# *On the structure of a vector measure* [1]

BY BALTASAR RODRIGUEZ-SALINAS

Recibido: 9 de mayo de 1990

*Académico Numerario*

## Abstract

The Rosenthal's lemma and the Diestel-Faires theorem are generalized. As a consequence the subspace of $l_\infty(I)$ are studied and it is proved that this space is primary when $I$ as an infinite set.

We will denote by $z(I)$ the subspace of $l_\infty(I)$ formed by the elements having countable support. A characterización of Banach spaces containing an isomorphic copy of $l_\infty(I)$ and $z(I)$ is stablished. Clearly it follows from the Hahn-Banach theorem that such copies of $l_\infty(I)$ are complemented subspaces.

## 1. Lemma

*(Generalization of the Rosenthal's lemma). Let $\Sigma$ be a $\sigma$-algebra of subsets of a set $\Omega$. Let $(\mu_i)_{i\in I}$ be a uniformly bounded infinite family of finite additive scalar-valued measures defined on $\Sigma$. Then, if $(E_i)_{i\in I}$ is a disjoint family of members of $\Sigma$ and $\epsilon > 0$, there is a sub-family $(E_i)_{i\in J}$ of $(E_i)_{i\in I}$ such that* card $J =$ card $I$ *and*

$$|\mu_i|\left(\bigcup_{\substack{j\in\Delta\\ j\neq i}} E_j\right) < \epsilon$$

*for all countable subsets $\Delta$ of $J$ and all $i \in J$.*

*If $\Sigma = 2^\Omega$ then the family $(E_i)_{i\in J}$ may be chosen such that*

$$|\mu_i|\left(\bigcup_{\substack{j\in\Delta\\ j\neq i}} E_j\right) < \epsilon$$

*for all $i \in J$.*

*If $\Sigma$ is an algebra the first result holds for all finite subset $\Delta$ of $J$.*

[1] Supported in part by DGICYT PB88-0141.

**Proof.**

Being card $(I \times I) = \operatorname{card} I$ we can proved like in [1] taking a partition of $I$ formed by a disjoint family $(M_i)_{i \in I}$ of subsets such

$$\bigcup_{i \in I} M_i = I \quad \text{and} \quad \operatorname{card} M_i = \operatorname{card} I$$

for all $i \in I$, and remarking that for every $\mu_i$ there exists an extension $\tilde{\mu}_i$ on $2^\Omega$ such that $\|\tilde{\mu}_i\| = \|\mu_i\|$ and

$$|\mu_i|(A) \leqslant |\tilde{\mu}_i|(A) \quad \text{for all} \quad A \in \Sigma.$$

(See [3]).

## 2. Theorem

*(Generalization of the Diestel-Faires theorem). Let $\Sigma$ be $\sigma$-algebra of subsets of a set $\Omega$ and $G : \Sigma \longrightarrow X$ be a Banach valued bounded measure. If $(E_i)_{i \in I}$ is a non finite and disjoint family of sets $E_i \in \Sigma$ such that*

$$\|G(E_i)\| > \epsilon \quad (\epsilon > 0)$$

*for every $i \in I$, then there exists an isomorphism $T : z(I) \longrightarrow X$ and a subfamily $(E_i)_{i \in J}$ of $(E_i)_{i \in I}$ with* card $J =$ card $I$ *such that*

$$T(e_A) = G(E_{j(A)})$$

*for every subset $A$ of $I$ with $\chi_A \in z(I)$ (in this case $A$ is countable), being*

$$e_A = \chi_A (= 1_A)$$

$$E_{j(A)} = \sum_{i \in A} E_{j(i)}$$

*and $j : I \longrightarrow J$ a bijection.*

*In the case of being $\Sigma = 2^\Omega$, the last result holds also if $z(I)$ is replaced by $l_\infty(I)$.*

*If $\Sigma$ is only an algebra then the result holds only if $z(I)$ is replaced by $c_0(I)$.*

**Proof.**

We can proceed like in [1] using the Lemma 1.

## 3. Definition

Let $J$ be a set, we say that a set $I$ is *J-primary* if

$$I = \bigcup_{j \in J} I_j$$

implies the existence of some $I_j$ having the same cardinal than $I$.

Whatever the infinite set $J$ be, there exists $J$-primary sets. In fact, if

$$\alpha \geqslant \operatorname{card} J \quad \text{and} \quad \operatorname{card} I = \alpha + 1,$$

we have that $I$ is a $J$-primary set.

**Remark.**

If the set $I$ is $N$-primary then the condition

$$\|G(E_i)\| > \epsilon$$

can be substituted by $G(E_i) \neq 0$. In fact, there exists $n \in N$ such that the set

$$\{i \in I : \|G(E_i)\| > 1/n\}$$

has the same cardinal than $I$.

If

$$I = \bigcup_{n \in N} I_n$$

with $\operatorname{card} I_n < \operatorname{card} I$ for every $n \in N$, then the vector measure

$$G : 2^I \to l_\infty(I)$$

defined by

$$\begin{aligned} G(A)(i) &= 1/n \quad \text{if} \quad i \in A \cap \left(I_n - \bigcup_{k=1}^{n-1} I_k\right) \\ G(A)(i) &= 0 \quad \text{if} \quad i \notin A. \end{aligned}$$

verifies that $G(\{i\}) \neq 0$ for every $i \in I$ and, nevertheless, there is not any disjoint family $(E_i)_{i \in I}$ of subsets of $I$ verifying that

$$\|G(E_i)\| > \epsilon$$

for some $\epsilon > 0$ and every $i \in I$.

## 4. Corollary

*Let $I$ be a $N$-primary set and $\Gamma$ a total subset of the dual space $X^*$ of the Banach space $X$. If $\Sigma_{i\in I} x_i$ is a (formal) series in $X$ with $x_i \neq 0$ for every $i \in I$, and such that every countable subseries is $\Gamma$-convergent, in the sense that for every countable subset $A$ of $I$ there exists $x_A \in X$ such that*

$$\sum_{i\in A} x^* x_i = x^* x_A$$

*for every $x^* \in \Gamma$, then the space $X$ contains an isomorphic copy of $z(I)$.*

*If in addition every subseries of $\Sigma_{i\in I} x_i$ is $\Gamma$-convergent, then $X$ contains an isomorphic copy of $l_\infty(I)$.*

**Proof.**

If we define $G : \Sigma \to X$ such that $G(A) = x_A$ for every $A \in \Sigma$, being $\Sigma$ the $\sigma$-ring of the countable subsets of $I$ or $2^I$, we can apply the Theorem 2 since it follows from the Dieudonné-Grothendieck theorem that the vector measure $G$ is bounded.[2]

## 5. Corollary

*Let $I$ be an $N$-primary set. If $\Sigma_{i\in I} f_i$ is a (formal) series in $C(K)$ of functions $f_i \neq 0$ such that every countable subseries is pointwise convergent on a dense subset $H$ of $K$, to some function of $C(K)$, then $C(K)$ contains an isomorphic copy of $z(I)$.*

*If in addition every subseries of $\Sigma_{i\in I} f_i$ is pointwise convergent on $H$ to some function of $C(K)$, then $C(K)$ contains an isomorphic copy of $l_\infty(I)$.*

**Remark.**

If $G : \Sigma \to X$ is a bounded vector measure (for instance with $X = C(K)$) and $\mu : \Sigma \to R$ is a non negative finetely additive measure, then

$$G_x(A) = G(A) + \mu(A)x \quad (A \in \Sigma,\ x \in X)$$

defines a bounded vector measure $G_x : \Sigma \to X$. If $(E_i)_{i\in I}$ is a disjoint family of non void sets $E_i \in \Sigma$ the measure $\mu$ can be chosen such that $\mu(E_i) = 0$ for every $i \in I$ and

$$\mu\left(\bigcup_{i\in J} E_i\right) = 1$$

for some countable subset $J \subset I$. Therefore, there exists vector measures $G_x \neq G$ such that

[2]Theorem 2 holds for measures defined on a $\sigma$-ring $\Sigma$ since they can be extended to a $\sigma$-algebre taking $G(A) = -G(A^c)$ when $A^c \in \Sigma$ if $\Omega \notin \Sigma$.

$$G_x(E_i) = G(E_i)$$

for every $i \in I$.

## 6. Definition

A family $(X_i)_{i\in I}$ of complemented subspaces of a Banach space $X$ is said to be a *projective representation* of $X$ when $P_i x = 0$ for every $i \in I$ implies that $x = 0$, being $P_i : X \to X_i$ the proyection associated to $X_i$. An important particular case appears when $P_i P_j = 0$ if $i \neq j$.

A Banach space $X$ is said to be *primary* if for every direct sum $Y \oplus Z = X$ we have that $X$ is isomorphic to $Y$ or $Z$. In a similar way, if $J$ is a set, we say that the Banach space $X$ is *J-primary* if for every projective representation $(X_j)_{j\in J}$ of $X$ there is an $X_j$ isomorphic to $X$.

## 7. Theorem

*Let be $X = l_\infty(I)$ or $X = z(I)$, where $I$ is a $J$-primary set and $J$ is an infinite set. If $(X_j)_{j\in J}$ is a projective representation of $X$, then there exists an $X_j$ containing an isomorphic copy of $X$.*

*Therefore, if $X = l_\infty(I)$, this $X_j$ is isomorphic to $X$ and $l_\infty(I)$ is a $J$-primary Banach space.*

**Proof.**

Let us suppose that $X = l_\infty(I)$. Then there exists a bounded vector measure $G : 2^I \to X$ and a disjoint family $(E_i)_{i\in I}$ in $2^I$ such that $\|G(E_i)\| > \epsilon$ for every $i \in I$ and some $\epsilon > 0$. Let be $G_j = P_j G$, being $P_j$ the projection associated to $X_j$, then there exists an $X_j$ such that

$$I_j = \{i \in I : \; G_j(E_i) \neq 0\}$$

has the same cardinal than $I$. Therefore, since $G_j : 2^I \to X_j$ is a bounded measure, it follows from the Theorem 2 that the space $X_j$ has a subspace which is isomorphic to $l_\infty(I)$ and, therefore, complemented (in $X_j$).

Being $X = X_j \oplus Y$ and $X_j \approx X \oplus Z$ it follows from Pelczyński's decomposition method that

$$X \oplus X_j \approx X \oplus (X \oplus Z) \approx (X \oplus X) \oplus Z \approx X \oplus Z \approx X_j$$

and

$$\begin{aligned} X \oplus X_j &\approx (X \oplus X \oplus \ldots)_\infty \oplus X_j \approx ((Y \oplus X_j) \oplus (Y \oplus X_j) \oplus \ldots)_\infty \oplus X_j \\ &\approx (Y \oplus Y \oplus \ldots)_\infty \oplus (X_j \oplus X_j \oplus \ldots)_\infty \oplus X_j \\ &\approx (Y \oplus Y \oplus \ldots)_\infty \oplus (X_j \oplus X_j \oplus \ldots)_\infty \end{aligned}$$

$$\begin{aligned} &\approx ((Y \oplus X_j) \oplus (Y \oplus X_j) \oplus \ldots)_\infty \\ &\approx X \end{aligned}$$

and, therefore, $X_j \approx X$.
In a similar way we can proceed when $X = z(I)$ (or $X = c_o(I)$) for the first part.

**Remark.**

If $I$ is not $J$-primary the Theorem 7 does not hold since $(l_\infty(I_j))_{j \in J}$ is a proyective representation of $l_\infty(I)$ if

$$I = \bigcup_{j \in J} I_j.$$

The same happens for $X = z(I)$.

## 8. Theorem

*If $I$ is an infinite set then $l_\infty(I)$ is a primary space.*

**Proof.**

It is enough to proceed like in the Theorem 7 with some evident modifications.

**Remark.**

The Theorem 2 can also be used for proving that $l_\infty$ is a prime space, in particular for proving the Lemma 2.a.8 of [2].

## REFERENCES

[1] DIESTEL, J. AND UHL, J. J. JR.: *Vector Measures.* A.M.S. Providence, R.I., 1977.

[2] LINDENSTRAUSS, J. AND TZAFRIRI, L.: *Classical Banach Spaces, I.* Springer, Berlin Heidelberg New York, 1977.

[3] ROSENTHAL, H. P.: *On relatively disjoint families of measures, with some applications in Banach space theory.* Studia Math. 37 (1970).