About mathematical model or some diffuse processes in the Mediterranean and exterior Poincare problem for the Helmholtz equation

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1. Introduction

Among the linear elliptic boundary–value problems, Poincare problem is of great importance, side by side with Dirichlet and Neumann problems. Numerous processes ocurring in the continuum (for example, sea tides, see Poincare, 1910, as well as Hilbert, 1912) can be simulated in terms of this problem. This problem differs essentially form Dirichlet and Neumann problems in the fact that it is normally solvable according to Noether under rather general assumptions (i.e. Noether theorems known in the theory of singular integral equations are valid for it, see Vekua, 1943; Gakhow, 1963; Carleman, 1939; Muskhelishvili, 1962; Noether, 1921; Pogorzelski, 1939; Khvedelidze, 1943; Yanushauskas, 1985).

In the proces of the investigation of Dirichlet and Neumann problems for external regions, certain difficulties arose due to the complex behaviour of the solutions of elliptic equations at infinity. This fact was revealed on the example of Helmholtz equation simulating wave processes in the linear formulation. The irradiation principle is of great importance for the theory of wave processes investigated in physics, technology, ecology and natural science. The existence, uniqueness and stability of the solutions of the mentioned problems for Helmholtz equation in infinite regions with boundary components representing closed Lyapunov's surfaces (in three–dimensional case) and closed Lyapunov's curves (in two–dimensional case) were established in the class of functions satisfying the condition of Sommerfeld's irradiation at infinity (Sommerfeld, 1912; Rellich, 1943; Freudenthal, 1968).

We shall only mention a few original applications arising in the diffuse inflow of water into the sea (for example in the Mediterranean). In the following we briefly describe the mathematical model of above processes which was discussed in the article of Legovic, Limic, 1989. In this article the transport of identifiable chemical species in a two–dimensional basin D

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with a flat bottom was considered. Let α be a dispersion coefficient, V and E –current in two–dimensional basin and extinction function respectively. Let q be imput. Using as the basic model the Steady–state Fick law for the concentration u of nutrients in two dimensions

$$-\alpha\,\Delta u + \operatorname{div}(V u) + E(u) = q,$$

$$\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

the authors by linearizing of this equation, set the following optimal control problem

I. $\varepsilon(u)$ is a minimum,

$$\varepsilon(u) = \sum_{i=1}^{n} [u(z_i) - u_i]^2$$

is least squares error, u_i is the measured values of concentration at z_i points in the basin, $i = 1, \ldots, n$.

II. $-\alpha\,\Delta u + ku = 0,$

III. $u\,|S_0 = u_n, \frac{\partial_u}{\partial_v}|\,S_c = 0$

IV. k is nonegative number,

where S_0 and S_c are the "open" and the "coastal" boundaries respectively and v denotes the outward normal on the S_c (Legovic, Limic, 1989).

Here we introduce Hankel's and McDonald's functions defined by the formulas:

$$H_0^{(1)}(\eta) = J_0(\eta) + i\,N_0(\eta)$$

$$H_0^{(2)}(\eta) = J_0(\eta) - i\,N_0(\eta),$$

where $J_0(\eta)$ is Bessel function, $N_0(\eta)$ is Neumann function.

$$J_0(\eta) = \sum_{n=0}^{\infty} (-1)^n \frac{(\eta/2)^{2n}}{(n!)^2}$$

$$N_0(\eta) = \frac{2}{\pi}\left\{ J_0(\eta)\log\frac{(\eta)}{2} - \sum_{n=0}^{\infty}(-1)^n \frac{(\eta/2)^{2n}}{(n!)^2}\left(-c* + \sum_{j=1}^{n}\frac{1}{j}\right)\right\};$$

$c*$–Euler's constant, $K_0\,(\eta)$ McDonald's function

$$K_0\,(\eta) = -I_0(\eta)\log\frac{\eta}{2} + \sum_{n=0}^{\infty}\frac{(\eta/2)^{2n}}{(n!)^2}\left(-c* \sum_{j=1}^{n}\frac{1}{j}\right).$$

and $I_0(\eta)$ is modified Bessel function

$$I_0(\eta) = \sum_{n=0}^{\infty} \frac{(\eta/2)^{2n}}{(n!)^2}$$

2. Theoretical considerations and results

Herinafter we denote by S a closed Lyapunov's curve of $C^{1,h}$ class lying in E^2 plane of complex variable $z = z + iy$, and by D^+ and D^- the internal (containing the point $z = 0$) and external (cointaining an infinitely distant point of the plane) regions of this plane, respectively.

Function $u(x,y) = u(z), z = x + iy$, defined by formula

$$u(z) = \int_S \varepsilon(\lambda|t - z|)\,\mu(t)ds,\ t = t(s) \in S, \qquad [1]$$

where $\varepsilon(\lambda\,|t - z|)$ elementary (fundamental) solution of Helmhotlz equation

$$\Delta u + c_0 u = 0,\ c_0 = const \qquad [2]$$

and $\mu(t)$ is a given real function on S, is called a metaharmonic potential of a simple layer. Function

$$u(z) = \int_S \sigma(t)\frac{d\,\varepsilon(\lambda|t - z|)}{d\nu_t}ds,\ t \in S, z \in D^- \qquad [3]$$

where $\sigma(t)$ is a given real function on S and $\nu(t)$ is a normal to S outward with respect to D^+ at the point $t \in S$, is called a metaharmonic potential of a double layer. Let us adopt symmbol $u(z) \equiv v(z;\mu)$ for the metaharmonic potential of a simple layer and $u(z) \equiv w(z;\sigma)$ for the metaharmonic potential of a double layer respectively.

If the region D coincides with D^-, an additional requirement is specified for metaharmonic functions concerning their behavoir at infinity when considering the boundary value problems of Dirichlet and Neuman (Vekua, 1943).This requirement is known in literature under the name of the irradiation principle (of Sommerfeld), and in case of real $c_0 = \lambda^2, \lambda > 0$, is mathematically written as

$$\frac{du}{dr} - i\lambda u = \bar{0}\ (r^{-1/2}), r = |t - z|, \quad \text{at} \quad r \to \infty \qquad [4]$$

Below we assume that c_0 is a real non–zero constant. Since the structural and qualitative properties of equation (2) are essentially different depending on whether $c_0 > 0$ or $c_0 < 0$, we consider below two cases of equation (2), i.e.

$$\Delta u + \lambda^2 u = 0$$

and

$$\Delta u - \lambda^2 u = 0$$

We are interested in such elementary solutions of above equations which attenuate in a definite way at $|t - z| \longrightarrow \infty$. In cases when $c_0 = \lambda^2$ and $c_0 = -\lambda^2, \lambda > 0$, function $\varepsilon(t, z)$ are given by formulas

$$\varepsilon_1(\lambda\,|t - z|) = c_1 H_0^{(1)}(\lambda |t - z|),$$

$$\varepsilon_2(\lambda\,|t - z|) = c_2 K_0(\lambda |t - z|).$$

respectively, where $H_0^{(1)}(\eta)$ Hankel function of the first kind, $K_0\,(\eta)$ McDonald's function and $c_1 = i/4,\ c_2 = 1/2\pi$.

The behaviour of the functions $H_0^{(1)}(z)$ and $H_0^{(2)}(z)$ at $z \longrightarrow \infty$ is well known. Namely.

$$H_0^{(1)}(\eta) = \sqrt{\frac{2}{\pi n}} \exp\ |i(n - \frac{\pi}{4})|\ \{1 + \underline{0}(\eta^{-1})\},$$

at $-\pi < \arg\eta < 2\pi$, and

$$H_0^{(2)}(\eta) = \sqrt{\frac{2}{\pi n}} \exp\ | - i(n - \frac{\pi}{4}| \{1 + \underline{0}(\eta^{-1})\},$$

at $-2\pi < \arg\eta < \pi$

Hence, at $r \longrightarrow \infty,\ r = |t - z|$

$$\varepsilon_1(\lambda |t - z|) = K_1 \exp(i\lambda r) r^{-1/2}\ \{1 + \underline{0}(r^{-1})\}$$

where $\lambda > 0$, and

$$K_1 + \frac{i}{4}\ \frac{2}{\pi\lambda} \exp(-i\frac{\pi}{4})$$

Hence at $r \longrightarrow \infty$

$$\frac{d\varepsilon_1}{dr} - i\lambda\varepsilon_1 = \underline{0}(r^{-3/2})$$

To elucide the behavoir of $\varepsilon_2(\lambda|t - z|), \lambda > 0$, at $|t - z| \longrightarrow \infty$, we use integral representation

$$K_0(n) = \int_0^\infty \frac{\exp(\eta t)}{\sqrt{t^2 - 1}} dt,$$

and we become convinced that at $r \longrightarrow \infty$

$$K_0(\lambda|t-z|) = \underline{0}(\exp(-\lambda r)),\ \lambda > 0$$

By virtue of the above formulas it is clear that when $c_0 = \lambda^2$ and $|t-z| \longrightarrow 0$ function $\varepsilon(\lambda|t-z|)$ has an logarithmic singularity, but if $|t-z| \longrightarrow \infty$ then

$$\varepsilon(\lambda|t-z|) = 0(|t-z|^{-1/2}).$$

In the case $c_0 = -\lambda^2$ it follows that if $|t-z| \longrightarrow \infty$ then

$$\varepsilon(\lambda|t-z|) = \underline{0}\exp(-\lambda|t-z|).$$

Below we assume that density μ of metaharmonic potential of a simple layer (1) is a continuous function according to Helder.

Funther we use well–known properties of metaharmonic potential of simple and double layers presented below. If $\mu(t)$ is a continuous function of the point t of the boundary S, then: a) the simple layer potential $v(z,\mu)$ defined by formula (1) represents a function continuous over the whole space according to Helder, i.e. $v \varepsilon C^{o,h}(E^2)$; b) there exist limiting values

$$\lim_{\substack{z \to t_0 \\ z \in D^-}} \operatorname{grad} v(z;\mu)\cdot v_{t_0} \equiv \frac{dv^-}{dv_{t_0}}; (t_0;\mu),$$

$$\lim_{\substack{z \to t_0 \\ z \in D^+}} \operatorname{grad} v(z;\mu)\cdot v_{t_0} \equiv \frac{dv^+}{dv_{t_0}}(t_0;\mu),\ t_0 \varepsilon S$$

with $\frac{dv^-}{dv_{t_0}}$ and $\frac{dv^+}{dv_{t_0}}$ and representing continuous function in S; c) the following equality is values (Vekua, 1943);

$$\frac{dv^-}{dv_{t_0}} = -\frac{1}{2}\mu(t_0) + \int_S \mu(t)\frac{d}{dv_{t_0}}\varepsilon(t_0,t)ds_t \qquad [5]$$

If $\sigma(t)$ is a continuous function of the point t of the boundary S, then for the metaharmonic potential of a double layer (3): 1) there exists limiting value:

$$w^+(t_0;\sigma) \equiv \lim_{\substack{z \to t_0 \\ z \in D^+}} w(z;\sigma)$$

and 2)w^+ is a continuous functions in S, 3) the function w^* defined by the formula

$$w^* = \begin{cases} w(z;\sigma), & z \in D^+ \\ w^+(z;\sigma), & z = t_0 \in S \end{cases}$$

belongs to the class $C^{0,0}(D^+ \cup S)$.

It follows directly from above that if $\mu(t) \in C^{0,h}(S)$ then

$$\frac{dv}{dv_{t_0}} \in C^{0,h}(S).$$

Further, when coming across the functions of the variable t_0, in order to simplify the representation, we will denote the tangent and normal derivatives of the functions with respect to t_0 by

$$\frac{d}{d_{s0}} \text{and} \frac{d}{d_{v0}},$$

respectively, i.e.

$$\frac{d}{d_{s_{t_0}}} \equiv \frac{d}{d_{s0}} \text{ and } \frac{d}{d_{v_{t_0}}} \equiv \frac{d}{d_{v0}}.$$

Funther we are intereseted exclusively in the functions of $C^{1,h}(D^- \cup S)$ class metaharmonic in D^- and satisfying the condition (4) at infinity, which may be represented in the form of the simple layer metaharmonic potential.

The following theorem is valid: for the function $u(z)$ of $C^{1,h}(D^- \cup S)$ class metaharmonic potentials in D^- and satisafying the condition (4) to be representable in the form of metaharmonic potentials of a simple layer (1), the fulfillment of the conditions

$$\int_S f\psi_i\, ds = 0, i = 1\ldots, m,$$

is necesary and sufficient, where $f(t)$ denotes the limiting of the normal derivative $u(z)$ function when the point $z = x + iy$ form the region D^- approaches the boundary point $t \in S$

$$\frac{du^-}{dv_t} \equiv f(t), t \in S \quad [6]$$

and $\psi_i, i = 1, \ldots, m$ are the densities of all linearly independent potentials $w_i(z)$ of the double layer metharmonic in D^+ and satisfying the uniform Dirichlet's condition.

$$w^+(t, \psi_i) = 0, t \in S, i = 1, \ldots, m$$

From this theorem it is clear that function $u(z)$ in D^- will be representable as a simple layer potential (1) if and if the function $v(z; \mu)$ defined by this formula represents a solution of Neumann's problem.

$$\frac{dv^-}{dv} = f(t), t \in S,$$

where $f(t)$ is given by formula (6).

It is well known that the characteristic numbers, $c_0 = \lambda_i, i = 1, 2, \ldots,$ of the homogenous Dirichlet's problems $u^+(t) = 0, t \in S$, for equation (2) within D^+ region are all positive, and their set is countable

$$0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots, \text{ and } \lambda_n \to \infty \text{ at } n \to \infty$$

(Tamarkin, 1927). It follows directly from this fact that if the parameter c_0 in equation (2) does not coincide with the mentioned characteristic number, then a function of $C^{1,h}(D^- \cup S)$ class metaharmonic in D^- and satisfying the condition (4) may be always represented in the form of a simple layer metaharmonic potential. Evidently, in case of equation (2) when $c_0 = -\lambda^2$ where λ is real constant, such a representation is always possible.

Below we suppose everywhere, that $c_0 \neq \lambda_i, i = 1, 2, \ldots$ Let us introduce function $Q(\lambda|t - z|)$ by formula.

$$Q(\lambda|t-z|) = -\left\{[J_0(\lambda|t-z|) - 1]\log(\lambda|t-z|) - J_0(\lambda|t-z|)\log 2\right\} +$$

$$+\sum_{k=0}^{\infty} (-1)^k \left(\frac{\lambda}{2}\right)^{2k} \frac{|t-z|^{2k}}{(k!)^2} \left(-c* + \sum_{j=1}^{k} \frac{1}{j}\right), t \in S, z \in D^- \qquad [7]$$

when $c_0 = \lambda^2, \lambda > 0, c*$–Euler's constant, and

$$Q(\lambda|t-z|) = -\left\{[I_0(\lambda|t-z|) - 1]\log(\lambda|t-z|) - I_0(\lambda|t-z|)\log 2\right\} +$$

$$+\sum_{k=0}^{\infty} \left(\frac{\lambda}{2}\right)^{2k} \frac{|t-z|^{2k}}{(k!)^2} \left(-c* + \sum_{j=1}^{k} \frac{1}{j}\right) \quad t \in S, \; z \in D^- \qquad [8]$$

in the case $c_0 = -\lambda^2, \lambda > 0$.

In (7) and (8) formulas $J_0(n)$ denotes Bessel function, $J_0(n)$ modified Bessel function. By virtues (7) and (8) for the expression of elementary function we obtain

$$\varepsilon(\lambda|t-z|) = \frac{1}{2\pi}\log\lambda|t-z| + \frac{1}{2\pi}Q(\lambda|t-z|),$$

and respectively expression (1) for metaharmonic potential $v(z; \mu)$ acquires the form

$$v(z;\mu) = -\frac{1}{2\pi}\int_S \log\lambda|t-z|\mu(t)ds_t +$$

$$+\frac{1}{2\pi}\int_S Q(\lambda|t-z|)\mu(t)ds_t, \; t \in S, z \in D^- \qquad [9]$$

The following theorem is true: if $\mu(t)$ is continuous according to Helder, then the tangent derivative of a simple layer metaharmonic potential $v(z, \mu)$

exists for all $z = t_0 \in S$ values, ramains continuous at the point z transition form external region D^- into D^+ region, and is given at $z = t_0 \in S$ by the formula

$$\frac{dv}{ds_0} = \frac{t'_0}{2\pi} \int_S \frac{\mu(t)\overline{t'}}{t - t_0} dt + \frac{1}{2\pi} \int_S \left\{ \frac{d}{ds_0} Q(t, t_0) + \right.$$

$$\left. + i \frac{d}{ds_0} \Theta(t, t_0) \right\} \mu(t) \bar{t}\, dt, \quad t, t_0 \in S, \qquad [10]$$

where $Q(t, t_0) \equiv Q(\lambda |t - t_0|)$ is defined by the formula (7) or (8) depending on whether

$$c_0 = \lambda^2, \ \lambda > 0 \text{ or } c_0 = -\lambda^2, \ \lambda > 0 \text{ and } \Theta(t, t_0) = \arg(t - t_0)$$

By virtue of (7), the function $\frac{d}{ds_0} Q(t, t_0)$ is continuous according to Helder at $t \neq t_0$ while at $t = t_0$ has singularity of $\log |t - t_0|$ type. Besides, it is well known that (Muskhelishvili, 1962),

$$\frac{\partial}{\partial s_0} \Theta(t, t_0) = \frac{H^*(t, t_0)}{|t - t_0|^h}, \ 0 < h < 1,$$

where the function $H^*(t, t_0)$ is continuous according to Helder with respect to the set of variables t, t_0. Taking into account these circumstances in the right–hand side of (10) represent an ordinary improper integral, while in the first summand the integral is meant in the sense of Cauchy's principal value.

We are aimed at the investigation of the external problem of the oblique derivative in the following formulation: to find the solution $u(x, y)$ of the equation (2) regular in D^- region, possesing first derivatives in $D^- \cup S$, continuous in the sense of Helder and and satisfying the boundary condition

$$(l \operatorname{grad} u)^- = f(t_0), \ t_0 \in S \qquad [11]$$

where $l(t_0) \equiv (l_1, l_2)$ is a unit vector of $C^{0,h}(S)$ class given in S, f is a real function given in S and continuous in the sense of Helder, and

$$(l \operatorname{grad} u)^- = l. \lim_{z \to t_0} \operatorname{grad} u(z)$$

Below we present the investigation of this problem in the class of functions representable as metaharmonic potentials of a simple layer which satisfy the irradiation condition (4) at infinity. On the basis of the above results, we will prove that the problem (2), (11) may be reduced to and equivalent singular integral equation.

In fact, writing the boundary condition (11) in the form

$$g_1(t_0) \frac{du}{dv_0} + g_2(t_0) \frac{du}{ds_0} = f(t_0), \ t_0 \in S \qquad [12]$$

where

$$g_1 = l_1 \cos v_0\hat{x}_0 + l_2 \cos v_0\hat{y}_0, \qquad [13]$$
$$g_2 = l_1 \cos v_0\hat{y}_0 - l_2 \cos v_0\hat{x}_0,$$

are real functions given in S and continuous in the sense of Helder, and substituting the function $u(z,y)$ defined by formula (9) into the boundary condition (12), by virtue (5), (10) we obtain

$$T_\mu \equiv A(t_0)\mu(t_0) + \frac{B(t_0)}{\pi i}\int_s \frac{\mu(t)}{t-t_0}dt + \frac{1}{\pi}\int_S k_0(t,t_0)\mu(t)dt = \qquad [14]$$
$$= F(t_0),\ t, t_0 \in S,$$

where

$$A(t_0) = g_1(t_0), B(t_0) = ik_1(t_0,t_0), F(t_0) = -2f(t_0) \qquad [15]$$
$$k_1(t,t_0) = -t'_0\bar{t}'g_2(t_0), k_0(t,t_0) = R_0(t,t_0) + k_2(t,t_0)$$
$$R_0(t,t_0) = R_e\{R_1(t,t_0) + R_2(t,t_0)\}$$
$$k_2(t,t_0) = (1 - t'_0,\bar{t}')\frac{g_2(t_0)}{t-t_0},$$
$$R_1(t,t_0) = -\bar{t}'\left\{g_1(t_0)\frac{d}{dv_0}Q(t,t_0) + g_2(t_0)\frac{d}{ds_0}Q(t,t_0)\right\}.$$
$$R_2(t,t_0) = \bar{t}'\left\{g_1(t_0)\frac{d}{dv_0}\log|t-t_0| - ig_2(t_0)\frac{d}{ds_0}\Theta(t,t_0)\right\}$$

Taking into account that S denote a closed Lyapunov's curve and by formula

$$\frac{d}{ds_0}\log|t-t_0| = -\left(\frac{t_0}{t-t_0} + i\frac{d}{ds_0}\Theta(t,t_0)\right)$$

on the basis (10) we conclude that the function $k_0(t,t_0)$ defined by formula (15) at $t \neq t_0$ is continuous according to Helder with respect t_0 of the variable t and t_0 everywhere in S, while at $t = t_0$ it has a logarithmic singularity and a singularity of the form $|t-t_0|^{h_0}$, $0 < h_0 < 1$.

Thus, we come to the conclusion that equality (14) with respect to the function μ represents a singular integral equation with Cauchy's nucleus, the last summand in its lefthand side being quite a continuous operator.

As we known (Bitzadze, 1981; Muslhelishvili, 1962), equation (14) is normally solvable if the conditions

$$A(t_0) + B(t_0) \neq 0,\ A(t_0) - B(t_0) \neq 0$$

are fulfilled everywhere in S. By virtue (15) these conditions are equivalent to the condition

$$g_1(t_0) + g_2(t_0) \neq 0,\ t_0 \in S \qquad [16]$$

Below we will assume that the coefficients g_1 and g_2 in the boundary condition (12) satisfy the condition (16) everywhere in S.

Let $T'\psi = 0$ be a homogeneous equation allied with (14) it is known from the theory of one–dimensional singular integral equation that in case of the fulfillment of condition (16) Neother's theorems are valid: 1) numbers m and m' of lineary independent solutions of the respective (14) homogeneous equation $T_\mu = 0$ and its allied homogeneous equation $T'\psi = 0$ are finite; 2) nonhomogeneous equation (14) is solvable if and only is

$$\int_S F\psi_k dt = 0, \; k = 1, \ldots, m' \qquad [17]$$

where $\{\psi_k\}$ represents all linearly independent solutions of equation $T'\psi = 0$ and if the condition (17) are fulfilled the general solution equation (14) is of the following form:

$$\mu = \sum_{k=1}^{m} \beta_k \mu_k + \mu_0,$$

where $\{\mu_k\}$ represents all linearly independent solutions of equation $T_\mu = 0$; β_k; are arbitrary real constants, and μ_0 partial solution of the same equations; 3) the index $k = m - m'$ if the integral equation (14) (or operator T) is defined by the formula

$$k = \frac{1}{2\pi}\left[\arg\frac{g_1 - ig_2}{g_1 + ig_2}\right] S,$$

where $[\ldots]S$ denotes the increment of the function on square brackets at a one fold tracing of the point t around the path S in the positive direction (Bitsadze, 1981; Muskeloskvili, 1962).

On the basis of these theorems, we come to the conclusion that: 1) in the absence of nontrivial solution of the equation $T'\psi = 0$, the problem (2), (11) is always solvable, and the number of its linearly independent solutions equals $m = k$, 2) if $k = 0$, and the homogeneous problem corresponding to (2), (11) has only a trivial solution, then the nonhomogeneous problem (2), (11) always has a solution which is the only one.

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