Boundaries of convex sets

POR MANUEL VALDIVIA*
Académico Numerario

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Summary

Let X be a Banach space. Let E and F be closed subspaces of X and X^*, respectively, such that $E^{\perp}$ is an orthogonal complement of F in $F+E^{\perp}$. We show that if there is a quasi-Baire mapping ψ from X to X^* such that $\|\psi(x)\| = 1, < x, \psi(x) >= \|x\|, x \in X$, and $\psi(x)$ is in F whenever x belongs to E, then $E^{\perp}$ is an orthogonal complement of F in X^*. As a consequence, we obtain that if X is a Banach space such that there is a quasi-Baire mapping ψ from X to X^* with $\|\psi(x)\| = 1$, $< x, \psi(x) >= \|x\|$, $x \in X$, then happens to be an Asplund space.

The linear spaces used in this paper are assumed to be defined over the field **R** of real numbers.

The norm in a Banach space X is denoted by $\|.\|$; B (X) is the closed unit ball of X. We write X^* for the Banach space conjugate of X; X^{**} is the conjugate of X^*, and we identify X with a subspaces of X^{**} via the canonical embedding. For a subset A of X, $A^{\perp}$ is the subspace or X^* wich is orthogonal to A. The usual duality between X and X*, and betwen X^* and X^{**}, is represented by <.,.>. Given two closed subspaces X_1 and X_2 of X , we say that X_1 is an orthogonal complement of X_2 in $X_1 + X_2$ whenever $X_1 \cap X_2 = \{0\}$ and the projection of $X_1 + X_2$ onto X_2 along X_1 is of norm one. A space X is said to be Asplund provived every closed separable subspace Y of X has separable dual Y^*.

* Departamento de Análisis Matemático. Facultad de Matemáticas. Burjasot. Valencia.
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For a set A,| A| denotes its cardinal number. The density character of a topological space M is the smallest cardinal number λ for which there is a dense subset B of M with $|B| = \lambda$. We then write dens $M=\lambda$.

Let S and T be two Hausdorff topological spaces. We say that a mapping f from S to T is quasi-Baire if there is a countable set L of continuous functions from S to T such that f is in the closure of L in the topological space T^S.

Let A be a convex closed and bounded subset in the conjugate X^* of a Banach space X and let B be a subset of A. We say, after G Godefroy, [1], that B is a boundary of A if for each x in X there is v in B such that

$$<x,v>= \sup\{<x,u>: u \in A\}.$$

The following result is proved in [1]: a) *Let A be convex closed and bounded subset of the conjugate X* of a Banach space X. Let B denote a boundary of A. Assume that if C is a closed and bounded subset of X and w is an element of X** in the closure of C with respect to the topology of pointwise convergence over B there is a sequence* (x_n) *in C such that w is the limit of* (x_n) *for such topology. Then A is wak-star compact and it is the closed convex hull of B in X*.*

LEMMA . *Let X be a Banach space. Let* A_o *and* B_o *be two infinite subsets of X and X*, respectively, such that* $|A_o| = |B_o|$. *If L is a family of continuous functions from X to X* with* $|L| \leq |A_o|$, *then there are two closed subspaces E and F of X and X*, respectively, satisfying the following conditions:*

(a) *dens* $E \leq |A_o|$, *dens* $F \leq |B_o|$, $A_o \subset E$, $B_o \subset F$.

(b) $E^\perp$ *is an orthogonal complement of F in* $F+E^\perp$.

(c) For each x in E and each g in L, g (x) lies in F.

Proof. For each u in X* and each positive integer n, we select in X an element x(u,n) such that

$$\|x(u,n)\| = 1, <x(u,n)u> \geq \|u\| - \frac{1}{n}.$$

Proceeding inductively, we assume that, for a non-negative integer m, we have already found subsets

$$A_m \subset X, B_m \subset X^*, |A_o| = |A_m| = |B_m|.$$

Let C_m and D_m denote the linear spans over the field of rationals of A_m and B_m, respectively. We then set

$$A_{m+1} := C_m \cup \{x(u,n): u \in D_m, n = 1,2...\},$$
$$B_{m+1} := D_m \cup \{g(x): x \in C_m, g \in L\}.$$

Now, let E and F be the closures of $\bigcup_{m=o}^{\infty} A_m$ and $\bigcup_{m=o}^{\infty} B_m$ in X and X^*, respectively. Clearly, E and F are Banach spaces for which

$$\text{dens } E \leq |A_o|, \ \text{dens } F \leq |B_o|, \ A_o \subset E, \ B_o \subset F.$$

Take v in $E^{\perp}$, w in F and $\varepsilon > o$. We find a positive integer m such that $\frac{1}{m} < \varepsilon$ and there is an element u in B_m with $\|w-u\| < \varepsilon$. Then

$$\begin{aligned}\|w\| &\leq \|w-u\| + \|u\| < \varepsilon + < x(u,m), u > + \frac{1}{m} \\ &\leq 2\varepsilon + < x(u,m), u+v > \\ &\leq 2\varepsilon + |< x(u,m), u-w >| + |< x(u,m), v+w >| \\ &\leq 2\varepsilon + \|u-w\| + \|v+w\| \leq 3\varepsilon + \|v+w\|,\end{aligned}$$

and hence

$$\|w\| \leq \|v+w\|,$$

we thus have that $E^{\perp}$ is an orthogonal complement of F in $F+E^{\perp}$.

Take now x in E and g in L. We may find a sequence (x_m) in X convergent to x and so that x_m is in $A_m, m = 1,2,...$ Then

$$g(x_m) \in F, m = 1,2,...$$

and thereby

$$g(x) = \lim_m g(x_m) \in F. \qquad \text{q.e.d.}$$

THEOREM 1. *Let X be a Banach space. Let M be a convex closed and bounded subset of X*. Let* ψ *be a quasi-Baire map from X to X* such that*

$$\psi(x) \in M, < x, \psi(x) >= \sup\{< x, u >: u \in M\}, x \in X.$$

Then M is weak-star compact and it is coincides with the closed convex hull in X of*

$$\{\psi(x): x \in X\}.$$

Proof. Let us assume that the above stated property is not true. Let P denote the weak-star closure of M in X* and let Q be the closed convex hull of $\{\psi(x): x \in X\}$ in X^*. Take an element u_o in P not in Q. Let L be a countable set of continuous mappings from X to X* such that ψ is in the closure of L respect to the pointwise convergence topology. We get hold of two countably infinite subsets A_o and B_o of X and X*, respectively, with u_o in B_o. The former lemma applies yielding two closed subspaces E and F of X and X*, respectively, with the properties there mentioned. It then comes plain that $\psi(x)$ is in *F* for each *x* in *E*. We identify, in the usual fashion, *E** with $X^*/E^{\perp}$. Let *f* be the canonical mapping from *X** onto $X^*/E^{\perp}$. We have that $A := f(M \cap F)$ is a convex closed and bounded subsets of $X^*/E^{\perp}$. If φ denotes the restriction of ψ to *E*, then

$$\varphi(x) \in A, < x, \varphi(x) >= \sup\{< x, u >: u \in A\}, x \in E.$$

Set

$$B := \{\varphi(x): x \in E\}$$

Clearly, *B* is a boundary of *A*. Take now a convex bounded subset *C* of *E* and an element *w* of *E*** in the closure of *C* respect to the topology of pointwise convergence over *B*. Let $\{y_i : \in I, \geq\}$ be a net in *C* which converges to w in each point of *B*. Let *v* in *E*** be a weak-start cluster point of the former net. Since *F* is separable, we may find a countable set $\{u_n : n = 1, 2, \ldots\}$ dense in F. For each positive integer n, we choose i_n in *I* such that

$$\left|< y_{i_n} - v, u_j >\right| < \frac{1}{n}, j = 1,2,...,n.$$

We put $x_n := y_{i_n}, n = 1,2,...,$ and show that the sequence (x_n) converges to w in every point of B. Being C a bounded subset of E, there is a positive integer r such that $rB(E) \supset C$. Take $\varepsilon > o$ and u in B. We find a positive integer m for which

$$\frac{1}{m} < \frac{1}{2}\varepsilon, \|u_m - u\| < \frac{1}{4r}\varepsilon.$$

If $n \geq m$, we have

$$\frac{1}{2}\varepsilon > \frac{1}{n} > \left|< x_n - v, u_m >\right| \geq \left|< x_n - v, u >\right| - \left|< x_n - v, u_m - u >\right|$$

$$\geq \left|< x_n - v, u >\right| - \|x_n - v\|.\|u_m - u\| \geq \left|< x_n - v, u >\right| - \frac{\varepsilon}{2},$$

hence

$$\left|< x_n - w, u >\right| = \left|< x_n - v, u >\right| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon, m \geq m.$$

The formely mentioned result a) can now be applied to conclude that A is weak-star compact and $f(u_o)$ is in the convex hull in $X^*/E^{\perp}$ of

$$\{\varphi(x) : x \in E\}.$$

Thus, u_o belongs to the closed convex hull in F of $\{\psi(x) : x \in E\}$, a contradiction

q.e.d.

THEOREMA 2. *Let X be a Banach space. Let* ψ *be a quasi-Baire mapping from X to X* such that*

$$\|\psi(x)\| = 1, < x, \psi(x) > = \|x\|, x \in X,$$

Let also E and F be two closed subspaces of X and X, respectively, such that* $E^{\perp}$ *is an orthogonal complement of F in* $F + E^{\perp}$*. If* $\psi(x)$ *is in F for every x in E then* $E^{\perp}$ *is an orthogonal complement of F in* X^*

Proof. We identify, as done before, E^* with $X^*/E^{\perp}$. Let f denote the canonical mapping from X^* onto $X^*/E^{\perp}$. Let φ represent the restriction of $f \circ \psi$ to E. Then

$$\varphi: E \rightarrow X^* / E^{\perp}$$

is quasi-Baire and

$$\|\varphi(x)\| = 1, < x, \varphi(x) > = \|x\|, x \in E,$$

hence, in light of our previous theorem, $B(X^* / E^{\perp})$ is the closed convex hull in $X^* / E^{\perp}$ of

$$\{\varphi(x): x \in E\},$$

Thus, if M denotes the closed convex hull in F of

$$\{\psi(x): x \in E\},$$

it follows that $f(M) = B(X / E^{\perp})$. Besides, $M \subset B(F)$ and $f(B(F))$ is contained in $B(X^* / E^{\perp})$. Therefore, $f(B(F)) = B(X^* / E^{\perp})$, and we have that $E^{\perp}$ is an orthogonal complement of F in X^*.

q.e.d.

Corrollary. *Let X be a Banach space. If there is a quasi-Baire mapping* ψ *from X to X* such that*

$$\|\psi(x)\| = 1, < x, \psi(x) > = \|x\|, x \in X,$$

then X is an Asplund space

Proof. Let Y be a closed separable subspace of X of infinite dimension. We take a countable dense subset A_o of Y. Let B_o be a subset of X^* with $|B_o| = |A_o|$. Let L be a countable family of continuous functions from X to X^* such that ψ belongs to the pointwise closure of L. An application of the initial lemma gives two subspaces E and F of X and X^*, respectively, with the properties there stated. Then $E^{\perp}$ is an orthogonal of F in $F + E^{\perp}$ and ψ (x) is in F for every x of E. The former theorem applies and we have that $E^{\perp}$ is an orthogonal complement of F in X^*. Since F is separable and isometric to E^*, Y^* is also separable and the result follows.

q.e.d.

It is shown in [2] that if φ is an upper semicontinuous map from an Asplund space X on the weak-star compact subsets of X^*, then a selector ψ of φ of the first Baire class from X to X^* exists. By means of this result, we have that if X is an Asplund space, there is a mapping ψ of the first Baire class from X to X^* such that

$$\|\psi(x)\| = 1, < x, \psi(x) >= \|x\|, x \in X.$$

Our previous corollary produces therefore a converse of this result.

REFERENCES

1. GODEFROY,G.: *Boundaries of a convex set and interpolation sets.* Math. Ann. **27**, 173-184 (1987).

2. JAYNE, J.E. and ROGERS, C.A.: *Borel selector for upper semicontinuous set-valued maps*, Acta Math. **155**, 41-49 (1985).