $h_n\in +B$ with $J(|f-h_n|)\to 0$; then $f_m:=f\wedge h_m\in +\bar{B}\cap\mathbf{R}^X$ by 1.- p. 82 of [5], $J(|f-f_m|)\to 0$; one can assume $f_m\le f_{m+1}\le f$. With (5) and $g=b_e+r$ one gets $f_m = g_m + p_m$ with $g_m \in +L^1 \cap \overline{B} \cap \mathbf{R}^X, p_m \in \overline{B}_n \cap \mathbf{R}^X$; with an analogue to Lemma 2 one can assume $g_m \le g_{m+1}, m \in \mathbf{N}$. Then $g_m \to: g \in L^1$ by the Monotone Convergence Theorem for L^1 ([1] p. 450) and Lemma 3. One has $f \le g + p$ with $p := (f-g)_+, p \le (f - f_m) + |p_m| =: q_m \in L$, so

$\overline{I}_B(|p|) \le J(q_m) \to 0, p \in +L_n$. With (14) one gets $0 \le f \le (1+q+r) + p \le |1| +$

$+|q| + |r| + p =: a + b + c$ with $a \in +\overline{B}, b \in +L_n, c \in +L_{1,n}$.

With $d := f - f \wedge a$ we have $f = f \wedge a + d, f \wedge a \in \overline{B}, 0 \le d \le b + c$ so if $h \in +B, d \wedge h \le b \wedge h + c \wedge h$ with $d \wedge h \in \overline{B}, b \wedge h \in \overline{B}_n, c \wedge h \in L_n^1$; Lemma 3 gives $0 = I_D(-c \wedge h) \le \overline{I}(b \wedge h - d \wedge) = -\overline{I}(d \wedge h) \le 0$, then $\overline{I}_B(|d|) = 0$,

$d \in +L_n$, or $+L \subset (+\overline{B}) + (+L_n)$.

$L = \overline{B} + L_n$ follows from $f = f_+ - f_-$, since the supports of $f_\pm$ are disjoint. Though we did not need it, let us remark that one can show

(18) $\overline{B} + (L_n + L_{1,n}) = (\overline{B} + L_n) + L_{1,n} = (\overline{B} + L_{1,n}) + L_n$.

Case (15): We assume first only $g_e := \sum_{s \in e} b_s \to 1(\overline{I}, B), b_s \in +B_{(+)} \cap \mathbf{R}^X$, $g_e \le 1$, with $l \in +\overline{\mathbf{R}}^X$; then we will show

(19) $f \wedge 1 \in (+\overline{B}) + (+L_n)$ if $f \in +L$:

$\overline{I}_B(|f \wedge 1 - f \wedge g_e|) \to 0$ and $f \wedge 1 \in L, f \wedge g_e \in \overline{B}$ by (3), the Legesgue Convergence Theorem for L of [5], p. 82 and 1.- p.82. If only finitely many e's are needed, $f \wedge l - f \wedge g_{e_o} \in L_n$, (19) follows. Else there are pairwise different $s_m \in S$ with $g_n := b_{s_1} + ... + b_{s_n} \in B_{(+)}$ and $\overline{I}_B(|f \wedge l - f \wedge g_n|) \to 0$.

If $g_M := \sum_{s \in M} b_s := \sup\{\sum_{s \in e} b_s : finite \subset M\}$ pointwise, then to $h \in +B$ and $\varepsilon \rangle 0$ there is e_ε with

(20) $\overline{I}(g_{S-e\varepsilon} \wedge h) = \overline{I}((g_S - g_{e_\varepsilon}) \wedge h) \le \overline{I}(|l - g_{e_\varepsilon}| \wedge h) \langle \varepsilon$;

also there is $n_\varepsilon \in \mathbf{N}$ with $s_n \notin e_\varepsilon$ if $n \rangle n_\varepsilon$.

Since $0 \le k_n := f \wedge g_n - f \wedge g_{n-1} \le b_{s_n}$ by (3) and $B_{(+)}$ is $\wedge$-closed ([3] p. 248), there exist $t_n \in +B_{(+)}$ with $k_n \le t_n \le b_{s_n}$ and $\bar{I}(t_n) \langle \bar{I}(k_n) + 2^{-n}$;

$t := \sum_1^\infty t_n \in B^+$. (20) and $\left| t - \sum_1^n t_m \right| = \sum_{n+1}^\infty t_m \le \sum_{n+1}^\infty s_m \le g_{S-e_\varepsilon}$

if $n \ge n_\varepsilon$ give $\sum_1^n t_m \to t\,(\bar{I}, B)$; since $\bar{I}\left(\sum_1^n t_m \right) \le \sum_1^{n-} \bar{I}(k_m) + 1 =$

$\bar{I}(f \wedge g_n) + 1 \le J(f) + 1$ for $n \in \mathbf{N}, t \in B^+ \cap \bar{B}(= B_{(+)})$ by Theorem 2 of [9].

But then $f \wedge t \in \bar{B}([5] 1.-p.82)$. Since $0 \le t \le 1, 0 \le f \wedge t - f \wedge \left(\sum_1^n t_m \right) \le$

$\le f \wedge l - f \wedge \left(\sum_1^n k_m \right) = f \wedge g_n$, so

$J(f \wedge l) \ge \bar{I}(f \wedge t) = lim\, \bar{I}\left(f \wedge \sum_1^n t_m \right) \ge lim\, \bar{I}(f \wedge g_n) = J(f \wedge l)$,

or $p := f \wedge l - f \wedge t \in +L_n . f \wedge t + p$ gives (19).
If now (15) holds and $0 \le f$ bounded $\le r, f \in L$, then $l = rX$ in (19) gives $f \in (+\bar{B}) + (+L_n)$. Then $L = \bar{B} + L_n$ with (12)$\Rightarrow$ (8) of Proposition 2.

If $I|B$ satisfies additionally $I\left(h \wedge \frac{1}{n} \right) \to 0\ for\ h \in +B$, then $\sum_e b_s \to 1X(\bar{I}, B)$ in (15) can be replaced by

(21) $\bar{I}\left(h \wedge \sum_e b_s \right) \to I(h)$ for each $h \in B$ with $0 \le h \le 1$.

Case (16): If $r := I^+(1X) \langle \infty$, there are $h_n \in +B$ with $h_n \le h_{n+1} \le 1, I(h_n) \to r$. Then $h_n \to 1X(\bar{I}, B)$, since if $\bar{I}((1X - h_n) \wedge h_o) \wedge \varepsilon_o \rangle 0$ for $n \in \mathbf{N}$, with $(1X - h_n) \wedge h_o = (h_n + h_o) \wedge 1 - h_n$ one would get a contradiction.

(15) holds with $S := \mathbf{N}, b_n := h_n - h_{n-1}$. Without Prop. 2, $1X \in B_{(+)} \subset \bar{B}$ by Theorem 2 of [9], $sc|f| \wedge n \in \bar{B}$ if $f \in L$ by |5|, 1.- P. 82; Theorem 3 of [9] gives $L = \bar{B}$.

Case (17): Then $+\overline{R}^X \subset B^+$, so $+L \subset L \cap B^+ = B_{(+)} \subset \overline{B}$ by Theorem 9 of [6], $L = \overline{B} = B_{(+)} - B_{(+)}.\overline{B} \subset R_1$ follows from (20) of [9] and prop. 1.4 of [10b], at least for $I_\mu | B_\Omega$.

$L = R_1 \Rightarrow L = \overline{B} + R_{1,n}$ follows from [10b] p. 45 (see [6], 38).-
With suitable examples (see [9]) one can show that the 'c' in Cor. III and IV are in general strict; no part of the asumptions in Prop. 3 and its corollaries can be omitted, e.g. $b_s \in +\overline{B}$ instead of $b_s \in +B_{(+)}$ in (15) does not give $L = \overline{B} + L_n(b_s = 1\{s\} \times I \; and \; 1\{(0, s-1)\}), S = (0,2]$, in the example of section 3). (10)$\Rightarrow$ (8) however is open.

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