A projective description of the simple scalar function space

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Recibido el 4 de Diciembre de 1991

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Abstract

In this note we deal with the space l_0^∞ (I) generated by the indicator functions of all subsets of some infinite index set I and provided with the supremum norm. As it is known, this space is not ultrabornological. Here we show that, despite this fact, this space still can be represented as a projective limit of ultrabornological non–complete normed spaces.

1980 Mathematical Subject Classification : 46A07, 28C99

Resumen

En esta nota consideramos el espacio l_0^∞ (I) generado por las funciones características de todos los subconjuntos de cierto conjunto infinito de índices I provisto con la norma supremo. Como es sabido, este espacio no es ultrabornológico. Aquí probamos que, a pesar de este hecho, este espacio todavía puede representarse como un límite proyectivo de espacios ultrabornológicos normados no completos.

If I is an infinite index set, we shall denote by $l_0^\infty(I)$ the subspace of the Banach space l^∞ (I) generated by the indicator functions e(P) of all subsets P of I. Let **U** represent the family of all ultrafilters on I and let $\mathcal{U}$ be any member of **U**. For each $U \in \mathcal{U}$ let us denote

$$L(U) := \{x \in l_0^\infty(I) : \; x(w) = constant \; \forall w \in U\}$$

It is obvious that $L(U)$ is a closed linear subspace of $l^\infty(I)$. On the other hand, given that $\mathcal{U}$ is a filter on I,

$$L(\mathcal{U}) := \cup\{L(U), \; U \in \mathcal{U}\}$$

is a linear subspace of $l^\infty(I)$.

If the ultrafilter $\mathcal{U}$ contains the filter $\mathcal{F}$ of the cofinite subsets of I, it is plain that each $U \in \mathcal{U}$ is infinite. Thus, in this case, each element x of $L(\mathcal{U})$ has infinitely many coordinates which are equal. Assume for example that $\mathrm{card}(I) = \aleph_0$ and $\mathcal{F} \subset \mathcal{U}$. Then pick a bounded injective scalar mapping φ

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defined on I and a bijective sequence (w_n) in I with $\varphi(w_n) = \frac{1}{n}$ for every $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, define $x_n(w_{n+i}) = \varphi(w_i)$ and $x_n(w_i) = 0$ for $1 \leqslant i \leqslant n$. Then, (x_n) is a sequence of linearly independent vectors of $l^\infty(I)$ whose linear combinations do not belong to $L(\mathcal{U})$. Hence, $L(\mathcal{U})$ is an infinite codimensional subspace of $l^\infty(I)$. In particular, if $\mathcal{U}$ is some ultrafilter on $\mathbb{N}$ containing the Fréchet filter on $\mathbb{N}$, then $L(\mathcal{U})$ is a linear subspace of l^∞ of infinite codimension. Finally, we conclude these observations noticing that for each $\mathcal{U} \in \mathbf{U}$, then $l_0^\infty(I)$ is contained in $L(\mathcal{U})$. Indeed, given $x \in l_0^\infty(I)$, there exists a finite partition $\{A_1, \ldots, A_n\}$ of I such that x is constant in every A_i for $1 \leqslant i \leqslant n$; as $\mathcal{U}$ is an ultrafilter on I, there is some $j \in \{1, \ldots, n\}$ such that $A_j \in \mathcal{U}$. Thus $x \in L(A_j) \subseteq L(\mathcal{U})$.

In what follows we shall see first that for each $\mathcal{U} \in \mathbf{U}$ the linear space $L(\mathcal{U})$ is the locally convex hull of the family $\{L(U), U \in \mathcal{U}\}$ of subspaces of $l^\infty(I)$. Then we shall prove that $l_0^\infty(I)$ coincides with the subspace $\bigcap\{L(\mathcal{U}), \mathcal{U} \in \mathbf{U}\}$. This will show that $l_0^\infty(I)$ is a projective limit of ultrabornological spaces although, [1], $l_0^\infty(I)$ is not itself an ultrabornological space.

Main Lemma

$L(\mathcal{U})$ is an ultrabornological space for each $\mathcal{U} \in \mathbf{U}$.

Proof.

We are going to show that $L(\mathcal{U})$ is the locally convex hull of the family $\{L(U), U \in \mathcal{U}\}$ of Banach subspaces of $l^\infty(I)$. Let V be an absolutely convex subset of $L(\mathcal{U})$ such that $V \cap L(U)$ is a neighbourhood of the origin in $L(U)$ for each $U \in \mathcal{U}$. We shall prove first that V absorbs the family $\{e(A), A \subseteq I\}$ of the indicator functions of the subsets of I.

Suppose that the aforementioned property does not hold and let $n_1 \in \mathbb{N}$ be such that $e(I) \in (1/6)n_1 V$. By hypothesis there is some $P_1 \subseteq I$ such that $e(P_1) \notin (4/3)n_1 V$. Moreover, given that $e(P_1) = e(I) - e(I - P_1)$, it follows that $e(I - P_1) \notin n_1 V$, since otherwise $e(P_1) \in (7/6)n_1 V$, which is a contradiction. Therefore

$$e(P_1) \notin n_1 V,\ e(I - P_1) \notin n_1 V$$

Let $n_2 > 2n_1$ be such that $e(P_1) \in (1/12)n_2 V$. A fortiori,

$$e(I - P_1) = e(I) - e(P_1) \in (1/6)n_1 V + (1/12)n_2 V \subseteq (1/6)n_2 V.$$

Hence,

$$e(P_1) \in (1/6)n_2 V,\ e(I - P_1) \in (1/6)n_2 V$$

Since $e(A) = e(A \cap P_1) + e(A \cap [I - P_1])$ for each $A \subseteq I$, it follows that either V does not absorb the family $\{e(A), A \subseteq P_1\}$ or does not absorb the family $\{e(A), A \subseteq I - P_1\}$. In the first case we set $A_1 := I - P_1$ and $B_1 := P_1$ and in the second case we put $A_1 := P_1$ and $B_1 := I - P_1$.

Now, there is some $P_2 \subseteq B_1$ such that $e(P_2) \notin (4/3)n_2V$. Given that $e(P_2) = e(B_1) - e(B_1 - P_2)$ and $e(B_1) \in (1/6)n_2V$, it follows that $e(B_1 - P_2) \notin n_2V$. Therefore

$$e(P_2) \notin n_2V, \ e(B_1 - P_2) \notin n_2V$$

If $n_3 > 2n_2$ is such that $e(P_2) \in (1/12)n_3V$ then $e(B_1 - P_2) \in (1/6)n_3V$.

Hence

$$e(P_2) \in (1/6)n_3V, \ e(B_1 - P_2) \in (1/6)n_3V$$

Reasoning as above, either V does not absorb $\{e(A), A \subseteq P_2\}$ or does not absorb $\{e(A), A \subseteq B_1 - P_2\}$. In the first case we set $A_2 := B_1 - P_2$ and $B_2 := P_2$ and in the second case we put $A_2 := P_2$ and $B_2 := B_1 - P_2$.

Proceeding by induction we obtain a strictly increasing sequence (n_i) in $\mathbb{N}$ and a sequence $\{A_i, i \in \mathbb{N}\}$ of pairwise disjoint subsets of I, such that

$$e(A_i) \notin n_iV$$

for each $i \in I$.

Let $A_0 := I - \bigcup\{A_i, i = 1, 2, \ldots\}$ and define

$$M_1 := \cup\{A_{2i}, i = 0, 1, 2, \ldots\}, \ M_2 := \cup\{A_{2i-1}, i = 1, 2, \ldots\}$$

Since U is an ultrafilter, either $M_1 \in \mathcal{U}$ or $M_2 \in \mathcal{U}$.

If $M_2 \in \mathcal{U}$ then $e(A_{2i}) \in L(M_2)$ for $i = 0, 1, 2, \ldots$. As $V \cap L(M_2)$ is a neighbourhood of the origin in $L(M_2)$ and $\{e(A_{2i}), i = 0, 1, 2, \ldots\}$ is a bounded set in $L(M_2)$ there is some $p \in \mathbb{N}$ with

$$\{e(A_{2i}), i = 0, 1, 2, \ldots\} \subseteq pV$$

If $j \in \mathbb{N}$ is such that $n_{2j} \geqslant p$, then $e(A_{2j}) \in n_{2j}V$, a contradiction. If $M_1 \in \mathcal{U}$ we also obtain a similar contradiction.

So we have shown that there exists some $\lambda > 0$ so that

$$\lambda e(A) \in V$$

for each $A \subseteq I$.

If B stands for the closed unit ball of $l^\infty(I)$ we show next that $\lambda B \cap L(\mathcal{U}) \subseteq 5V$. So V is a neighbourhood of the origin in $L(\mathcal{U})$ and we are done.

Pick $x \in \lambda B \cap L(\mathcal{U})$ and let $U \in \mathcal{U}$ be such that $x \in \lambda B \cap L(U)$. As $B \cap l_0^\infty(I)$ is dense in B there is a sequence (x_n) in $\lambda B \cap l_0^\infty(I)$ converging to x in $l^\infty(I)$. Thus, given $\varepsilon > 0$ there exists some $n_0 \in \mathbb{N}$ such that

$$\sup\{|x_n(w) - x(w)|, w \in I\} < \varepsilon$$

for each $n \geqslant n_0$. In particular,

$$\sup\{|x_n(w) - x(w)|, w \in I - U\} < \varepsilon$$

for each $n \geqslant n_0$. As $x \in \lambda B \cap L(U)$ there is some scalar μ with $|\mu| \leqslant \lambda$ such that $x(w) = \mu$ for each $w \in U$. Now for each n define $z_n(w) = x_n(w)$ if $w \notin U$ and $z_n(w) = \mu$ if $w \in U$. Clearly, $z_n \in \lambda B \cap l_0^\infty(I) \cap L(U)$ for each positive integer n and besides $z_n \to x$ in $L(U)$, since

$$\sup\{|z_n(w) - x(w)|, \ w \in I\} = \sup\{|x_n(w) - x(w)|, \ w \in I - U\} < \varepsilon$$

for each $n \geqslant n_0$.

If Q stands for the absolutely convex cover of $\{e(A), \ A \subseteq I\}$, it is shown in [4] that $B \cap l_0^\infty \subseteq 4Q$. Hence $z_n \in 4\lambda Q$ for each $n \in \mathbb{N}$. As $\lambda e(A) \in V$ for each $A \subseteq I$ and V is absolutely convex, it follows that $\lambda Q \subseteq V$, so

$$z_n \in 4V \cap L(U)$$

for each $n \in \mathbb{N}$. Now, as $V \cap L(U)$ is a neighbourhood of the origin in $L(U)$, it follows that

$$x \in \overline{4V \bigcap L(U)} \subseteq 4V \bigcap L(U) + V \bigcap L(U) \subseteq 5V$$

This shows that

$$\lambda B \bigcap L(\mathcal{U}) \subseteq 5V$$

Remark 1.

A Hausdorff locally convex space E is said to be unordered Baire–like, [3], if given a sequence of closed absolutely convex subets of E covering E, there is one of them which is a neighbourhood of the origin. If $\mathcal{U}$ is any ultrafilter on $\mathbb{N}$ containing the filter $\mathcal{F}$ of the cofinite subsets of $\mathbb{N}$, then $L(\mathcal{U})$ is not unordered Baire–like. If $\{S_n, \ n = 1, 2, \ldots\}$ is the sequence of all the ordered pairs (i,j) with $i < j$, then for each $n \in \mathbb{N}$ define E_n as the subspace of $L(\mathcal{U})$ formed by those x taking the same value in the two points of S_n. Clearly $E_n \neq L(\mathcal{U})$ for each $n \in \mathbb{N}$ since if $S_n = (i,j)$, then $e(I - \{i\}) \in L(\mathcal{U}) - E_n$. As every set of $\mathcal{U}$ contains infinitely many points of $\mathbb{N}$, it follows that $\{E_n, \ n = 1, 2, \ldots\}$ covers $L(\mathcal{U})$.

Proposition.

$$l_0^\infty(I) = \bigcap\{L(\mathcal{U}), \mathcal{U} \in \mathbf{U}\}.$$

Proof.

As we noticed above $l_0^\infty(I) \subseteq \bigcap\{L(\mathcal{U}), \mathcal{U} \in \mathbf{U}\}$. Suppose that there exists some x which belongs to $\bigcap\{L(\mathcal{U}), \mathcal{U} \in \mathbf{U}\}$ but $x \notin l_0^\infty(I)$.

Let $\{A_d, d \in D\}$ be a partition of I formed by infinitely many non–empty subsets of I (if I is uncountable, D may be uncountable too) with the property that x is constant in each set A_d and takes different values in each one of those sets. Let $\mathcal{B}$ represent the filter base formed by the complements in I of the finite unions of A_d. In other words, $E \in \mathcal{B}$ if there is some positive integer p and some indexes $d_1, d_2, \ldots d_p$ in D such that

$$E = I - \bigcup\{A_{d_i}, 1 \leqslant i \leqslant p\}$$

Let $\mathcal{G}$ be an ultrafilter which refines $\mathcal{B}$. It is plain that $A_d \notin \mathcal{G}$ for any $d \in D$. Since $x \in L(\mathcal{G})$, exists some $M \in \mathcal{G}$ such that x is constant on M. As $A_d \neq M$ for each $d \in D$, either there is some $r \in D$ such that $M \subset A_r$ or there are $i,j \in E$ with $i \neq j$ such that M meets both A_i and A_j. The first posibility cannot happen since A_r does not belong to $\mathcal{G}$ and the second implies that x takes the same value in both A_i and A_j, a contradiction.

Corollary

The space $l_0^\infty(I)$ is topologicaly isomorphic to a closed linear subspace of the product $\prod\{L(\mathcal{U}), \mathcal{U} \in \mathbf{U}\}$.

Proof.

The mapping $T: l_0^\infty(I) \to \prod\{L(\mathcal{U}), \mathcal{U} \in \mathbf{U}\}$ such that

$$Tx = (x, x, \ldots, x, \ldots)$$

is clearly a topological isomorphism from $l_0^\infty(I)$ into $\prod\{L(\mathcal{U}), \mathcal{U} \in \mathbf{U}\}$ of closed range.

Theorem.

The space $l_0^\infty(I)$ is the projective limit of the family $\{L(\mathcal{U}), \mathcal{U} \in \mathbf{U}\}$ provided with the canonical inclusion maps.

Proof.

This is an obvious consequence of the previous results.

Remark 2.

It is worth noticing that the space $l_0^\infty(I)$ is not ultrabornological, [1]. But nevertheless, according to our Theorem, it can be represented as a projective limit of ultrabornological spaces. Besides, excluding all those $L(\mathcal{U})$ that coincide with $l^\infty(I)$, the remainders ultrabornological spaces are non–complete. On the other hand, if the cardinal of $\mathbf{U}$ is smaller than the first strongly inaccesible cardinal, the product $\prod\{L(\mathcal{U}), \mathcal{U} \in \mathbf{U}\}$ is ultrabornological, ([2], p.72). Hence, in this case, $l_0^\infty(I)$ is (isomorphic to) a non–ultrabornological closed linear subspace of an ultrabornological space.

Open Problem.

A Hausdorff locally convex space E in called totally barrelled, [5], if given a sequence of linear subspaces of E covering E, there is one of them which is barrelled and its closure is of finite codimension in E. We do not know whether or not $L(\mathcal{U})$ is totally barrelled whenever $\mathcal{U}$ is an ultrafilter wich contains the cofinite subsets of I.

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