Ultrabarrelled spaces and dense vector subspaces

POR J. KAKOL [1] POZNAN

Recibido: 9 de mayo de 1990

Presentado por el Académico Numerario D. Manuel Valdivia

INTRODUCTION

Let $(E.\tau)$ be a Hausdorff topological vector space (tvs) over the field $K \in (\mathbb{R}\mathbb{C})$. Let M be a vector subspace of the *algebraic* dual E^* of E. If $\mathfrak{B}(\tau)$ is a O-basis for τ, then the sets

$$U \cap (X \in E : |f_i(x)|\langle 1, \quad i = 1, 2, \ldots n).$$

where $U \in \mathfrak{B}(\tau)$, $n \in N$, $f_i \in M$, form a O-basis for the weakest vector topology $\tau[M]$ on E which is finer than τ and which makes the elements of M continuous. Clearly

$$\tau[M] = \sup(\tau, \sigma(E, M)).$$

Moreover $(E, \tau[M])' = E' + M$, where $E' = (E, \tau)'$ denotes the *topological* dual of E, [6], Lemma 1. If dim $M \leqslant \aleph_0$ and $M \cap E' = 0$, then $\tau[M]$ is called a *countable extension* of τ, [6]. In [6] Popoola and Tweddle considered the question of ultrabarrelledness (resp. barrelledness) under the topology $\tau[M]$, when E is given to be ultrabarrelled (resp. barrelled) under its original topology. It is easy to see that for an ultrabarrelled (resp. barrelled) tvs (E, τ) the space $(E, \tau[M])$ is ultrabarrelled (resp. barrelled) for any finite extension $\tau[M]$ of τ, [6]. Theorem 1. In [6]. Theorem 3, it was shown that for any barrelled tvs (E, τ) containing a dense $c = 2^{\aleph_0}$ codimensional barrelled subspace there exists a countable infinite extension $\tau[M]$ of τ under which E is barrelled. In the same paper the authors asked whether this theorem remains true when barrelledness is replaced by ultrabarrelledness.

In the present paper we answer this question in the positive; for *F-spaces* (i. e. metrizable and complete tvs) we obtain even a stronger result. Moreover, we discuss the problem of the existence of dense subspaces with large codimension in ultrabarrelled spaces.

[1]This work was undertaken with the support of the AVH.

A sequence $(U_n)_{n\in N}$ of subsets of E is called a *string* [1] if every U_n is *balanced* and *absorbing* and

$$U_{n+1} + U_{n+1} \subset U_n, \quad n \in N.$$

A string $(U_n)_{n\in N}$ in a tvs (E, τ) is [1]:

(a) *closed*, if every U_n is τ–closed;

(b) *topological*, if every U_n is a τ–neighbourhood of zero.

A tvs (E, τ) is *ultrabarrelled* if *every closed* string in E is topological [1]. The following conditions are equivalent for a tvs (E, τ):

[i]. *(E, τ) is ultrabarrelled*

[ii]. *Every linear map from (E, τ) into an F-space with closed graph is continuous.*

[iii]. *Every Hausdorff vector topology θ on E which is τ-polar, i.e. θ has a O-basis consisting of τ-closed sets, is coarser than τ, [1], p. 32, p. 44.*

–Results: First we prove the following two lemmas;

Lemma 1.

Let (E, τ) be an ultrabarrelled tvs containing a dense c-codimensional ultrabarrelled subspace. Then E admits a countably infinite extension $\tau[M]$ of τ under which E is ultrabarrelled. Moreover, if (E, τ) is metrizable, then there exists a sequence $(\tau[M_n])_{n\in N}$ of countably infinite extensions of τ such that:

[i]. *$(E, \tau[M_n])$ is metrizable and ultrabarrelled, $n \in N$.*

[ii]. $\tau = \inf\{\tau[M_n], \tau[M_m]\}$ *for all $n, m \in N$, $n \neq m$.*

Proof.

Let G be a dense c-condimensional ultrabarrelled subspace of E. let $(x_a)_{\alpha\in A}$ be a Hamel basis of an algebraic complement H to G in E. Consider a partition (A_n) of A such that

$$\text{card } A_n = \text{card } A, \; n \in N, \quad A = \cup(A_n : n \in N).$$

Let

$$F_n = \text{lin}\{x_\alpha : \alpha \in \cup A_i, 1 = 1, 2 \ldots n\}, \quad n \in N.$$

Then

$$E = \cup(G + F_n), \quad F_n \subset F_{n+1}$$

and

$$\text{codim}(G + F_n) = c, \quad n \in N.$$

Clearly every $E_n = G + F_n$ is τ-dense and ultrabarrelled. Let $n = 1$. Choose on $W = E/E_{n_1}$ a (metrizable and ultrabarrelled) vector topology α_1 such that (W, α) is isomorphic to

$$\left(K^N, \sigma(K^N, K^{(N)})\right).$$

If $Q : E \to W$ is the quotient map, then

$$M_1 = \{f \; o \; Q : f \in (W, \alpha_1)'\}$$

is a countably infinite dimensional subspace of E^* and $M \cap E' = 0$. Let τ_1 be the initial vector topology on E with respect to the maps

$$\mathrm{id} : E \to (E, \tau) \quad \text{and} \quad Q : E \to (W, \alpha_1).$$

Then

$$\tau_1 | E_{n_1} = \tau | E_{n_1}, \tau_1 / E_{n_1} = \alpha_1, \; \tau < \tau_1, \; \text{cf.[8]}.$$

The sets

$$\begin{aligned} &U \cap Q^{-1}\{x \in W : |f_i(x)| < 1, \; i = 1, 2 \ldots n\} = \\ &= U \cap \{y \in E : |f_i \; o \; Q(y)| < 1, \; i = 1, 2, \ldots n\}, \end{aligned}$$

where

$$U \in \mathfrak{B}(\tau), \; n \in N, \; f_i \in (W, \alpha_1)',$$

compose a O-basis for τ_1. By [8]. Theorem 2.6 (E, τ_1) is ultrabarrelled (and metrizable when τ is metrizable). Clearly $\tau_1 = \tau[M_1]$. Now assume that (E, τ) is metrizable. Since (E, τ_1) is metrizable and ultrabarrelled, then

$$\bar{E} = \cup(\bar{E}_n : n \in N),$$

where the closure is taken in the completion of $(E, \tau[M_1])$, [1], (11), p. 89 and (7), p. 87. Hence there exists a number $n_2 > n_1$ such that E_{n_2} is $\tau[M_1]$-dense; clearly $(E_{n_2}, \tau | E_{n_2})$ is ultrabarrelled. This enables us to construct on E a countably infinite extension $\tau[M_2]$ of τ such that $(E, \tau[M_2])$ is metrizable and ultrabarrelled and

$$\tau | E_{n_2} = \tau[M_2] | E_{n_2}.$$

By simple induction we construct.

1. $(E_{n_m})_{m \in N}$ a strictly increasing subsequence of $(E_n)_{n \in N}$.
2. $(M_m)_{m \in N}$ a sequence of vector subspaces of E^* such that $E' \cap M_m = 0, \; \dim M_m = \aleph_0$.
3. $(\tau[M_m])_{m \in N}$ a sequence of countably infinite extensions of τ such that

[i]. $(E, \tau[M_m])$ is metrizable and ultrabarrelled;

[ii]. E_{n_m} is $\tau[M_{m-1}]$-dense, $m \geqslant 2$;

[iii]. $\tau[M_m] \,|E_{n_m} = \tau \,|E_{n_m}, \; \tau < \tau[M_m]$.

Fix $p, m \in N, \; p \neq m$. Assume $p < m$. Then $E_{n_{p+1}} \subset E_{n_m}$. Since

$$\tau|E_{n_m} = \inf\{\tau[M_p], \tau[M_m]\}|E_{n_m} = \tau[M_m] \,|E_{n_m}$$

then

$$\tau|E_{n_{p+1}} = \inf\{\tau[M_p], \tau[M_m]\} \,|E_{n_{p+1}}.$$

Since $E_{n_{p+1}}$ is $\tau[M_p]$-dense, then

$$\tau/E_{m_{p+1}} = \inf\{\tau[M_p], \tau[M_m]\}/E_{n_{p+1}} = \tau[M_p]/E_{n_{p+1}}$$

Therefore

$$\tau = \inf\{\tau[M_p], \tau[M_m]\},$$

by [3], Lemma 1.

Lemma 2.

Let (E, τ) be the inductive limit space of an increasing sequence $(E_n, \tau_n)_{n \in N}$ of F-spaces. If $E' \neq E^*$, then E contains a dense c-codimensional ultrabarrelled subspace.

Proof.

If $E^* \neq E'$, then for some $n \in N$ dim $E_n > \aleph_0$. Hence dim $E_n \leqslant c$, [5]. Without loss of generality we may assume that $n = 1$. Using the Baire category theorem one shows that (E_1, τ_1) contains a c-codimensional dense ultrabarrelled subspace L. Let F be an algebraic complement to E_1 in E. Then $W = F + L$ is τ-dense. Let $(U_n)_{n \in N}$ be a closed string in $(W, \tau|W)$. Then $(\bar{U}_n \cap L)_{n \in N}$ is topological in $(L, \tau_1|L)$, where the closure is taken in τ. Since L is dense in E_1 and (E_1, τ_1) is ultrabarrelled, then $(\bar{U}_n \cap E_1)_{n \in N}$ is topological in (E_1, τ_1). Let $x \in E$, then $x = y + z, \; y \in F, \; z \in E_1$. Fix $n \in N$. There exist $l_n \in L$ and $\lambda_n > 1, \; \lambda_n \in K$, such that

$$l_n - z \in \bar{U}_{n+1} \cap E_1 \quad \text{and} \quad l_n + y \in \lambda_n \bar{U}_{n+1}.$$

Hence $x \in \lambda_n \bar{U}_n$. By the ultrabarrelledness of (E, τ) we derive that $U_n = \bar{U}_n \cap W$ is a $\tau|W$-neighbourhood of zero for all $n \in N$.

Using Lemmas 1 and 2 we note the following.

Proposition 1.

Let (E, τ) be an infinite dimensional F-space (or metrizable (LF)-space). Then there exists a sequence $(M_n)_{n \in N}$ of vector subspaces of E^* such that for every $n \in N$:

(i) $M_n \cap M_m = 0, \quad \dim M_n = \aleph_0, \quad E' \cap M_n = 0, \quad n \neq m.$

(ii) $(E, \sup\{\tau, \sigma(E, M_n)\}$ is metrizable and ultrabarrelled.

(iii) $\sup\{\tau, \sigma(E, M_n)\}$ and $\sup\{\tau, \sigma(E, M_m)\}$ are not comparable, $n \neq m$.

Corollary 1.

Let (E, τ) be an infinite dimensional F-space. Then

$$\tau = \inf\{\tau[M] : M \subset E^*, \ M \cap E' = 0\}.$$

In [7] (cf. also [4]) it was shown that every locally convex barrelled space E with $E^* \neq E'$ contains a dense denumerable-codimensional subspace, necessarily barrelled by [10]. We extend this result to ultrabarrelled spaces. The proof of the Proposition 2 uses some ideas found in [7].

Proposition 2.

Every ultrabarrelled tvs (E, τ) with $E' \neq E^*$ contains a dense infinite-codimensional subspace.

Proof.

Let $\{(x_i)_{i \in B}\}$ be a Hammel basis for E and $\{(x_i, f_i)_{i \in B}\}$ the corresponding biorthogonal system. Assume that $D = \{i \in B : f_i \notin E'\}$ is finite. Then $Z = \text{lin}\{x_i : i \in B \setminus D\}$ is an ultrabarrelled finite-codimensional subspace of E, [2]. Clearly Z' separates points of Z. Let $\mu(Z, Z')$ be the finest locally convex topology on Z weaker than $\tau|Z$. The convex $\tau|Z$-neighbourhoods of zero compose a 0-basis for $\mu(Z, Z')$. Hence $(Z, \mu(Z, Z'))$ is a Hausdorff barrelled space. By Theorem 1 of [7] $Z' = Z^*$. Let φ be the finest vector topology on Z. Then $\mu(Z, Z') \leqslant \tau|Z \leqslant \varphi$. By [9], 6.8 (e), the topology φ is $\mu(Z, Z^*)$-polar. Hence φ is $\tau|Z$ *– polar*. Since $\tau|Z$ is ultrabarrelled, then $\varphi = \tau|Z$ (cf. Introduction). Therefore Z is τ-closed and hence $E' = E^*$, a contradiction. This proves that D is infinite. Let

$$M = \text{lin}\{x_i : i \in B \setminus I\},$$

where I is a countably infinite subset of D. By $\bar{M}$ and H we denote the closure of M in τ and an algebraic complement to $\bar{M}$ in E, respectively. Assume that $\dim(\bar{M}/M) < \aleph_0$. Then $M + F = \bar{M} + F$, where

$$F = \text{lin}\{x_i : i \in T\}$$

for some finite subset T of I. Then

$$(E, \tau) = (M + F) + N$$

(topologically), where

$$N = \text{lin}\{x_i : i \in I \backslash T\},$$

[2], (4), p. 29. Hence every f_i, $i \in I \backslash T$, belongs to E', a contradiction with $I \subset D$. Therefore $M + H$ is a dense infinite-codimensional subspace in (E, τ).

Corollary 2.

Let E be an ultrabarrelled tvs. Then E has its finest vector topology if and only if every subspace of E has infinite codimension in its closure.

REFERENCES

[1] N. ADASCH, B. ERNST, D. KEIM.: *Topological vector spaces,* Springer-Verlag, Berlin 1978.

[2] N. ADASCH, B. ERNST.: *Teilräume gewisser topologischer Vektorräume,* Collect. Math. 24(1073), 27-39.

[3] S. DIEROLF, U.SCHWANENGEL.: *Examples of locally compact non-compact minimal topological groups,* Pacif. J. Math. 82(1979), 349-355.

[4] S. DIEROLF.: *Über Verabbarkeitseingeschaften in topologischen Vektorräumen, Dissertation, München 1974.*

[5] J. O. POPOOLA, I. TWEDDLE.: *On the dimension of a complete metrizable topological vector space,* Canad. Math. Bull. 20(1977), 271-2.

[6] J. O. POPOOLA, I. TWEDDLE.: *Stability of barrelledness and related concepts in topological vector spaces,* Proc. Edinburgh Math. Soc. 27(1984), 321-325.

[7] W. J. ROBERTSON, S. A. SAXON, A. P. ROBERTSON.: *Barrelled spaces and dense vector subspaces,* Bull. Austr. Math. Soc. 37(1988), 383-388.

[8] W. ROELCKE, S. DIEROLF.: *On the three-space-problem for topological vector spaces,* Collect. Math. 32(1981), 3-25.

[9] W. ROELCKE.: *Topologische Vektoräume I, Vorlesung.* München 1976.

[10] S. A. SAXON, M. LEVIN.: *Every countable-codimensional subspace of a barrelled space is barrelled.* Proc. Amer. Math. Soc. 29(1971), 91-96.

Institute of Mathematics
A. Mickiewicz University
60-769 Poznan
Matejki 48/49
Poland

Mathematisches Institut
der Universität
D-8000 München 2
Theresienstr. 39
Bundesrepublik Deutschland