On the behaviour of generalized solutions of the Cauchy problem for essentially nonautonomous quasilinear first order equations

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§1. INTRODUCCION

This paper deals with the evolution of generalized solutions of equations in the form

$$u_t + [A(t,x,u)]_X + B(t,x,u) = 0. \qquad (1.1)$$

Here $A(t,x,u), B(t,x,u)$ are continuous functions on their arguments, sufficiently smooth in $t,x; A(t,x,0) = B(t,x,0) = 0$; $A_u \geqslant 0, A(t,x,u)$ is convex in u and $B(t,x,u)$ is nondecreasing in u. The essential dependence $A(t,x,u), B(t,x,u)$ on t,x may be present. For example, these functions may tend to zero or infinity as t converges to $+\infty$.

It is well known, that there is an intersection of characteristic's projections on the plane (t,x) for the equation (1.1) even for very smooth initial data. So, there is no classical global solutions and one needs the concept of generalized solutions. The main peculiarity of such solutions is the occurence of discontinuities even for smooth initial data. It is the same difficulty than with gas dynamics system. So, (1.1) can be regarded as the simplest model of gas dynamics equations.

Beginning with the paper [7] several particular cases of (1.1) were studied by different authors (see for example [15],[22]). The complete theory of equations in the form (1.1) with $A(t,x,u)$ convex in u was constructed by O.A.Oleinik in [16],[17]. There the physically relevant definition of generalized solution to Cauchy problem was given and its correctness was proved. Some properties of such solutions were studied later. In [23] the Cauchy problem for (1.1) with nonconvex $A(t,x,u)$ in u was solved, provided an initial function with bounded variation. Finally, S.N.Kruzhkov [11] constructed a complete theory of Cauchy problem for equation (1.1) and its multydimensional generalization with an arbitrary smooth $A(t,x,u)$ and measurable

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bounded initial datum. The improvements that followed were connected with the discontinuity of A_u [10] and with locally unbounded initial function [12], [2], [4]. One can become acquainted with the theory of discontinuous solutions of (1.1) and its generalizations using the surveys [17], [6], the lectures notes [13] and books [18], [21].

Many authors have studied the behaviour of generalized solutions to Cauchy problem for equations in the form (1.1) as t converges to $+\infty$. For example one can indicate [14], [8], [11], [1], [9] and their references. As a rule those authors supposed that $A = A(u)$, $B = 0$, the initial function converges to some constants as x converges to $\pm\infty$ and they compared the generalized solution with some self–similar (or some particular) solutions.

This work involves another set of questions connected with the Cauchy problem for (1.1). With the help of comparison theorems we investigate several phenomena such as finite extinction time of generalized solutions and the localization of perturbations. Formerly these effects have been studied mainly for second order quasilinear degenerate parabolic equations. But the used methods are invalid for first order quasilinear equations, because one must use discontinuous comparison functions to obtain precise results.

We shall consider the above mentioned phenomena for the model equation

$$L_1 u = u_t + A_1(1+t)^p (u^m)_x + A_2(1+t)^q u^n = 0, \qquad (1.2)$$

where A_1 =const > 0, A_2 =const > 0, $m > 1, n > 0, n \neq 1, p \in \mathbb{R}, q \in \mathbb{R}$. The case $n = 1$ has been investigated in [20]. One can easily generalize the obtained results to the equations (1.1) provided appropriate estimates on nonlinear terms hold true.

In §2 the definition of generalized solution of Cauchy problem for (1.1) is given and a comparison theorem is proved.

§3 is devoted to the investigation of the finite extinction time property.

Definition 1.1

Suppose $u(t,x)$ is a generalized solution of Cauchy problem for (1.2). One says that extinction occurs, if there exists some $T > 0$ such that $u(t,x) \equiv 0$ for $t \geqslant T$. One says that extinction does not occur, if for every $T > 0$ we have $u(t,x) \not\equiv 0$ for $t \geqslant T$.

This question have been dealt with in [5], where sufficient conditions for the existence of a finite extinction time have been given in the case $p = q = 0$. But in general the situation is more complicate because of the essential dependence on t. So, in the case $q < -1, 0 < n < 1$ the extinction presence depends on the L_∞ –norm of the initial function. In case the extinction does not occur the localization phenomenon is investigated (see below Definition 1.2). Estimates for the extinction time will appear in a forthcoming paper.

In §4 questions concerning the presence and absence of localization for (1.2) in the case $n > 1$ are studied.

Definition 1.2

Suppose $u(t,x)$ is a generalized solution of Cauchy problem for (1.2). One says that spatial localization occurs, if there exists some $X > 0$ such that $u(t,x) \equiv 0$ for $|x| \geqslant X$. One says that localization does not occur, if for every sufficiently large $x_* > 0$ there exists some $t_* > 0$ such that $u(t,x) \not\equiv 0$ in a neighbourhood of (t_*, x_*).

In [5] sufficient conditions for the presence of localization for (1.2) are given for the autonomous case. In §4 the general case $p \in \mathbb{R}$, $q \in \mathbb{R}$ is investigated and sufficient conditions for the presence or absence of localization are given. These conditions contain all meanings exponents $m > 1, n > 1, p \in \mathbb{R}, q \in \mathbb{R}$.

Our technique allows to consider some results in the case of essential dependence on x in (1.1). For example in §5 we consider the Cauchy problem for the equation

$$L_2 u = u_t + (u^m)_x + B(x)u^n = 0 \tag{1.3}$$

where

$$m > 1, 0 < n < 1; \quad B(x) = A(1+x)^s, \quad s < -1, \text{ for } x \geqslant 0; \tag{1.4}$$

$B(x)$ is bounded and sufficiently smooth for $x \leqslant 0$. In this case sufficient conditions for the extinction property will be stated. Unlike (1.2) here the initial function is supposed to be bounded from below by some power function with fixed parameters.

§2. BASIC DEFINITIONS. A COMPARISON THEOREM

First of all let us consider the following equation (more general than (1.1))

$$Lu \equiv u_t + [A(t,x,u)]_x + B(t,x,u) = H(t,x), (t,x) \in \mathbb{R}_+ \times \mathbb{R} \tag{2.1}$$

with the initial data

$$u(0,x) = u_0(x), \quad x \in \mathbb{R} \tag{2.2}$$

Here $A(t,x,u)$ and $B(t,x,u)$ are continuous functions $A(t,x,0) = B(t,x,0) = 0$; $B(t,x,u)$ is monotonically increasing in u; $A(t,x,u)$ is a twicely continuously differentiable function in t,x; A, A_X, A_{XX}, B, B_X are bounded for bounded values of u, t; $A_u \in C(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R})$; $A_u \geqslant 0$; $B(t,x,u) + A_x(t,x,u) \geqslant 0$; $H(t,x)$ is a measureble function, bounded for bounded values of t; $u_0 \in L^\infty(\mathbb{R})$, $u_0(x) \geqslant 0$.

Definition 2.1

A mesurable function $u(t,x)$ bounded for bounded t, is called a generalized solution (in abbreviation g.s.) of equation (2.1) in $\mathbb{R}_+ \times \mathbb{R}$ if for every constant s and every $\omega(t,x) \geqslant 0, \omega \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R})$ the following inequality holds

$$\iint_{\mathbb{R}\times\mathbb{R}} \Big\{ |u(t,x)-s|\,\omega_t + \mathrm{sign}(u(t,x)-s)[A(t,x,u)-A(t,x,s)]\,\omega_x -$$

$$-\mathrm{sign}(u(t,x)-s)[A_x(t,x,s)+B(t,x,u)-H(t,x)]\,\omega \Big\}\, dt\, dx \geqslant 0 \quad (2.3)$$

Definition 2.2

A measurable function $u(t,x)$ bounded for bounded t, is called a g.s. of the problem (2.1), (2.2) in $\mathbb{R}_+ \times \mathbb{R}$ if the inequality (2.3) holds and there exists a set $E \subset [0,+\infty]$, meas $E = 0$, such that for $t \in [0,+\infty]/E$ the function $u(t,x)$ is defined for almost every $x \in \mathbb{R}$ and for each segment $[a,b] \subset \mathbb{R}$

$$\lim_{\substack{t\to 0\\ t\in[0,+\infty)/E}} \int_a^b |u(t,x)-u_0(x)|\, dx = 0.$$

Remark 2.1

If a function $u(t,x)$ is a g.s. of (2.1) and is piecewise continuous then the definition 2.1 implies (see [11]) the Hugoniot condition at the line of discontinuity $x = y(t)$ of $u(t,x)$

$$\dot{y} = [A(t,y(t),u^+) - A(t,y(t),u^-)]/(u^+ - u^-) \quad (2.4)$$

and the stability condition

$$\mathrm{sign}(u^+ - u^-)[A(t,y(t),\mu\bar{u} + (1-\mu)\,u^+) -$$

$$\mu A(t,y(t),u^-) - (1-\mu)\,A(t,y(t),u^+)] \geqslant 0 \quad (2.5)$$

for every $\mu \in (0,1)$; here $u^- = u(t,y(t)-0)$, $u^+ = u(t,y(t)+0)$.

The existence theorem for the problem (2.1), (2.2) with appropriate $u_0(x)$ and restrictions on A, B, H listed above can be proved by analogy with [11].

Lemma 2.1

Suppose $u(t,x)$ is a g.s. of the problem (2.1), (2.2). Then for every constant k the following inequality holds

$$\iint_{\mathbb{R}_+\times\mathbb{R}} \mathrm{sign}_+(u(t,x)-k)\Big\{(u-k)\omega_t + [A(t,x,u)-A(t,x,k)]\,\omega_x -$$

$$-[B(t,x,u)+A_x(t,x,k)-H(t,x)]\,\omega\Big\}\, dt\, dx \geqslant 0, \quad (2.6)$$

where $\omega(t,x) \geqslant 0$, $\omega \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R})$; $s_+ = s$ for $s > 0$, $s_+ = 0$ for $s \leqslant 0$.

Proof.

Let us follow the scheme of [2]. Suppose $p_m(s) \geqslant 0$, $p_m(s) \in C_0^\infty(\mathbb{R})$ $(m = 1, 2, \ldots)$,

$$\int_K^r p_m(s)\,ds \Rightarrow \operatorname{sign}_+(r-k), \quad m \longrightarrow +\infty \tag{2.7}$$

Let us multiply (2.3) on $p_m(s)$ and integrate with respect to s from $-\infty$ to $+\infty$. Changing the integration order one gets

$$\int_{\mathbb{R}_+\times\mathbb{R}} \left\{ \int_{-\infty}^{u(t,x)} [(u-s)\,\omega_t + +(A(t,x,u) - A(t,x,s))\omega_X - (B(t,x,u) + \right.$$

$$+A_X(t,x,s) - H(t,x))\,\omega] p_m(s)\,ds - \int_{u(t,x)}^{+\infty} [(u-s)\,\omega_t + (A(t,x,u) - A(t,x,s))\omega_x -$$

$$\left. (B(t,x,u) + A_X(t,x,s) - H(t,x))\omega] p_m(s)\,ds \right\} dt\,dx \geqslant 0$$

It follows from (2.7) that

$$\int_{-\infty}^r \Phi(S) p_m(s)\,ds \longrightarrow \Phi(K)\operatorname{sign}_+(r-k),$$

$$\int_r^{+\infty} \Phi(S) p_m(s)\,ds \longrightarrow \Phi(k)(1 - \operatorname{sign}_+(r-k))$$

as m goes to $+\infty$, where $\Phi(s)$ is an arbitrary continuous function. Passing to the limit as $m \longrightarrow +\infty$ one obtains

$$\int\!\!\int_{\mathbb{R}_+\times\mathbb{R}} \left\{ [2\operatorname{sign}_+(u(t,x) - k) - 1][(u-k)\omega_t + (A(t,x,u) - A(t,x,k))\,\omega_x - \right.$$

$$\left. (B(t,x,u) + A_x(t,x,k) - H(t,x))\omega] \right\} dt\,dx \geqslant 0$$

Let us substitute into (2.3) at first $s = \sup\, u(t,x)$ and then $s = \inf\, u(t,x)$. It follows from the two arising inequalities that $u(t,x)$ satisfies (2.1) in the sense of distributions. Using this fact one gets the desired result.

Theorem 2.1

Suppose $|A(t,x,u)| \leqslant a_0(t)\,a(u)$, where $a_0(t)$, $a(u)$ are continuous functions. Suppose $h(t,x)$, $g(t,x)$ are measurable functions bounded at bounded t. Suppose $w(t,x)$ is the g.s. of the Cauchy problem for the equation $Lw = h(t,x)$ in $\mathbb{R}_+ \times \mathbb{R}$ with initial data $w(0,x) = w_0(x) \in L^\infty(\mathbb{R})$, and $v(t,x)$ is the g.s. of the Cauchy problem for the equation $Lv = g(t,x)$ in $\mathbb{R}_+ \times \mathbb{R}$ with initial data $v(0,x) = v_0(x) \in L^\infty(\mathbb{R})$. Suppose $w_0(x) \leqslant v_0(x)$ almost everywhere in $\mathbb{R}_+ \times \mathbb{R}$. Then $w(t,x) \leqslant v(t,x)$ almost everywhere in $\mathbb{R}_+ \times \mathbb{R}$.

The proof of this theorem is similar to [11]. One must use inequality (2.6) instead of (2.3) and the monotonicity $B(t,x,u)$ in u.

In particular the uniqueness theorem follows from theorem 2.1.

§3. FINITE EXTINCTION TIME

Below in §§ 3,4 $u(t, x)$ will denotes the g.s. of the problem (1.2), (2.2) in $\mathbb{R}_+ \times \mathbb{R}$.

Suppose we have $n < 1$ in (1.2). Let us carry out some auxillary constructions. One can seek a family of particular solutions for (1.2) in the form $v(t, x) = (1+t)^\alpha \lambda(\xi)$, where $\xi \equiv (x+a)(1+t)^{-\beta}$, $\alpha = (q+1)/(1-n)$, $\beta = p+1+\alpha(m-1)$, a =const, $\lambda(\xi)$ is a piecewise continuosly differentiable function which will be defined later. Substituting $v(t, x)$ into (1.2) we get for $\lambda(\xi)$:

$$\frac{d\lambda}{d\xi} = \frac{\alpha\lambda + A_2\lambda^n}{\beta\xi - mA_1\lambda^{m-1}} \tag{3.1}$$

Or in another form

$$\frac{d\xi}{d\lambda} = \frac{\beta}{\alpha\lambda + A_2\lambda^n}\xi - \frac{mA_1\lambda^{m-1}}{\alpha\lambda + A_2\lambda^n} \tag{3.2}$$

Suppose $\alpha \neq 0$. Let us denote $M_0 \equiv (A_2/|\alpha|)^{1/(1-n)}$; $k \equiv A_1 M_0^{m-1}$;

$$\psi(\lambda) \equiv (mA_1/A_2)\int_0^\lambda s^{m-n-1}\Big(1+(s/M_0)^{1-n}\operatorname{sign}\alpha\Big)^{-1-\beta/(q+1)} ds;$$

$$\varphi(\lambda, \eta_1) \equiv (mA_1/A_2)\int_{\eta_1}^\lambda s^{m-n-1}\Big((s/M_0)^{1-n} - 1\Big)^{-1-\beta/(q+1)} ds,$$

$$\eta_1 = \text{const} \geqslant M_0, \lambda \geqslant M_0;$$

$$K_1(M) \equiv [m(m-1)/|\beta|]\int_0^M s^{m-2}\Big(1+(s/M_0)^{1-n}\operatorname{sign}\alpha\Big)^{-\beta/(q+1)} ds$$

It is easy to see that functions

$$\zeta_1(\lambda) \equiv (C - \psi(\lambda))(1+(\lambda/M_0)^{1-n}\operatorname{sign}\alpha)^{\beta/(q+1)}$$

for $\alpha > 0$ or $\alpha < 0$, $\lambda \leqslant M_0$ and

$$\zeta_2(\lambda, \eta_1) \equiv \varphi(\lambda, \eta_1)((\lambda/M_0)^{1-n} - 1)^{\beta/(q+1)}$$

for $\alpha < 0$, $\lambda \geqslant M_0$ are solutions of the equation (3.2) for any constants C and $\eta_1 \geqslant M_0$. Suppose that the functions $\xi = \zeta_i(\lambda)$ $(i = 1, 2)$ increase on some set G. Then there exist the inverse functions $\lambda = \zeta_i^{-1}(\xi)$ on $\zeta_i(G)$ $(i = 1, 2)$ satisfying (3.1). Now let us consider a monotonically increasing solution $\lambda = \lambda(\xi)$ of (3.1) containing zero in its range.

Suppose $\lambda(\xi_0) = 0$, $\lambda(\xi_1) = N$ =const> 0. Let us define $f(\xi)$ in the following way: $f(\xi) = 0$ for $\xi \leqslant \xi_0$; $f(\xi) = \lambda(\xi)$ for $\xi_0 \leqslant \xi \leqslant \xi_1$; $f(\xi) = N$ for $\xi \geqslant \xi_1$. Let us denote by $y(t)$ the solution of the Cauchy problem

$$\dot{y} = A_1(1+t)^{\beta-1} f^{m-1}((y+a)(1+t)^{-\beta}),\ y(0) = y_0, \tag{3.3}$$

where a =const, $y_0+a > \xi_0$. Let us denote by (τ, η) the intersection point of the lines $x = y(t)$ and $x = z_1(t) \equiv \xi_1(1+t)^\beta - a$; suppose $z_0(t) \equiv \xi_0(1+t)^\beta - a$.

Suppose $\rho_1(t, x; \lambda, N, \xi_0, \xi_1, y_0, a) = (1+t)^\alpha f(\xi)$ for $x \leqslant y(t)$ and $\rho_1(t, x; \lambda, N, \xi_0, \xi_1, y_0, a) = 0$ for $x > y(t)$. It is easy to see that at the discontinuity line $x = y(t)$ of the function $\rho_1(t, x; \lambda, N, \xi_0, \xi_1, y_0, a)$ relations (2.4), (2.5) hold with $A(t, x, u) = A_1(1+t)^p u^m$. The function $\rho_1(t, x; \lambda, N, \xi_0, \xi_1, y_0, a)$ will be used below as a comparison function.

Now let us turn to the investigation of the g.s. $u(t, x)$ to problem (1.2), (2.2).

Theorem 3.1

Suppose $0 \leqslant u_0 \leqslant M$. Then the finite extinction time property occurs for the problem (1.2), (2.2) if one of the following conditions holds:

$$1)\quad q \geqslant -1; \qquad 2)\quad q < -1,\ M < M_0.$$

Proof.

Let us compare $u(t, x)$ with the function $\sigma(t, x)$:

$$\sigma(t, x) = \left\{[M^{1-n} + A_2(1-n)(1-(1+t)^{q+1})/(q+1)]_+\right\}^{1/(1-n)}$$

for $q \neq -1$;

$$\sigma(t, x) = \left\{[M^{1-n} - A_2(1-n)\ln(1+t)]_+\right\}^{1/(1-n)}$$

for $q = -1$.

It is easy to see that $\sigma(t, x)$ is the g.s. of the problem (1.2), (2.2) with $u_0(x) = M$. So we have the desired result by virtue of Theorem 2.1.

Theorem 3.2

Suppose $0 \leqslant u_0(x) \leqslant M_0$, supp $u_0(x) \subset [x_0, x_1]$, $u_0(x) = M_0$ for $x \in [x_2, x_3] \subset [x_0, x_1]$, $\alpha < 0$, $\beta \neq 0$. Then the following statements hold true

1) If $\beta > 0$ then the finite extinction time property occurs in the problem (1.2), (2.2).
2) If $\beta < 0$, $\beta/(q+1) < 1$, $x_1 - x_0 < k(m-1)/|\beta|$ then the finite extinction time property occurs.
3) If $\beta < 0$, $\beta/(q+1) < 1$, $x_3 - x_2 > K_1(M_0) + k/\beta$ then the finite extinction time does not occur and $u(t, x) > 0$ for

$$x_2 + K_1(M_0)[1-(1+t)^\beta] < x < x_3 + k[1-(1+t)^\beta]/|\beta|;$$

but localization occurs and $u(t,x) = 0$ for

$$x \leqslant x_0 + mk/|\beta| - K_1(M_0)(1+t)^\beta,$$

$$x > x_* + k[1-(1+t)^\beta]/|\beta|, \qquad x_* = \max\{x_1, x_0 + (m-1)k/|\beta|\}.$$

Proof.

1) Suppose $\beta > 0$. Then $\psi(M_0) < +\infty$. Let us choose $C = \psi(M_0)$ and compare $u(t,x)$ with

$$\rho_1(t,x;\zeta_1^{-1}, M_0, C, \zeta_1(M_0), x_1, \zeta_1(M_0) - x_0).$$

In the points where

$$\rho_1(t,x;\zeta_1^{-1}, M_0, C, \zeta_1(M_0), x_1, \zeta_1(M_0) - x_0)$$

is smooth the inequality $L_1\rho_1 \geqslant 0$ holds while at the line of discontinuity $x = y(t)$ conditions (2.4), (2.5) are valid. Besides

$$u_0(x) \leqslant \rho_1(0,x;\zeta_1^{-1}, M_0, C, \zeta_1(M_0), x_1, \zeta_1(M_0) - x_0),$$

so

$$u(t,x) \leqslant \rho_1(t,x;\zeta_1^{-1}, M_0, C, \zeta_1(M_0), x_1, \zeta_1(M_0) - x_0)$$

in $\mathbb{R}_+ \times \mathbb{R}$ by virtue of Theorem 2.1.

From the relations (3.3) with $y_0 = x_1$, $\xi \geqslant \zeta_1(M_0)$ it follows that

$$\dot{y} = k(1+t)^{\beta-1}, \qquad y(0) = x_1.$$

As $\zeta_1(M_0) = mk/\beta$, we have the expressions for τ and η

$$\begin{aligned} 1+\tau &= [1 + (x_1 - x_0)[(m-1)k/\beta]^{-1}]^{1/\beta} \\ \eta &= mk[(1+\tau)^\beta - 1]/\beta + x_0 \end{aligned} \tag{3.4}$$

At the point (τ,η) we have $\dot{y} = k(1+\tau)^{\beta-1}$. Since $m > 0$ so $f < M_0$ for $t > \tau$. By virtue of Theorem 3.1 one gets the desired result.

2) Suppose $\beta < 0$, $\beta/(q+1) < 1$. Let us take $C = -K_1(M_0)$ and rewrite $\psi(\lambda)$ integrating by parts

$$\psi(\lambda) = \frac{m(m-1)A_1}{\beta} \int_0^\lambda s^{m-2}\left(1 - (s/M_0)^{1-n}\right)^{-\beta/(q+1)} ds -$$

$$-\frac{mA_1}{\beta}\lambda^{m-1}\left(1 - (\lambda/M_0)^{1-n}\right)^{-\beta/(q+1)}$$

Now, compare $u(t,x)$ with the function

$$\rho_1(t,x;\zeta_1^{-1},M_0,C,\zeta_1(M_0),x_1,\zeta_1(M_0)-x_0)$$

by analogy with the part 1). When $x_1 - x_0 < k(m-1)/|\beta|$ holds formulas (3.4) are true, so the result follows.

3) Suppose $\beta < 0$, $\beta/(q+1) < 1$, $x_3 - x_2 > K_1(M_0) + k/\beta$. Firstly we shall prove the absence of extinction. Let us take $C = -K_1(M_0)$ and obtain a lower estimate for $u(t,x)$. In the points where

$$\rho_1(t,x;\zeta_1^{-1},M_0,C,\zeta_1(M_0),x_3,C-x_2)$$

is smooth the inequality $L_1\rho_1 \leqslant 0$ holds and at the line of discontinuity $x = y(t)$ conditions (2.4), (2.5) are valid. Besides

$$u_0(x) \geqslant \rho_1(0,x;\zeta_1^{-1},M_0,C,\zeta_1(M_0),x_3,C-x_2),$$

$$\text{so} \quad u(t,x) \geqslant \rho(t,x;\zeta_1^{-1},M_0,C,\zeta_1(M_0),x_3,C-x_2),$$

in $\mathbb{R}_+ \times \mathbb{R}$ by virtue of Theorem 2.1.

Under the above hypotheses $\dot{y} = k(1+t)^{\beta-1}$, $y(0) = x_3$. Further

$$y(t) - z_1(t) \geqslant x_3 + k[1-(1+t)^\beta]/|\beta| + mk(1+t)^\beta/|\beta| - x_2 - K_1(M_0) \geqslant$$
$$\geqslant x_3 - x_2 - (K_1(M_0) - k/|\beta|) > 0$$

under our conditions. Hence there is no intersection of lines $x = y(t)$ and $x = z_1(t)$, so there is no extinction. As a consequence $u(t,x) > 0$ for $z_0(t) < x < x_3 + k[1-(1+t)^\beta]/|\beta|$.

The assertion on the localization follows from the estimate

$$u(t,x) \leqslant \rho_1(t,x;\zeta_1^{-1},M_0,-K(M_0),\zeta_1(M_0),x_*,\zeta_1(M_0)-x_0)$$

established in section 2) and the support structure of the majorant.

Theorem 3.3

Suppose $0 \leqslant u_0(x) \leqslant M$, $M > M_0$, $\operatorname{supp} u_0(x) \subset [x_0,x_1]$, $u_0(x) = M$ for $x \in [x_2,x_3] \subset [x_0,x_1]$, $\alpha < 0$, $\beta > 0$, $m+n \geqslant 2$. Then the following statements hold true.

1) If $\beta > |\alpha|(m+n-1)$ then the finite extinction time property occurs for the problem (1.2), (2.2).
2) If $\beta < |\alpha|$, $x_3 - x_2 > A_1M^{m-1}/\beta - \psi(M_0)$,

$$\left(\frac{M}{M_0}\right)^{1-n} \geqslant \left[1 - \left(\frac{\beta+|\alpha|(m-1)}{m|\alpha|}\right)^{(1-n)/(m-1)}\right]^{-1} \tag{3.5}$$

then localization does not occur and $u(t,x) > 0$ for

$$x_2 + \psi(M_0)[(1+t)^\beta - 1] < x < x_3 + A_1M^{m-1}[(1+t)^\beta - 1]/\beta \tag{3.6}$$

Proof.

1) Let us introduce the function $\zeta(\xi) : \zeta(\xi) \equiv \zeta_1^{-1}(\xi)$ for $\xi \leqslant mk/\beta$ and $\zeta(\xi) \equiv \zeta_2^{-1}(\xi)$ for $\xi \geqslant mk/\beta$. Let us compare $u(t,x)$ with the function

$$\rho_1(t,x;\zeta,+\infty,\psi(M_0),+\infty,x_1,\zeta_2(M,M_0)-x_0), \quad \eta_1 = M_0.$$

From Theorem 2.1 it is easy to show that

$$u(t,x) \leqslant \rho_1(t,x;\zeta,+\infty,\psi(M_0),+\infty,x_1,\zeta_2(M,M_0)-x_0).$$

Further, integrating by parts in the expression for $\zeta_2(f)$ one finds

$$\xi = \zeta_2(f,M_0) = mA_1\beta^{-1}\{f^{m-1} - (m-1)[(f/M_0)^{1-n} - 1]^{\beta/(q+1)} \times \int_{M_0}^{f} s^{m-2}[(s/M_0)^{1-n} - 1]^{-\beta/(q+1)}\,ds\}.$$

Hence

$\xi \geqslant mA_1\beta^{-1}\{f^{m-1} - (m-1)|q+1|^{-1}[(f/M_0)^{1-n} - 1]^{\beta/(q+1)} f m + n - 2 \times \int_{M_0}^{f} [(s/M_0)^{1-n} - 1]^{-\beta/(q+1)} d(|\alpha|s^{1-n}) = mA_1\beta^{-1}[f^{m-1} - (m-1)f^{m+n-2} \times (|\alpha|f^{1-n} - A_2)/(|\alpha|(1-n)+\beta)] \geqslant mA_1\beta^{-1}[1-|\alpha|(m-1)/(\beta+|\alpha|(1-n))]f^{m-1}.$

From this estimate and (3.3) it follows that

$\dot{y} \leqslant \beta(m\gamma)^{-1}(y+a)(1+t)^{-1}, y(0) = x_1, \gamma \equiv 1-|\alpha|(m-1)/(\beta+|\alpha|(1-n)).$

Consequently we have the estimates for coordinates (τ_0,η_0) of the point of intersection for curves $x = y(t)$ and $x = mk(1+t)^{\beta}/\beta - a$:

$1+\tau_0 \leqslant [\beta(x_1-x_0+\zeta_2(M,M_0))(mk)^{-1}]^{mk/(\beta(m\gamma-1))}, \ \eta_0 = mk(1+\tau_0)^{\beta}/\beta - a.$

Because of $\beta > |\alpha|(m+n-1)$ we have $m\gamma > 1$. So the point (τ_0,η_0) really exists. For $t \geqslant \tau_0$ at the line $x = y(t)$ one has $f \leqslant M_0$ and consequently Theorem 3.2 can be applied.

2) Suppose inequalities (3.5) hold. Let us take $\eta_1 = M_0$. It is easy to see that

$$u(t,x) \geqslant \rho_1(t,x;\zeta,M,\psi(M_0),\zeta_2(M,M_0),x_3,\psi(M_0)-x_2)$$

in $\mathbb{R}_+ \times \mathbb{R}$. From de relations (3.3) one infers $\dot{y} = A_1M^{m-1}(1+t)^{\beta-1}$, $y(0) = x_3$. Further,

$$\zeta_2(M,M_0) \leqslant mA_1\beta^{-1}\{M^{m-1} - (m-1)(M^{1-n} - M_0^{1-n})^{\beta/(q+1)} \times \int_{M_0}^{M} s^{-n}(s^{1-n} - M_0^{1-n})^{(m+n-2)/(1-n)-\beta/(q+1)} ds\} = mA_1\beta^{-1}[M^{m-1} - (M^{1-n} - M_0^{1-n})^{(m-1)/(1-n)}|\alpha|(m-1)/(\beta+|\alpha|(m-1))] \leqslant A_1M^{m-1}\beta^{-1}$$

by virtue of (3.5). It means that there is no intersection for the curves $x = y(t)$ and $x = z_1(t) = \zeta_2(M,M_0)(1+t)^{\beta} - a$. Consequently

$$\rho_1(t,x;\zeta,M,\psi(M_0),\zeta_2(M,M_0),x_3,\psi(M_0)-x_2) > 0$$

for t,x from (3.6). This ends the proof.

Theorem 3.4

Suppose $0 \leqslant u_0(x) \leqslant M$, $M > M_0$, $\operatorname{supp} u_0(x) \subset [x_0, x_1]$, $u_0(x) = M$ for $x \in [x_2, x_3] \subset [x_0, x_1]$, $\alpha < 0$, $\beta < 0$, $|\beta| < |\alpha|(1-n)$. Then the following statements hold true.

1) If $p < -1$, then localization in the problem (1.2), (2.2) occurs and $u(t,x) = 0$ for

$$x \leqslant x_0 - K_1(M_0)(1+t)^{\beta},\ x \geqslant x_0 + [(x_1 - x_0)^{(p+1)/\beta} +$$

$$A_1(\zeta_2^{-1}(x_1 - x_0, M))^{m-1}(x_1 - x_0)^{-\alpha(m-1)/\beta}(1 - (1+t)^{p+1})/|\beta|]^{\beta/(p+1)};$$

if in addition $x_3 - x_2 > K_1(M_0)$ then there is no finite extinction time in the problem (1.2), (2.2) and $u(t,x) > 0$ for

$$x_3 - (x_3 - x_2)(1+t)^{\beta} < x < x_3 + [k\,\varphi(+\infty, M)^{-\alpha(m-1)/\beta}(1-(1+t)^{p+1})/|\beta|]^{\beta/(p+1)}.$$

2) If $p > -1$, $x_3 - x_2 > K_1(M_0)$ there is no localization in the problem (1.2), (2.2) and there exists such $M_2 > M_0$ that $u(t,x) > 0$ for

$$x_3 - \zeta_2(M, M_2) - (x_3 - x_2 - \zeta_2(M, M_2))(1+t)^{\beta} < x < x_3 - \zeta_2(M, M_2)[1 - (1+t)^{k_1}],$$

$$k_1 \equiv (p+1)m^{-1}[1 - (M_0/M_2)^{1-n}]^{1+\beta/(q+1)}.$$

3) If $p = -1$, $x_3 - x_2 > K_1(M_0)$ there is no localization in the problem (1.2), (2.2) and there exists such $M_3 > M_0$ that $u(t,x) > 0$ for

$$x_3 - \xi_2(M, M_3) - (x_3 - x_2 - \xi_2(M, M_3))(1+t)^{\beta} < x < z(t),$$

where $z(t)$ satisfies the inequality

$$\left(\frac{z(t)+a}{x_3+a}\right)^{|\beta|/\delta_1} \frac{\delta_1/|\beta| + \ln[(x_3+a)/\delta_2]}{\delta_1/|\beta| + \ln[(z(t)+a)(1+t)^{|\beta|}/\delta_2]} \geqslant 1, \qquad (3.7)$$

where

$$\delta_1 = |\alpha|(m-1)m^{-1}[1 - (M_0/M_3)^{1-n}]^{(m-n)/(1-n)},$$

$$\delta_2 = \varphi(M, M_3)[(M_3/M_0)^{1-n} - 1]^{(m-1)/(1-n)},\ a = \zeta_2(M, M_3) - x_3.$$

Proof

1) Suppose $p < -1$. With the help of Theorem 2.1 it is easy to show that

$$u(t,x) \leqslant \rho_1(t, x; \zeta, +\infty, -K_1(M_0), +\infty, x_1, -x_0),\ \eta_1 = M.$$

From (3.3) with $y_0 = x_1$ it follows that $\dot{y} > 0$ so $y(t) > x_1$ for $t > 0$. Thus $f(x+a) > f(x_1+a) = \zeta_2^{-1}(x_1 - x_0, M)$ for $t > 0$, $x > x_1$. Then

$$\xi = \zeta_2(f, M) \geqslant f^{\beta/\alpha}[M_0^{n-1} - (\zeta_2^{-1}(x_1 - x_0, M))^{n-1}]^{\beta/(q+1)}(x_1 - x_0)\times$$

$$[(\xi_2^{-1}(x_1 - x_0, M)/M_0)^{1-n} - 1]^{-\beta/(q+1)} = (x_1 - x_0)(f/\zeta_2^{-1}(x_1 - x_0, M))^{\beta/\alpha}.$$

From the relations (3.3) with $y_0 = x_1$ it follows that

$$\dot{y} \leqslant A_1(\xi_2^{-1}(x_1 - x_0, M))^{m-1}(x_1 - x_0)^{-\alpha(m-1)/\beta}(1+t)^p(y+a)^{\alpha(m-1)/\beta}, y(0) = x_1$$

Hence

$$y(t) + a \leqslant [(x_1 - x_0)^{(p+1)/\beta} - A_1(\zeta_2^{-1}(x_1 - x_0, M))^{m-1}(x_1 - x_0)^{-\alpha(m-1)/\beta}\times$$

$$[1 - (1+t)^{p+1}]/\beta]^{\beta/(p+1)}.$$

But

$$\rho_1(t, x; \zeta, +\infty, -K_1(M_0), +\infty, x_1, -x_0) \equiv 0$$

for $\xi \leqslant -K_1(M_0)$ so localization occurs.

Now let us prove the second part of statement 1). One can easy establish the inequality

$$u(t, x) \geqslant \rho_1(t, x; \zeta, x_2 - x_3, +\infty, x_3, -x_3), \quad \eta_1 = M.$$

By analogy with the above arguments one gets formulas

$$\xi = \zeta_2(f, M) \leqslant (f/M_0)^{\beta/\alpha}\ \varphi(+\infty, M),$$

$$\dot{y} \geqslant k\ \varphi(+\infty, M)^{-\alpha(m-1)/\beta}(1+t)^p(y+a)^{\alpha(m-1)/\beta}, \quad y(0) = x_3$$

Hence

$$y(t) \geqslant [k\ \varphi(+\infty, M)^{-\alpha(m-1)/\beta}(1 - (1+t)^{p+1})/|\beta|]^{\beta/(p+1)}$$

and making use of the support structure of

$$\rho_1(t, x; \zeta, +\infty, x_2 - x_3, +\infty, x_3, -x_3)$$

one ends the proof of the second part 1).

2) Suppose $p > -1$. Let us choose M_2 such that $\zeta_2(M, M_2) < x_3 - x_2$. With the help of Theorem 2.1 we obtain that

$$u(t, x) \geqslant \rho_1(t, x; \zeta, +\infty, \zeta_2(M, M_2) - x_3 + x_2, +\infty, x_3, \zeta_2(M, M_2) - x_3).$$

Further,

$$\xi = \zeta_2(f, M_2) \leqslant (f/M_0)^{\beta/\alpha}\varphi(f, M_2) = mA_1A_2^{-1}M_0^{1-n}f^{\beta/\alpha}\times$$

$$\int_{M_2}^{f} s^{m-2-\beta/\alpha}[1-(M_0/s)^{1-n}]^{-1-\beta/(q+1)}ds \leqslant mA_1A_2^{-1}M_0^{1-n}f^{\beta/\alpha}\times$$

$$[1-(M_0/M_2)^{1-n}]^{-1-\beta/(q+1)}[f^{m-1-\beta/\alpha}-M_2^{m-1-\beta/\alpha}]/(m-1-\beta/\alpha) \leqslant$$

$$mA_1f^{m-1}(p+1)^{-1}[1-(M_0/M_2)^{1-n}]^{-1-\beta/(q+1)}.$$

So from the relations (3.3) with $y_0 = x_3$ it follows that

$$\dot{y} \geqslant k_1(y+a)/(1+t), \quad y(0) = x_3. \tag{3.8}$$

Hence, $y(t) \geqslant x_3 - \zeta_2(M, M_2)(1-(1+t)^{k_1})$. Because of the minorant support structure one infers the statement 2).

Suppose $p = -1$. In comparison with the proof of statement 2) changes will be presented only in the estimate for ξ and formulas (3.8). Suppose $M_3 = M_2$ for $|\alpha|(m-1) = |\beta|$. Thus

$$\xi = \zeta_2(f, M_3) \leqslant (f/M_0)^{m-1}\varphi(f, M_3) = mA_1A_2^{-1}M_0^{1-n}f^{m-1}\times$$

$$\int_{M_3}^{f} s^{-1}[1-(M_0/s)^{1-n}]^{-1-(m-1)/(1-n)}ds \leqslant mA_1A_2^{-1}M_0^{1-n}\times$$

$$[1-(M_0/M_3)^{1-n}]^{-1-(m-1)/(1-n)}f^{m-1}\ln(f/M_3).$$

On the other hand

$$\xi = \zeta_2(f, M_3) \geqslant (f/M_0)^{m-1}[1-(M_0/M_3)^{1-n}]^{(m-1)/(1-n)}\varphi(M, M_3).$$

So

$$A_1f^{m-1} \geqslant |\alpha|m^{-1}[1-(M_0/M_3)^{1-n}]^{(m-n)/(1-n)}\xi\ln^{-1}(f/M_3) \geqslant \delta_1\xi\ln^{-1}(\xi/\delta_2).$$

Instead of (3.8) we have

$$\dot{y} \geqslant \delta_1(y+a)(1+t)^{-1}\ln^{-1}[(y+a)\delta_2^{-1}(1+t)^{-\beta}], \quad y(0) = x_3.$$

Let us denote $\eta(t) \equiv (y(t)+a)(1+t)^{-\beta}/\delta_2$; then

$$\dot{\eta} = \dot{y}\delta_2^{-1}(1+t)^{-\beta} - \beta\eta(1+t)^{-1} \geqslant \delta_1\eta(1+t)^{-1}\ln^{-1}\eta - \beta\eta(1+t)^{-1} =$$

$$\eta(\delta_1\ln^{-1}\eta - \beta)(1+t)^{-1}, \quad \eta(0) = \zeta_2(M, M_3)/\delta_2.$$

Further,

$$\int_{\eta(0)}^{\eta(t)} \frac{\ln s}{s(\delta_1 - \beta\ln s)}ds \geqslant \ln(1+t),$$

$$\frac{\eta(t)}{\eta(0)}\left[\frac{\delta_1/|\beta| + \ln\eta(0)}{\delta_1/|\beta| + \ln\eta(t)}\right]^{\delta_1/|\beta|} \geqslant (1+t)^{|\beta|}$$

Hence, $y(t) > z(t)$, where $z(t)$ satisfies (3.7).

§4. LOCALIZATION

Suposse $n > 1$ in (1.2). Let us introduce the notation:

$$\zeta_3(\lambda, M) \equiv [mA_1\beta^{-1}M^{m-1-\beta/\alpha}(\alpha + A_2M^{n-1})^{-\beta/(q+1)} +$$
$$mA_1 \int_\lambda^M s^{m-2-\beta/\alpha}(\alpha + A_2 s^{n-1})^{-1-\beta/(q+1)} ds]\,\lambda^{\beta/\alpha}(\alpha + A_2\lambda^{n-1})^{\beta/(q+1)}$$

where α, β are the same as at the beginning of §3.

Theorem 4.1

Suppose the following conditions hold:

$$u_0 \in C([x_0, x_1]), \quad u_0(x) \geqslant 0, \quad u_0(x) \not\equiv 0;$$

there exists such $\varepsilon > 0$ and $\eta \in (0,1)$ that $u_0(x) > \eta$ for $x \in [x_2, x_3] \subset [x_0, x_1]$;

$$\int_\varepsilon^{+\infty} (1+t)^p \left[\int_0^t (1+s)^q\, ds\right]^{-(m-1)/(n-1)} dt = +\infty .$$

Then there is no localizaton in the problem (1.2), (2.2).

Proof.

Let us denote:

$$g(t) \equiv \left[1 + (n-1)A_2 \int_0^t (1+s)^q\, ds\right]^{-1/(n-1)};$$

$$\gamma(t) \equiv mA_1 \int_0^t (1+\tau)^p g(\tau)^{m-1}\, d\tau; \qquad \gamma_1(t) \equiv x_2 + \eta^{m-1}\gamma(t).$$

Suppose $\gamma_2(t)$ is the solution of the equation

$$\dot\gamma_2 = A_1\eta^{m-1}(1+t)^p g(t)^{m-1}$$

with the initial data $\gamma_2(0) = x_3$.

If there exists $t_0 > 0$ such that $\gamma_1(t_0) = \gamma_2(t_0)$ then let us introduce $\gamma_3(t)$ as the solution of the equation $\dot\gamma_3 = A_1(1+t)^p(v_1(t))^{m-1}$, where

$$v_1(t) = g(t)[(\gamma_3(t) - x_2)/\gamma(t)]^{1/(m-1)}$$

with the initial data $\gamma_3(t_0) = \gamma_1(t_0)$. Define $\gamma_4(t) = \gamma_1(t)$ for $t \leqslant t_0$, $\gamma_4(t) = \gamma_3(t)$ for $t > t_0$; $\gamma_5(t) = \gamma_2(t)$ for $t \leqslant t_0$, $\gamma_5(t) = \gamma_3(t)$ for $t > t_0$. Define $\rho_2(t,x) = 0$ for $x \leqslant x_2$;

$$\rho_2(t,x) = g(t)[(x - x_2)/\gamma(t)]^{1/(m-1)}$$

for $x_2 \leqslant x < \gamma_4(t)$ and $t > 0$;

$$\rho_2(t,x) = \eta g(t) \text{ for } \gamma_1(t) \leqslant x < \gamma_2(t); \quad \rho_2(t,x) = 0 \text{ for } x > \gamma_5(t).$$

If there does not exist such t_0, then define $\rho_2(t,x) = 0$ for $x \leqslant x_2$;

$$\rho_2(t,x) = g(t)[(x - x_2)/\gamma(t)]^{1/(m-1)}$$

for $x_1 \leqslant x \leqslant \gamma_1(t)$ and $t > 0; \rho_2(t,x) = \eta g(t)$ for $\gamma_1(t) \leqslant x < \gamma_2(t)$; $\rho_2(t,x) = 0$ for $x > \gamma_2(t)$.

For $t \geqslant t_0$ (if t_0 exists) one finds

$$\dot{\gamma}_3 = A_1(1+t)^p g(t)^{m-1}(\gamma_3(t) - x_2)/\gamma(t) = \dot{\gamma}(t)(\gamma_3(t) - x_2)(m\gamma(t))^{-1}.$$

In $[\gamma_3(t) - x_2)/(\gamma_3(t_0) - x_2)] = m^{-1}\ln[\gamma(t)/\gamma(t_0)]$,

$$\gamma_3(t) = x_2 + \eta^{m-1}\gamma(t_0)^{1-1/m}\gamma(t)^{1/m}.$$

Then we conclude $v_1(t) < 1$. It is easy to see that $L_1\rho_2 \leqslant 0$ at the points where ρ_2 is smooth and at the lines of its discontinuity relations (2.4), (2.5) hold.

Now $\gamma_3(t) \longrightarrow +\infty$ as $t \longrightarrow +\infty$, if t_0 exists; $\gamma_1(t) \longrightarrow +\infty$ as $t \longrightarrow +\infty$, if t_0 does not exist. With the help of Theorem 2.1 one ends the proof.

Theorem 4.2

Suppose $0 \leqslant u_0(x) \leqslant M$, $\operatorname{supp} u_0(x) \subset [x_0, x_1]$, $q < -1$, $p < -1$, $\beta > 0$. Then localization in the problem (1.2), (2.2) occurs and $u(t,x) = 0$ for $x \leqslant x_0$ and

$$x > (m(m-1)/\beta)^{(p+1)/\beta}(1 - 1/m)\ [(m-1)A_1M^{m-1} +$$
$$\beta(x_1 - x_0)]^{(p+1)/\beta} + A_1\beta^{-1}k_1(M)(1+t)^{p+1},$$

where

$$k_1(M) \equiv [mA_1\beta^{-1}M^{m-1-\beta/\alpha}]^{-\alpha(m-1)/\beta}.$$

Proof.

Consider the function

$$\rho_1(t,x;\zeta_3^{-1},M,0,mA_1\beta^{-1}M^{m-1},x_1,mA_1\beta^{-1}M^{m-1} - x_0),$$

defined at the beginning of §3. It is easy to see that $L_1\rho_1 \geqslant 0$ at the points where ρ_1 is smooth. If $t \leqslant \tau$ relations (3.3) for discontinuity line $y(t)$ of ρ_1 take the form

$$\dot{y} \leqslant A_1M^{m-1}(1+t)^{\beta-1}, \quad y(0) = x_1. \qquad (4.1)$$

From this one obtains estimates for (τ,η):

$$(1+\tau) \leqslant \left(1+\frac{\beta(x_1-x_0)}{(m-1)A_1M^{m-1}}\right)^{1/\beta}$$
$$\eta + a = mA_1\beta^{-1}M^{m-1}(1+\tau)^{\beta} \tag{4.2}$$

Further, $\varepsilon = \zeta_3(f) \geqslant mA_1\beta^{-1}M^{m-1-\beta/\alpha}f^{\beta/\alpha}$. Hence from (3.3) we have

$$\dot{y} \leqslant A_1(1+t)^{\beta-1}[mA_1\beta^{-1}M^{m-1-\beta/\alpha}]^{-\alpha(m-1)/\beta}(y+a)^{\alpha(m-1)/\beta}(1+t)^{p},\ y(\tau)=\eta.$$

Now

$$(y+a)^{(p+1)/\beta} \geqslant (\eta+a)^{(p+1)/\beta} + A_1\beta^{-1}k_1(M)[(1+t)^{p+1}-(1+\tau)^{p+1}] =$$

$$(mA_1\beta^{-1}M^{m-1})^{(p+1)/\beta}(1-1/m)(1+\tau)^{p+1} + A_1\beta^{-1}k_1(M)(1+t)^{p+1}.$$

Applying estimate (4.2) we get the assertion of the theorem because $\rho_1 = 0$ for $x > y(t)$.

Theorem 4.3

Suppose $0 \leqslant u_0 \leqslant M$, $\operatorname{supp} u_0(x) \subset [x_0, x_1]$, $q < -1$, $\beta < 0$. Then localization in the problem (1.2) (2.2) occurs and $u(t,x) = 0$ for $x \leqslant x_0$ and for

$$x > x^* + A_1M^{m-1}[1-(1+t)^{\beta}]/|\beta|,$$

where

$$x^* > \max\{x_1, x_0 + (m-1)A_1|\beta|^{-1}M^{m-1}\}.$$

Proof.

The g.s. $u(t,x)$ of the problem (1.2), (2.2) vanishes for $x \leqslant x_0$ even when $A_2 = 0$ and so in our case. Further, let us consider the function

$$\rho_1(t,x;\zeta_3^{-1}, M, -\infty, mA_1\beta^{-1}M^{m-1}, x^*, mA_1\beta^{-1}M^{m-1} - x_0).$$

It is easy to see that $L_1\rho_1 \geqslant 0$ at the points where ρ_1 is smooth. If $x \geqslant x^*$ then the point (τ,η) does not exist and relations (3.3) with $y_0 = x^*$ have the form

$$\dot{y} \leqslant A_1M^{m-1}(1+t)^{\beta-1}, \quad y(0) = x^*,$$

$$\text{so} \qquad y \leqslant x^* + A_1\beta^{-1}M^{m-1}[(1+t)^{\beta}-1].$$

This ends the proof, because $\rho_1 = 0$ for $x > y(t)$.

Theorem 4.4

Suppose $0 \leqslant u_0(x) \leqslant M$, $\operatorname{supp} u_0(x) \subset [x_0, x_1]$, $q < -1$, $\beta = 0$. Then localization in the problem (1.2), (2.2) occurs and $u(t,x) = 0$ for $x \leqslant x_0$ as well as for

$$x \geqslant x_1 + \int_0^M \frac{mA_1 s^{m-2}}{\alpha + A_2 s^{n-1}}\, ds \equiv \hat{x}.$$

Proof.

Let us define the function $w(x)$ in the following way: $w(x) = 0$ for $x \geqslant \hat{x}$;

$$x = x_1 + \int_W^M \frac{mA_1 s^{m-2}}{\alpha + A_2 s^{n-1}}\, ds$$

for $x_1 \leqslant x \leqslant \hat{x}$; $w(x) = M$ for $x_0 \leqslant x \leqslant x_1$; $w(x) = (x - x_0 + \varepsilon)M/\varepsilon$ for $x_0 - \varepsilon \leqslant x \leqslant x_0$; $w(x) = 0$ for $x \leqslant x_0 - \varepsilon$; ε =const > 0. Further define $\rho_3(t,x) = (1+t)^\alpha w(x)$. It is easy to see that $L_1\rho_3 \geqslant 0$ at the points where ρ_3 is smooth for any $\varepsilon > 0$. One ends the proof letting ε tend to zero.

Let us define the function $\zeta_4(\lambda)$:

$$\zeta_4(\lambda) = -\int_\lambda^{M_0} mA_1A_2^{-1} s^{m-2-\beta/\alpha}(M_0^{n-1} - s^{n-1})^{-1-\beta/(q+1)} ds \times$$

$$\lambda^{\beta/\alpha}\, (M_0^{n-1} - \lambda^{n-1})^{\beta/(q+1)}$$

for $\lambda \leqslant M_0$ and

$$\zeta_4(\lambda) = -\int_{M_0}^{\lambda} mA_1A_2^{-1}\, s^{m-2-\beta/\alpha}(s^{n-1} - M_0^{n-1})^{-1-\beta/(q+1)} ds \times$$

$$\lambda^{\beta/\alpha}(\lambda^{n-1} - M_0^{n-1})^{\beta/(q+1)}$$

for $\lambda \geqslant M_0$; $\beta < 0$, $\alpha < 0$.

Theorem 4.5

Suppose $0 \leqslant u_0(x) \leqslant M$, $\operatorname{supp} u_0(x) \subset [x_0, x_1]$, $q > -1, \beta < 0$. Then localization in the problem (1.2), (2.2) occurs and $u(t,x) = 0$ for $x \leqslant x_0$ as well as for $x \geqslant x_1 - \zeta_4(M)$.

Proof.

Let us consider the function

$$w(t,x) = (1+t)^\alpha\, \zeta_4^{-1}((x - x_1 + \zeta_4(M))(1+t)^{-\beta}).$$

Let us introduce the function $\rho_4(t,x)$: $\rho_4(t,x) = 0$ for $x \geqslant x_1 - \zeta_4(M)$; $\rho_4(t,x) = w(t,x)$ for $x_1 - \zeta_4(M)[1-(1+t)^\beta] \leqslant x \leqslant x_1 - \zeta_4(M)$; $\rho_4(t,x) = M(1+t)^\alpha$ for $x_0 \leqslant x \leqslant x_1 - \zeta_4(M)[1-(1+t)^\beta]$; $\rho_4(t,x) = (1+t)^\alpha(x - x_0 + \varepsilon)M/\varepsilon$ for $x_0 - \varepsilon \leqslant x \leqslant x_0$; $\rho_4(t,x) = 0$ for $x \leqslant x_0 - \varepsilon$; $\varepsilon =$const > 0. It is easy to see that $L_1\rho_4 \geqslant 0$ at the points where ρ_4 is smooth, $u(0,x) \leqslant \rho_4(0,x)$; so $u(t,x) \leqslant \rho_4(t,x)$ for every $\varepsilon > 0$. Letting ε tend to zero one ends the proof.

Theorem 4.6

Suppose $0 \leqslant u_0(x) \leqslant M$, $\operatorname{supp} u_0(x) \subset [x_0, x_1]$, $\beta < 0$, $q = -1$. Then localization in the problem (1.2), (2.2) occurs and $u(t,x) = 0$ for $x \leqslant x_0$ as well as for

$$x > x^* + A_1|\beta|^{-1}M^{m-1}[1-(1+t)^\beta],$$

where

$$x^* > \max\{x_1, x_0 + (m-1)A_1|\beta|^{-1}M^{m-1}\}.$$

Proof.

Suppose

$$\zeta_5(\lambda, C) = \exp\{-\beta/[A_2(n-1)\lambda^{n-1}]\}\times$$

$$[C - mA_1A_2^{-1}\int_0^\lambda s^{m-n-1}\exp(\beta/[A_2(n-1)s^{n-1}])\,ds],\ C = \text{const} < 0.$$

The fact $u(t,x) = 0$ for $x \leqslant x_0$ is true for g.s. of the problem (1.2), (2.2) even when $A_2 = 0$ and so in our case. Let us choose $C < 0$ with $|C|$ so large that $\zeta_5(M,C) = mA_1\beta^{-1}M^{m-1}$. Let us consider the function

$$\rho_1(t,x;\zeta_5^{-1}, M, -\infty, mA_1\beta^{-1}M^{m-1}, x^*, mA_1\beta^{-1}M^{m-1} - x_0).$$

It is easy to check that $L_1\rho_1 \geqslant 0$ at the points where ρ_1 is smooth and $\rho_1 \geqslant u$ for $t = 0$. If $x \geqslant x^*$ then the point (τ,η) does not exist. Relations (3.3) with $y_0 = x^*$ have the form $\dot{y} \leqslant A_1M^{m-1}(1+t)^{\beta-1}$, $y(0) = x^*$; so

$$y(t) \leqslant x^* + A_1\beta^{-1}M^{m-1}[(1+t)^\beta - 1]$$

The desired assertion follows from this estimate.

Theorem 4.7

Suppose $0 \leqslant u_0 \leqslant M$, $\operatorname{supp} u_0(x) \subset [x_0, x_1]$, $q = p = -1$, $m > n$. Then localization in the problem (1.2), (2.2) occurs and $u(t,x) = 0$ for $x \leqslant x_0$ as well as for

$$x \geqslant x_1 + mA_1M^{m-n}[A_2(m-n)]^{-1} \equiv \hat{x}.$$

Proof.

Let us consider the function $\rho_5(x) : \rho_5(x) = 0$ for $x \geqslant \hat{x}$;

$$\rho_5(x) = [A_2(m-n)(mA_1)^{-1}(x_1 - x) + M^{m-n}]^{1/(m-n)}$$

for $x_1 \leqslant x \leqslant \hat{x}$; $\rho_5(x) = M$ for $x_0 \leqslant x \leqslant x_1$; $\rho_5(x) = (x - x_0 + \varepsilon)M/\varepsilon$ for $x_0 - \varepsilon \leqslant x \leqslant x_0$; $\rho_5(x) = 0$ for $x \leqslant x_0 - \varepsilon$. It is easy to see that $L_1\rho_5 \geqslant 0$ at the points where ρ_5 is smooth for any $\varepsilon > 0$ and $\rho_5(x) \geqslant u_0(x)$. Letting ε tend to zero one gets the desired result.

Remark

It follows from Theorem 4.1 that there is no localization for $q < -1$, $p \geqslant -1$, or $q > -1$, $\beta \geqslant 0$, or $q = -1$, $p > -1$, or $q = p = -1$, $m \leqslant n$. By virtue of Theorems 4.2–4.7 for any other combination of the paremeter values localization occurs.

§5. AN EXAMPLE OF AN EQUATION WITH ESSENTIAL DEPENDENCE ON THE SPACIAL VARIABLE

Let us denote by $u(t,x)$ the g.s. of the problem (1.3), (2.2) under the assumption (1.4).

Let us carry out some auxiliary constructions. We seek a family of particular solutions to (1.3) for $x \geqslant 0$ in the form $v(t,x) = (1+x)^{\alpha}\,\lambda(\xi)$, $\xi \equiv (t+a)(1+x)^{-\beta}$, where $\alpha = (s+1)/(m-n)$, $\beta = 1-\alpha(m-1)$, a =const, $\lambda(\xi)$ is a piecewise continuously differentiable function which will be defined later. By analogy with the beginning of §3 one finds the equation for $\lambda(\xi)$:

$$\frac{d\xi}{d\lambda} = \frac{m\beta\lambda^{m-1}\xi - 1}{\alpha m\lambda^m + A\lambda^n},$$

Hence,

$$\xi = \eta(\lambda) = (|\alpha|m)^{-1}(M_1^{m-n} - \lambda^{m-n})^{\beta/(s+1)}\int_{\lambda}^{M_1}\sigma^{-n}(M_1^{m-n} - \sigma^{m-n})^{-1-\beta/(s+1)}d\sigma =$$

$$\lambda^{1-m}(m\beta)^{-1} - (m-1)(m\beta)^{-1}(M_1^{m-n} - \lambda^{m-n})^{\beta/(s+1)}\int_{\lambda}^{M_1}\sigma^{-m}(M_1^{m-n} - \sigma^{m-n})^{-\beta/(s+1)}d\sigma$$

where

$$M_1 \equiv (A/(|\alpha|m))^{1/(m-n)}.$$

Then $\lambda(\xi) = \eta^{-1}(\xi)$. Let us denote $\xi_0 \equiv \xi(0)$,

$$\xi_1 \equiv \xi(M_1) = (m\beta M_1^{m-1})^{-1}.$$

Theorem 5.1

Suppose $\operatorname{supp} u_0(x) \subset [x_0, x_1]$, $x_0 > (m\beta M_1^{m-1}\xi_0)^{1/\beta} - 1$, $u_0(x) \leqslant M_1(1+x)^{\alpha}$. Then the finite extinction time property occurs in the problem (1.3), (2.2).

Proof.

Let us define the function $g(\xi)$ in the following way: $g(\xi) = 0$ for $\xi \leqslant \xi_0$; $g(\xi) = \lambda(\xi)$ for $\xi_0 \leqslant \xi \leqslant \xi_1$; $g(\xi) = M_1$ for $\xi \geqslant \xi_1$. Let us denote by $y(t)$ the solution of the Cauchy problem

$$\dot{y} = (1+y)^{1-\beta} g^{m-1}((t+a)(y+1)^{-\beta}), \quad y(0) = x_1, \tag{5.1}$$

where $a = (x_0+1)^{\beta}\xi_1$.

Suppose

$$\rho_6(t, x; \lambda, M_1, \xi_0, \xi_1, a) = (1+x)^{\alpha} g(\xi)$$

for $x \leqslant y(t)$ and

$$\rho_6(t, x; \lambda, M_1, \xi_0, \xi_1, x_1, a) = 0$$

for $x > y(t)$. It is easy to check that $L_2\rho_6 \geqslant 0$ at the points where ρ_6 is smooth while at the line of discontinuity $x = y(t)$ relations (2.4), (2.5) are fulfilled with $A(t, x, u) = u^m$. Moreover $u_0 \leqslant \rho_6$ at $t = 0$. By vitue of Theorem 2.1 $u \leqslant \rho_6$ almost everywhere in $\mathbb{R}_+ \times \mathbb{R}$.

Let us find coordinates (t_*, x_*) of the intersection point for the curves $x = y(t)$ and $(1+x)^{\beta} = m\beta M_1^{m-1}(t+a)$, where $g = M_1$. For $t \leqslant t_*$ relations (5.1) take the form

$$\dot{y} = (1+y)^{1-\beta} M_1^{m-1}, \quad y(0) = x_1.$$

Hence

$$t_* = \frac{(1+x_1)^{\beta} - (1+x_0)^{\beta}}{(m-1)\beta M_1^{m-1}}$$
$$(1+x_*)^{\beta} = m\beta M_1^{m-1}(t_*+a) \tag{5.2}$$

Further,

$$\xi \leqslant (m\beta)^{-1}\lambda^{1-m}, \quad \lambda \geqslant (\xi m\beta)^{-1/(m-1)}.$$

For $t \geqslant t_*$ one has

$$\dot{y} \leqslant (1+y)^{1-\beta}(m\beta\xi)^{-1} = (1+y)(t+a)^{-1}(m\beta)^{-1}, \quad y(t_*) = x_*.$$

Hence

$$y(t) \leqslant (m\beta M_1^{m-1})^{1/\beta}(t_*+a)^{(m-1)/(m\beta)}(t+a)^{1/(m\beta)} - 1 \tag{5.3}$$

The curve $x = \omega(t)$, where $g = 0$, will be defined by the equation

$$\omega(t) = \xi_0^{-1/\beta}(t+a)^{1/\beta} - 1.$$

In our case $m > 1$ so lines $x = y(t)$ and $x = \omega(t)$ have the intersection point $(\hat{t}, \hat{x})$. One ends the proof using (5.2) and (5.3).

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