# *Strongly proper optimums and maximal optimization in multiobjective programming*

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## Abstract

The main object of this paper is to give conditions under which a minimal solution of a problem of mathematical multiobjective programming can be transformed in a minimum solution in the usual sense of the order relations. One of the most important reasons for this study is that to measure the sensitivity of a minimum solution is in general easier than to measure the sensitivity of a minimal solution.

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## 1. INTRODUCTION

Along this work some properties of the solutions of a multiobjective program of the following type will be analized:

$$\left.\begin{array}{c}\text{Min}\, f(x) \\ x \in D\end{array}\right\}$$

where $D$ is a subset of a Banach space $X$ and $f$ is a function defined on $D$ taking values in an ordered Banach space. One of the characteristics which makes a distinction between the multiobjective programming and the scalar one, is that the searched solutions are minimal for the given objective and they are not minima in the usual sense (of the order relations). This fact makes hardly difficult to analize the behaviour of that solutions and to introduce satisfactory dual programs whose solutions can measure the primal sensitivity. Thus in several papers like [16] and [17], it can be remarked that if it is assumed that the solutions of the program are minima then it is possible to obtain results about the duality and the sensitivity of the program which are formally analogous to the usual ones for the scalar programming.

In [5] the concept of strongly proper optima is introduced, proving that for these it can be extended the usual results about the saddle points of the

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Lagrangean function and the existence of dual programs (is convex programming) verifying that the dual solutions have a real interpretation in terms of the marginal analysis, obtaining in this way a direct generalization of the known results for the scalar case. The proper optimums are defined in [6] stating that for proper optimus a dual program can be introduced (which in the particular case of the linear multiobjective programming it coincides with the Isermann's one ([8], [9]) and in the case of the scalar convex programming it coincides with the Luemberger's one (see [10])) which permits to measure the primal sensitivity in the context of the convex programming. A similar result can be found in [4] for the differentiable programming. In both cases, it is necessary to add a term, which lies in the kernel of the operator, to the dual solution. These results have a precedent in [3] where it is proved that for the linear programming and the dual stablished in [7] and [8], the primal sensitivity is not measured only by the dual solution, being necessary to add to the dual solution one of its directional derivatives.

As a natural consequence of the results stated in the papers mentionned before, it appears the question of finding conditions for the existence of the operator (which defines the proper (or strongly proper) optimum) and determining the "size" of its kernel. The present work is devoted to these questions giving in the theorem 3 a constructive proof of the existence of that operator (for proper optimums) and the conditions which assure that its kernel is the largest possible one. Also in the theorem 6 it is also given a constructive proof of the existence of that operator for strongly proper optimums.

## 2. PRELIMINARIES AND NOTATIONS

Let us consider two (real) Banach spaces $X$ and $Y$, $Y_+$ a pointed closed convex cone of $Y$, $D$ a subset of $X$, a function $f : D \longrightarrow Y$ and the program

$$\left.\begin{array}{c}\operatorname{Min} f(x)\\ x \in D\end{array}\right\} \quad (1).$$

If $u,\ v \in Y$ we will write $u > v$ (or $v < u$) iff $u - v \in Y_+ - \{0\}$ and $u >> v$ (or $v << u$) iff $u - v \in (Y_+)^0$, $(Y_+)^0$ being the interior of $Y_+$. Also $Y'$ will denote the dual space of $Y$, $Y'_+ = \{y' \in Y' :\ y'(y) \geqslant 0$ for every $y \in Y_+\}$ the dual cone of $Y_+$ (as it is well known, $Y'_+$ is a closed convex cone, non pointed in general, which is pointed if $(Y_+)^0 \neq \emptyset$ or if $Y_+$ separates points of $Y'$) and $\mathcal{Y}'_+ = \{y' \in Y' :\ y'(Y_+ - \{0\}) \subset (0, +\infty)\}$; clearly $\mathcal{Y}'_+ \subset Y'_+ - \{0\}$, in general this content being strict (the same notation will be used for $\mathcal{W}'_{1+}$).

### Definition 1

We say that $x_0 \in D$ is an optimum of the program (1) if and only if the condition $[f(x_0) - f(x)] \notin Y_+ - \{0\}$ holds for every $x \in D$ (i.e., there is no $x \in D$ verifying that $f(x) < f(x_0)$).

Let $W$ be a (real) Banach space, $W_+$ a pointed closed convex cone of $W$ and $T : Y \longrightarrow W$ a positive (i.e., $T(Y_+ - \{0\}) \subset W_+ - \{0\}$) linear and continuos operator such that its image $T(Y)$ is dense in $W$, then an element $x_0 \in D$ is said to be a proper $(T, W)$–optimum (of the program (1)) iff the inequality $Tf(x_0) \leqslant Tf(x)$ holds for every $x \in D$ (we write $Tf$ instead of $T \circ f$). If Moreover, $T$ is a linear homeomorphism, then we say that $x_0$ is an strongly proper $(T, W)$–optimum (of the program (1)). Clearly if $x_0 \in D$ and there exists $W$ and $T$ such that $x_0$ is a proper $(T, W)$–optimum (of the program (1)) then $x_0$ is an optimum of this program.

### Definition 2

Let be $x_0 \in D$, $W$ a Banach space ordered by a pointed closed convex cone $W_+$, such that $W$ is a closed subspace of a Banach lattice which induces on $W$ its initial order (i.e., the order defined by $W_+$) and $T : Y \longrightarrow W$ a positive linear and continous mapping such that $T(Y)$ is dense in $W$. We will say that $(x_0, T, W)$ is maximal (for the program (1)) if the following conditions are verified:

2.1. $x_0$ es a proper $(T, W)$–optimum (of the program (1)).

2.2. For every Banach space $W_1$, ordered by a pointed closed convex cone, verifying similar conditions than $W$ (i.e., it is a closed subspace of a Banach lattice which induces on $W_1$ its original order) and every positive linear and continuous mapping $T : Y \longrightarrow W_1$ such that $T(Y)$ is dense in $W_1$ and $x_0$ is a proper $(T_1, W_1)$–optimum, there exists a linear and continuous mapping $\pi_1 : W \longrightarrow W_1$ such that $T_1 = \pi_1 T$.

## 3. MAXIMAL OPTIMIZATION

### Theorem 3

If $Y$ is a separable Banach space, $x_0 \in D$ and there exists $y'_0 \in \mathcal{Y}'_+$ such that the inequality $y'_0[f(x_0)] \leqslant y'_0[f(x)]$ holds for every $x \in D$, then following the notation of the Definition 2, there exists $W$ and $T$ such that $(x_0, T, W)$ is maximal (for the program (1)). Moreover $W$ and $T$ are unique up to linear homeomorphisms (i.e., if $(x_0, W_1, T_1)$ is also maximal (for the program (1)) then there is a linear homeomorphism $\varnothing : W \longrightarrow W_1$ such that $T_1 = \varnothing T$).

### Proof

Let us prove first the uniqueness. In fact, if $(x_0, T_1, W_1)$ and $(x_0, T_2, W_2)$ are maximal then it follows from the definition 2 the existence of two linear continuous mappings $\pi_1 : W_1 \longrightarrow W_2$ and $\pi_2 : W_2 \longrightarrow W_1$ such that $T_1 = \pi_2 T_2$ and $T_2 = \pi_1 T_1$. Therefore,

$$T_1(y) = \pi_2 T_2 (y) = \pi_2 \pi_2 T_1 (y)$$

for every $y \in Y$, from where it is immediately deduced (since $T_1(Y)$ and $T_2(Y)$ are dense subsets of $W_1$ and $W_2$ respectively) that $\pi_1$ and $\pi_2$ are linear homeomorphisms and $\pi_i = \pi_j^{-1}$ for $i,j \in \{1,2\}$.

Let us prove now the existence. If

$$M = \{y' \in \mathcal{Y}'_+ : \|y'\| \leqslant 1 \text{ and } y'[f(x_0)] \leqslant y'[f(x)] \text{ for every } x \in D\}$$

then $\frac{y_0'}{\|y_0'\|} \in M \neq \emptyset$ and $M \subset B' = \{y' \in Y' : \|y'\| \leqslant 1\}$. It follows from the Alaoglu–Bourbaki theorem that B′ endowed with the weak* topology, is a compact topological space and so if $A'$ denotes the closure of $M$ in $(B', \sigma^*)$ we have that $(A', \omega^*)$ is a compact topological space. Let us consider $A'$ endowed with the weak* topology and the Banach lattice $\mathcal{C}(A')$ of the weak* continuous real functions defined on $A'$ with the supremum norm and the obvious order, and define $T : Y \longrightarrow \mathcal{C}(A')$ such that $T(y)(y') = y'(y)$ for every $y \in Y$ and every $y' \in A'$. Clearly $T$ is positive linear and continuous (with $\|T\| \leqslant 1$) since

$$\|T(y)\| = \sup\{|y'(y)| : y' \in A,\} \leqslant \sup\{|y'(y)| : y' \in B'\} = \|y\|.$$

Let be $W = \overline{T(Y)}$ and $W_+ = \mathcal{C}(A')_+ \cap W$, then the Banach space $W$ endowed with the supremum norm is a closed subspace of the Banach lattice $\mathcal{C}(A')$ and $W_+$ is a pointed closed convex cone (since $\mathcal{C}(A')_+$ is it also). Let us see that $T : Y \longrightarrow W$ is a positive operator, in fact, clearly $T(y)(y') \geqslant 0$ for every $y' \in M$ and every $y \in Y_+$ and therefore $T(y)(y') \geqslant 0$ for every $y' \in A'$ because $T(y) : A' \longrightarrow \mathbb{R}$ is continuous and $M$ is weak*–dense in $A'$, moreover $T(y)(y') > 0$ for every $y \in Y_+ - \{0\}$ and every $y' \in M$ and so $T(y) \neq 0$ for every $y \in Y_+ - \{0\}$.

Also $x_0$ is a proper $(T, W)$–optimum (for the program (1)). Since the inequality $y'(f(x_0)) \leqslant y'(f(x))$ holds for every $x \in D$ and every $y' \in M$ then we have that

$$T(f(x_0))(y') \leqslant T(f(x))(y') \tag{3.1}$$

for every $x \in D$ and every $y' \in M$, from where recalling that $M$ is weak*–dense in $A'$ it follows that the inequality (3.1) is verified for every $x \in D$ and every $y' \in A'$ and so $T(f(x_0) \leqslant Tf(x)$ for every $x \in D$. Let us prove now that $(x_0, T, W)$ is maximal. In fact, it $T_1$, $W_1$ are like in 2.2 and $x_0$ is a proper $(T_1, W_1)$–optimum let us consider $\pi : T(Y) \longrightarrow W_1$ such that $\pi T = T_1$, then it would be enough (since $T(Y)$ is dense in $W$) to see that $\pi$ is well defined and continuous. Let us see first that $\pi$ is well defined. If $y_1, y_2 \in Y$ and $T(y_1) = T(y_2)$ then $T_1(y_1) = T_1(y_2)$, otherwise we would have that there exists $w_1' \in \mathcal{W}_{1+}'$ such that $w'_1 T_1(y_1) \neq w_1' T_1(y_2)$ (recall that $\mathcal{W}_{1+}'$ separates points of $W_1$ since, $Y$ is separable, $T_1(Y)$ is dense in $W_1$ and so $W_1$ is also separable) and therefore $T_1' w_1'(y_1) \neq T_1' w_1'(y_2)$ where $T'_1$ denotes the transpose of $T_1$. Moreover, $T_1' w_1'(y) = w_1' T_1(y) > 0$ for every $y \in Y_+ - \{0\}$

and $T'_1 w_1' \in \mathcal{Y}'_+$, then since $T_1 f(x_0) \leqslant T_1 f(x)$ for every $x \in D$, we have that $w_1' T_1 f(x_0) \leqslant w_1' T_1 f(x)$ and so $T_1' w_1' f(x_0) \leqslant T_1' w_1' f(x)$ for every $x \in D$, which is a contradiction since the equality

$$y'(y_1) = T(y_1)(y') = T(y_2)(y') = y'(y_2)$$

holds for every $y' \in \mathcal{Y}'_+$ verifying that $y'(f(x_0)) \leqslant y'(f(x))$ for every $x \in D$.

Let us see now that the mapping $\pi$ is continuous. In fact, if $(y_n) \subset Y$ is such that

$$\|T(y_n)\| = \sup\{< y', y_n >: \ y' \in A'\} \xrightarrow[n \to \infty]{} 0$$

then, since $\dfrac{T'_1(w')}{\|T'_1(w')\|} \in A'$ if $w' \in \mathcal{W}'_{1+}$ (as we hare seen in the last paragraph) and

$$\|w_1\| \leqslant 2 \sup\{| < w', w_1' > | : \ \|w'\| \leqslant 1, \ w' \in \mathcal{W}'_{1+}\}$$

if $w_1 \in W_1$ (see [1] p.17 and [13] p.86), we have that

$$\begin{aligned} \|\pi T(y_n)\| &\leqslant 2 \sup\{| < w', T_1(y_n) >: \ \|w'\| \leqslant 1, \ w' \in \mathcal{W}'_{1+}\} \\ &= 2 \sup\{| < \frac{T'_1(w')}{\|T'_1(w')\|}, \ y_n > | \|T'_1(w')\| : \|w'\| \leqslant 1, \ w' \in \mathcal{W}'_{1+}\} \\ &\leqslant 2 \|T(y_n)\| \ \ \|T'_1\| \end{aligned}$$

and $\lim_n \|\pi \, T(y_n)\| = 0$.

**Remark 4**

Theorem 3 can be slightly generalized as it can be noticed from its proof. Thus, the assumption of being the Banach space $Y$ separable is used to assure that $\mathcal{W}'_{1+}$ separates points of $W_1$, but this holds also if it is assumed that $\mathcal{W}'_{1+} \neq \emptyset$, since $W'_{1+}$ separates points of $W_1$ (since $W_1$ is a closed subspace of a Banach lattice) and $W'_{1+} + \mathcal{W}'_{1+} \subset \mathcal{W}'_{1+}$. Let us remark that the condition $\mathcal{W}'_{1+} \neq \emptyset$ is not verified in general (for instance, it is not verified if $W_1 = C(K)$ and $K$ is the Alexandroff compactification of [0,1] endowed with the discrete topology) and also we can have $\mathcal{W}'_{1+} \neq \emptyset$ without $W_1$ being separable (for instance, if $W_1 = 1^\infty$ then $\mathcal{W}'_{1+} \supset l^1_+$ and $l^1_+$ separates points of $l^\infty$).

Also the assumption of $W_1$ being (and of course also $W$) a closed subspace of Banach lattice which induces on $W_1$ its original order, has been used to assure that $\|w_1\| \leqslant 2 \sup\{| < w', w_1 >: \|w'\| \leqslant 1, \ w' \in \mathcal{W}'_{\infty +}\}$, but this inequality is also verified changing 2 by some $\lambda > 0$ if $W_1$ (and $W$) is linear homeomorphic to a closed subspace of a separable Banach lattice.

### Theorem 5

If $x_0 \in D$ and the set $\{y' \in Y'_+ : y'(f(x_0)) \leqslant y'(f(x))$ for every $x \in D\}$ separates points of $Y$, then there exists a Banach space $W$ ordered by a pointed closed convex cone and a positive isometry $T: Y \longrightarrow W$ such that $x_0$ is a strongly proper $(T, W)$–optimum.

### Proof

As in the proof of the theorem 3, it follows from the Alaoglu–Bourbaki theorem that $B' = \{y' \in Y' : \|y'\| \leqslant 1\}$ endowed with the weak* topology is a compact topological space. Let us consider the mapping $T : Y \longrightarrow \mathscr{C}(B'), \mathscr{C}(B')$ being the space of the weak* continuos real functions defined on $B'$, endowed with the supremum norm, such that $T(y)(y') = y'(y)$ for every $y \in Y$ and $y' \in Y'$. Then, it is clear that $T$ is well defined and linear, and it follows from the Hahn–Banach theorem that it is also injective and isometric. Therefore, $W = T(Y)$ is a complete subspace of $\mathscr{C}(B')$. Let us define

$$W_+ = \{w \in W \ : \ w(y') \geqslant 0 \quad \text{for every} \quad y' \in A'\}$$

with

$$A' = \{y' \in Y'_+ \ : \ \|y'\| \leqslant 1 \ \text{ and } \ y'(f(x_0)) \leqslant y'(f(x)) \ \text{ for every } \ x \in D\},$$

then $W_+$ is a convex cone of $W$. Moreover, if $(w_n) \subset W_+$ is uniformly convergent on $B'$ to $h \in \mathscr{C}(B')$, then $h \in W$ (since $W$ is a complete subspace of $\mathscr{C}(B')$ and so it is closed) and obviously $h(y') \geqslant 0$ for every $y' \in A'$, and therefore, $W_+$ is closed. Let us see now that $W_+$ is pointed. In fact, if $w, -w \in W_+$ then since $T$ is a bijective mapping from $Y$ into $W$, there exists $y \in Y$ such that $w = T(y)$ and therefore, $T(y), T(-y) \in W_+$ and so $y'(y) = 0$ for every $y' \in A'$, from where it follows immediately that $y = 0$ and then $w = T(y) = 0$, since $A'$ separates points of $Y$.

Let us prove now that $T$ is positive. In fact, if $y \in Y_+$ then $T(y)(y') = y'(y) \geqslant 0$ for every $y' \in A' (\subset Y'_+)$ and $T(Y_+ - \{0\}) \subset W_+ - \{0\}$ since $T$ is injective.

Finally, for every $x \in D$ we have that $Tf(x_0) \leqslant Tf(x)$, since

$$T(f(x_0))(y') = y'(f(x_0)) \leqslant y'f(x) = T(f(x))(y')$$

for every $y' \in A'$.

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