*Analytic Convexity and Complex Interpolation**

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Resumen

El método de interpolación compleja de Calderón presenta dificultades al aplicarse a pares de espacios cuasi-Banach. Mostramos que estas dificultades desaparecen en el caso de pares de espacios con un espacio contenedor cuasinormable por una cuasinorma plurisubarmónica. Se hallan en esta situación los ejemplos más frecuentes.

1. INTRODUCCION

The first approach of complex interpolation of couples (X_0, X_1) of quasi–Banach spaces was made by Rivière and it has been developed by Cwikel, Milman and Sagher in [5]. They use the classical construction (see [2]) but, when defining the intermediate spaces, they work on the intersection space with quasi–seminorms that can be identically zero and, if they are genuine quasi–norms, the intersection need not be complete. Its inclusion in the sum space may also fail to be continuous and even if this latter problem does not arise, it is not clear whether the extension of the continuous inclusion to the completion is one to one. Another approach can be found in [6], without convexity in the interpolation theorem.

We present a method based on complete spaces of analytic functions and with convexity in the interpolation theorem. To do this we restrict the category of quasi–Banach pairs to a smaller class. All spaces will be quasi–Banach spaces and we refer to [1] for proofs.

For a given space $\mathcal{U}$, $A(\Delta,\mathcal{U})$ and $A(S,\mathcal{U})$ are the spaces of all $\mathcal{U}$–valued bounded analytic functions that are continuous up to the boundary of the disk Δ and of the strip S. We will consider spaces $\mathcal{U}$ which have an equivalent plurisubharmonic quasi–norm. They are called A–convex by Kalton [7] and are characterized by the existence of a constant C so that

$$\|f(0)\| \leqslant C \sup_{|z|=1} \|f(z)\| \quad \text{if} \quad f \in A(\Delta,\mathcal{U}).$$

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Any space $(X, \|\cdot\|)$ has an equivalent quasi–norm $|\cdot|$ so that, for certain $0 < r \leqslant 1$, $|\cdot|^r$ is subadditive and it is said to be an r–norm. All the quasi–norms are suposed to be r–norms for some r. On any A–convex space $\mathcal{U}$ there is an equivalent r–norm which is plurisubharmonic ([7]).

Let $(X, \|\cdot\|)$ be a quasi–Banach space and denote $\|x\|_A$ the biggest plurisubharmonic quasi–seminorm on X such that $\|\cdot\|_A \leqslant \|\cdot\|$. If we have a continuous embedding $X \subset \mathcal{U}, \mathcal{U}$ being A–convex,

$$\|x\|_{\mathcal{U}} \leqslant C\|x\|_A \quad \text{and} \quad \|\cdot\|_A$$

is a quasi–norm. The completion X_A of $(X, \|\cdot\|_A)$ is an A–convex space and X is continuously embedded in X_A.

2. INTERPOLATION PAIRS WITH AN A–CONVEX CONTAINING SPACE

The pair (X_0, X_1) has an A–convex containing space if there is an A–convex space $\mathcal{U}$ so that $X_0 + X_1 \subset U$ continuosly, and $\mathcal{H}(\mathcal{U})$ will consist of all functions $f \in A(S, \mathcal{U})$ such that $f(j + it) \in X$, and $f(j + it)$ defines a bounded continuous function f_j from $\mathbb{R}$ to $X_j, j = 0, 1$. The three lines theorem can be proved in this case and

$$\|f\|_{\mathcal{H}(\mathcal{U})} = \max_{j=0,1} \sup_{t\in\mathbb{R}} \|f_j(t)\|_{X_j}$$

defines a complete quasi–norm on $\mathcal{H}(\mathcal{U})$.

Let $\mathcal{F}(\mathcal{U})$ be the $\|\cdot\|_{\mathcal{H}(\mathcal{U})}$–closure of the class of all functions

$$\sum_{n=1}^{N} \varphi_n x_n : x_n \in X_0 \cap X_1$$

with $\varphi_n \in A(S, \mathbb{C})$, $N \in \mathbb{N}$. For $0 \leqslant \theta \leqslant 1$, $(X_0, X_1)_{[\theta],u}$ is the space of all $f(\theta)$, $f \in \mathcal{F}(\mathcal{U})$, with the complete quasi–norm

$$\|x\|_{[\theta],u} = \inf\Big\{ \|f\|_{\mathcal{H}(\mathcal{U})} : \ f \in \mathcal{F}(\mathcal{U}), \ \{(\theta) = \S\Big\}.$$

For any

$$0 < \theta < 1, \ (X, X)_{[\theta], X_A} = X_A$$

with equality of quasi–norms. In [1] we use this fact to prove that the interpolation spaces $(X_0, X_1)_{[\theta],u}$ can be different for different choices of the containing space.

But there is some independence of the choice of $\mathcal{U}$:

Theorem 1

Suppose that there exist two A–convex spaces $\mathcal{U}$ and $\mathcal{V}$, both contained with continuous inclusions in a Hausdorff topological vector space $\mathcal{A}$, and such that $X_0 + X_1 \subset \mathcal{U}$ and $X_0 + X_1 \subset \mathcal{V}$ with continuous inclusions. Then $\mathcal{F}(\mathcal{U}) = \mathcal{F}(\mathcal{V})$, and $(X_0, X_1)_{[\theta],u}$ and $(X_0, X_1)_{[\theta],v}$ are isometric and coincide as vector subspaces of $\mathcal{A}$.

Notation $(X_0, X_1)_{[\theta],u}$ can now be replaced by $(X_0, X_1)_{[\theta],A}$ or $(X_0, X_1)_{[\theta]}$, to indicate that these spaces do not depend on the containing A–convex spaces $\mathcal{U}$ such that

$$X_0 + X_1 \subset \mathcal{U} \subset \mathcal{A}.$$

3. SOME PROPERTIES OF THE SPACES $(X_0, X_1)_{[\theta]}$, AND EXAMPLES

Let (X_0, X_1) and (Y_0, Y_1) be two interpolation pairs with the containing A–convex spaces $\mathcal{U}$ and $\mathcal{V}$, and $0 < \theta < 1$.

Theorem 2

Let $T : \mathcal{U} \to \mathcal{V}$ be a bounded operator of type (X_j, Y_j) with constant M_j $(j = 0, 1)$. Then T is bounded from $(X_0, X_1)_{[\theta]}$ to $(Y_0, Y_1)_{[\theta]}$ with constant $M_0^{1-\theta} M_1^{\theta}$.

Basic properties of complex interpolation are extended in our setting and the following examples show a number of couples with a containing A–convex space.

- ***Example 1.*** If X_0 is continuosly contained in X_1 and X_1 is A–convex, then X_1 is a containing A–convex space. This is the case of the pairs (L^{p_1}, L^{p_0}) on a finite measure space, $(H^{p_1}(\Delta), H^{p_0}(\Delta))$ and (l^{p_0}, l^{p_1}), if $0 < p_0 < p_1 < \infty$.
- ***Example 2.*** (Hardy spaces on $\mathbb{R}^n$). For the pair $(H^{p_0}(\mathbb{R}^n), H^{p_1}(\mathbb{R}^n))$, the sum has a containing Banach space because it has a separating dual.
- ***Example 3.*** For every quasi–Banach pair (X_0, X_1) with an A–convex containing space $\mathcal{U}$, $(L^{p_0}(X_0), L^{p_1}(X_1))$ has an A–convex containing space $\mathcal{V}$.

Let $p = \min\{p_0, p_1, 1\}$ and let $\{B_n\}_n$ be an increasing sequence of measurable sets of finite measure such that $\Omega = \cup_{n=1}^{\infty} B_n$. For

$$f \in L^{p_0}(X_0) + L^{p_1}(X_1), \quad \|f \chi_{B_n}\|_{L^p(\mathcal{U})} \leqslant C(n) \|f\|_{L^{p_0}(X_0)+L^{p_1}(X_1)}$$

and $\mathcal{V}$ consists of all measurable $\mathcal{U}$–valued functions g such that

$$\|g\|_v = \sup_{n \geqslant 1} C(n)^{-1} \|g \chi_{B_n}\|_{L^p(u)} < +\infty .$$

Theorem 3

If $0 < p_0, p_1 < \infty$ and $1/p = (1-\theta)/p_0 + \theta/p_1$, then

$$\left(L^{p_0}(X_0),\ L^{p_1}(X_1)\right)_{[\theta],v} = L^p\left((X_0, X_1)_{[\theta],u}\right).$$

- ***Example 4.*** (Tent spaces of [4]). Let $0 < p, q < \infty$. The space T_q^p is the set of all measurable functions f on $\mathbb{R}_+^{n+1}$ such that

$$\|f\|_{p,q} = \left[\int_{\mathbb{R}^n}\left(\int_{\Gamma(x)} |f(y,t)|^q \frac{dy\,dt}{t^{n+1}}\right)^{p,q} dx\right]^{1/p} < +\infty,$$

where $\Gamma(x)$ is the cone of all $(y,t) \in \mathbb{R}_+^{n+1}$ such that $d(y,x) < t$.

We denote by L_*^q the space L^q on $\mathbb{R}_+^{n+1}$ with respect to the measure $dy\,dt/t^{n+1}$ and $Tf(x) = \chi_{\Gamma(x)} f \cdot T$ maps $T_{q_0}^{p_0} + T_{q_1}^{p_1}$ into $L^{p_0}(L_*^{q_0}) + L^{p_1}(L_*^{q_1})$. The set $\mathcal{W}$ of all measurable functions f on $\mathbb{R}_+^{n+1}$ so that $Tf \in \mathcal{V}$ (from *example 3*), with the induced quasi–norm by $\mathcal{V}$, is an A–convex space which contains continuosly the sum $T_{q_0}^{p_0} + T_{q_1}^{p_1}$, and

$$T\left((T_{q_0}^{p_0}, T_{q_1}^{p_1})_{[\theta]}\right) \subset L^p(L_*^q),$$

if $1/p = (1-\theta)/p_0 + \theta/p_1$ and $1/q = (1-\theta)/q_0 + \theta/q_1$.

Moreover, $(T_{q_0}^{p_0}, T_{q_1}^{p_1})_{[\theta]} \subset T_q^p$.

Interpolation of H^p spaces $(0 < p \leqslant \infty)$ and BMO works as in [3] and [6].

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