Resolution of identity in certain metrizable locally convex spaces

By M. Valdivia(*)

Recibido: 7 marzo 1989

Académico numerario

Abstract

In this article projectional resolutions of identity are constructed in a class of metrizable locally convex spaces that contains the weakly countably determined Fréchet spaces. It is also proved that if a Fréchet space E has $E''\,[\beta\,(E'', E')]/E$ separable then E is the topological direct sum of two subspaces F and G with $G''\,[\beta\,(G'', G')]$ separable and F reflexive.

Resumen

En este artículo se demuestra que existen resoluciones proyectivas del operador identidad en los espacios localmente convexos metrizables de una amplia clase que contiene a los espacios de Fréchet débil numerablemente determinados.

Se obtiene, como aplicación, que si E es un espacio de Fréchet tal que $E''\,[\beta\,(E'', E')]/E$ es separable entonces E es la suma directa topológica de dos espacios de Fréchet F y G, de manera que F es reflexivo y $G''\,[\beta\,(G'', G')]$ es separable.

1. INTRODUCTION AND NOTATIONS

All the topological spaces considered here will be Haudorff and completely regular topological spaces.

If X is a topological space $\{X\}^*$ denotes its Stone-Cech compactification. It is said for a sequence of subsets (A_n) of X that determines X if given any x in X there is a subsequence (A_{n_i}) of (A_n) such that

$$x \in \bigcap_{j=1}^{\infty} B_j \subset X,$$

where B_j denotes the closure of A_{n_j} in $\{X\}^*$, $j = 1, 2, \ldots$ It is said that X is countable determined if there is some sequence of subsets of it that determines it. It is said that X is K-analitic if it can be obtained through the Suslin operation with closed subsets of $\{X\}^*$. Obviously every K-analytic space is countably determined. If X is the continuous image of a K-analytic space it

(*) Supported in part by CAICYT.

is also K-analytic; every closed subspace of K-analytic space is K-analytic too, and the countable product of K-analytic space with the product topology is K-analytic, [2].

If Y is a subset of a topological space X and $\mathscr{T}$ is the topology of X, $Y[\mathscr{T}]$ denotes the topological space Y with the topology induced by $\mathscr{T}$.

N will be the set of positive integers. If A is a set, $|A|$ denotes its cardinal number. The first infinite ordinal will be ω. If α is an ordinal number, $|\alpha|$ denotes its cardinal number; particularly $|\omega| = \aleph_0$. If X is a topological space, we set $d(X)$ to denote the density character of X, we mean the first cardinal number such that there is a dense subset A of X with $d(X) = |A|$.

All of the vectors spaces we shall use here are defined over the field K of real or complex numbers; if K is real, H denotes the field of rational numbers, and if K is complex, H denotes the subfield of numbers $a+bi$ with a and b rational numbers.

Given a dual pair $\langle E, F\rangle$ of vector spaces with the bilinear from $\langle .,. \rangle$, we write $\sigma(E, F)$, $\mu(E, F)$ and $\beta(E, F)$ to denote the weak, Mackey and strong topologies on E, respectively.

If E is a locally convex space, E' is its topological dual and E'' its bidual, i.e., E'' is the topological dual of $E'[\beta(E', E)]$. $\check{E}$ denotes the algebraic dual of E'. We identify, as usually, E with a subspace of $\check{E}$. If A is a subset of E, $\tilde{A}$, $\check{A}$ and A^* are the closures of A in $E''[\sigma(E'', E')]$, $\check{E}[\sigma(\check{E}, E')]$ and $\{E[\sigma(E, E')]\}^*$, respectively; $\hat{A}$, and also $(A)\hat{}$, is the closure of A in the completion $\hat{E}$ of E. A^0 is the polar set of A in E', and $A^{\perp}$ is the subspace of E' which is orthogonal to A. If x belongs to E and u to E', we write $\langle x, u\rangle$ instead of $u(x)$. If P is a continuous projection in E, P' is the adjoint projection in E'.

If a locally convex space E is such that $E[\sigma(E, E')]$ is K-analytic (resp. countably determined) we say that E is weakly K-analytic (resp. weakly countably determined). It is easy to see that every reflexive Frèchet space is weakly K-analytic.

Let E be a metrizable locally convex space of infinite dimension. Let $\|.\|_n$, $n = 1, 2, \ldots$, be a family of continuous seminorms on E that defines its topology. Let μ be the first ordinal with $|\mu| = d(E)$. A resolution of identity in E, associated with $\|.\|_n$, $n = 1, 2, \ldots$, is a family

$$\{P_\alpha\colon \omega \leqslant \alpha \leqslant \mu\}$$

of continuous projections in E such that, for $\omega \leqslant \alpha \leqslant \beta \leqslant \mu$, the following conditions are satisfied:

1) $\|P_\alpha\|_n = 1, \quad n = 1, 2, \ldots,$

2) $P_\alpha \circ P_\beta = P_\alpha = P_\beta \circ P_\alpha,$

3) $d(P_\alpha(E)) \leqslant |\alpha|,$

4) P_μ is the identity in E,

5) if α is a limit ordinal, $\omega < \alpha$, the closure of

$$\bigcup \{P_\eta (E): \omega \leqslant \eta < \alpha\}$$

in E coincides with $P_\alpha (E)$.

If $(X, \|.\|)$ is a weakly countably determined Banach space, L. Vasák proves in [11] that there is a resolution of identity in X, associated with $\|.\|$. This result extends a previous one of D. Amir and J. Lindenstrauss for weakly compactly generated Banach spaces. In this paper we obtain resolutions of identity in a class of metrizable locally convex spaces that contains the weakly countably determined Fréchet spaces. On the other hand, our method to construct the projections applied to Banach spaces drastically simplifies the Amir-Lindenstrauss-Vasák way of doing it.

We shall need later the following two results that we have proved in [7] and [8], respectively:

a) Let E be a Fréchet space. Let F be a subspace of E'' such that $E'' = E + F$ and $F[\beta(E'', E')]$ is separable. If x is a point of E'' there is a sequence (x_n) in E which converges to x in $E''[\sigma(E'', E')]$.
b) If F is a closed subspace of a Fréchet space E, then $E + \tilde{F}$ is closed in $E''[\beta(E'', E')]$.

2. WEAKLY COUNTABLY DETERMINED METRIZABLE LOCALLY CONVEX SPACES

Let (A_n) be a sequence of subsets in a locally space E. We shall say that (A_n) is quasi-bounded if it is a decreasing sequence such that for every neighbourhood of the origin U in E there is a positive integer m such that

$$A_m \subset mU.$$

Proposition 1.– Let (A_n) be a quasi-bounded sequence in a locally convex space E. If

$$\bigcap_{n=1}^{\infty} \tilde{A}_n \subset E$$

then

$$\bigcap_{n=1}^{\infty} \tilde{A}_n = \bigcap_{n=1}^{\infty} \breve{A}_n.$$

Proof.– It is clear that

$$A := \bigcap_{n=1}^{\infty} \tilde{A}_n$$

is a weakly compact subset in E. Let us take a point x in $\check{E}$ which is not in A. Let V be a closed neighbourhood of x in $\check{E}[\sigma(\check{E}, E')]$ which does not meet A. Let us suppose that there are vectors

$$x_n \in A_n \cap V, \qquad n = 1, 2, \dots$$

The sequence (x_n) is bounded in E, so it has a cluster point x_0 in $E''[\sigma(E'', E')]$. Obviously,

$$x_0 \in \tilde{A}_n, \qquad n = 1, 2, \dots,$$

from where it follows that

$$x_0 \in A \cap V$$

which is a contradiction. Therefore, there is a positive integer m with

$$A_m \cap V = \emptyset.$$

Thus,

$$x \notin \check{A}_m$$

and we have

$$\bigcap_{n=1}^{\infty} \tilde{A}_n = \bigcap_{n=1}^{\infty} \check{A}_n.$$

q.e.d.

Proposition 2.– If (A_n) is a decreasing sequence of subsets in a locally convex space E, the following are equivalent:

1) (A_n) is a quasi-bounded sequence and $\bigcap_{n=1}^{\infty} \tilde{A}_n \subset E$.

2) $\bigcap_{n=1}^{\infty} A_n^* \subset E$.

Proof.– Let us take a point x in $\bigcap_{n=1}^{\infty} A_m^* \left(\bigcap_{n=1}^{\infty} \tilde{A}_n\right)$. We set

$$\{V_j : j \in J\}$$

to denote a neighbourhood basis of x in $\{E[\sigma(E, E')]\}^*$ $(E''[\sigma(E'', E')])$. For the product $N \times J$ we introduce the order relation:

$$(n_1, j_1),\ (n_2, j_2) \in N \times J,$$
$$(n_1, j_1) \leqslant (n_2, j_2) \quad \text{if} \quad n_1 \leqslant n_2 \quad \text{and} \quad V_{j_1} \supset V_{j_2}.$$

We choose for every $(n, j) \in N \times J$ a vector

$$x(n, j) \in A_n \cap V_j.$$

Obviously, the net

$$\{x(n,j)\colon (n,j)\in N\times J, \leqslant\} \tag{1}$$

converges to x in $\{E[\sigma(E,E')]\}^*$ $(E''[\sigma(E'',E')])$.

Firstly let us suppose that 1) holds.

Given any u of E', we could find a neighbourhood of the origin U in E such that

$$\sup\{|\langle y,u\rangle|\colon y\in U\}\leqslant 1.$$

The quasi-boundedness of (A_n) give us a positive integer m such that

$$A_m\subset mU.$$

We set $u=u_1+iu_2$ with u_1 and u_2 real forms. For $k=1,2$, $y\in E$, we write:

$$w_k(y)=\begin{cases} m, & \text{if } u_k(y)>m,\\ u_k(y), & \text{if } -m\leqslant u_k(y)\leqslant m,\\ -m, & \text{if } u_k(y)<-m.\end{cases}$$

Let us take x in $\bigcap_{n=1}^{\infty} A_n^*$ and a neighbourhood basis $\{V_j\colon j\in J\}$ of x in $\{E[\sigma(E,E')]\}^*$. Then, the net (1) converges to x in that space, hence, given any $\varepsilon>0$, we could find $(n_0,j_0)\in N\times J$ with $n_0>m$ such that

$$|w_k(x(n_1,j_1))-w_k(x(n_2,j_2))|<\frac{\varepsilon}{2}, \qquad k=1,2,$$

$$(n_1,j_1),\ (n_2,j_2)\in N\times J, \qquad (n_1,j_1),\ (n_2,j_2)\geqslant(n_0,j_0).$$

Obviously,

$$u_k(x(n_h,j_h))=w_k(x(n_h,j_h)), \qquad h,k=1,2,$$

and therefore

$$|u(x(n_1,j_1)-x(n_2,j_2))|\leqslant$$
$$\leqslant |u_1(x(n_1,j_1))-u_2(x(n_2,j_2))|+|u_2(x(n_1,j_1)-u_2(x(n_2,j_2))|<$$
$$<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

from where we see that (1) is a Cauchy net in $E[\sigma(E,E')]$, so it must be $E(\check{E},E')$-convergent to a point z of $\check{A}_n$, $n=1,2,\ldots$ If we apply the former proposition we obtain that z belongs to E and thus $z=x$.

Conversely, let us now suppose that 2) holds. If x belongs to $\bigcap_{n=1}^{\infty}\tilde{A}_n$

and $\{V_j: j \in J\}$ is a neighbourhood basis of x in $E''[\sigma(E'', E')]$, then the net (1) converges to x in that space. Oviously, this net has a cluster point z in $\{E[\sigma(E, E')]\}^*$ that belongs to A_n^*, $n = 1, 2, \ldots$, from where we know that z belongs to E and, consequently, $z = x$. Therefore we have

$$\bigcap_{n=1}^{\infty} \tilde{A}_n \subset E.$$

On the other hand, let us suppose that (A_n) is not a quasi-bounded sequence in E and so we have a neighbourhood of the origin U in E such that

$$A_m \notin mU, \qquad m = 1, 2, \ldots$$

We chosse

$$x_m \in A_m, \qquad x_m \notin mU, \qquad m = 1, 2, \ldots$$

(x_m) is an unbounded sequence in $E[\sigma(E, E')]$ and therefore there is some u in E' such that

$$\sup \{|\langle x_m, u\rangle|: m = 1, 2, \ldots\} = \infty$$

Taking a subsequence of (x_m), that we follow denoting it by (x_m), we have

$$|\langle x_m, u\rangle| > m + 1, \qquad m = 1, 2, \ldots$$

Let y be an element of $\bigcap_{n=1}^{\infty} A_n^*$ that is a cluster point of (x_n). We know from 2) that y belongs to E. Let us write

$$V := \{t \in E: |\langle t, u\rangle| < |\langle y, u\rangle| + 1\}$$

which is a $\sigma(E, E')$-neighbourhood of y from where we obtain a positive integer $p > |\langle y, u\rangle|$ such that x_p belongs to V. Nevertheless,

$$|\langle y, u\rangle| + 1 > |\langle x_p, u\rangle| > p + 1 > |\langle y, u\rangle| + 1,$$

which is a contradiction to finish the proof.

q.e.d.

Theorem 1.– If E is a metrizable locally convex space the following are equivalent:

1) E is weakly countably determined.
2) There is a sequence (U_n) of neighbourhoods of the origin in E such that, for every x in E, there is a quasi-bounded subsequence (U_{n_j}) of (U_n) with

$$x \in \bigcap_{j=1}^{\infty} \tilde{U}_{n_j} \subset E.$$

Proof.– 1) $\Rightarrow$ 2). Let (A_n) be a sequence of subsets in E that determines $E\,[(\sigma\,(E, E')]$. Without lose of generality we can suppose that (A_n) is closed by finite intersection of its elements and that the origin belongs to A_n, $n = 1, 2, \ldots$ Let

$$V_1 \supset V_2 \supset \ldots \supset V_n \supset \ldots$$

be a fundamental system of neighbourhoods of the origin in E. We set

$$U_n := A_n + V_n, \qquad n = 1, 2, \ldots$$

If x belongs to E, there is a subsequence (B_n) of (A_n) such that

$$x \in \bigcap_{n=1}^{\infty} B_n^* \subset E.$$

So we have a subsequence (A_{n_j}) of (A_n) such that

$$A_{n_j} = \bigcap_{n=1}^{m_j} B_n$$

where (m_j) is a strictly increasing sequence of positive integers. In accordance with the former proposition, (A_{n_j}) is a quasi-bounded sequence and

$$x \in A := \bigcap_{j=1}^{\infty} \tilde{A}_{n_j} \subset E.$$

If is immediate that (U_{n_j}) is a quasi-bounded sequence. Let z be any point of E'' that does not belong to A. Let W be a closed neighbourhood of z in $E''\,[\sigma\,(E'', E')]$ such that

$$W \cap \left(\bigcap_{j=1}^{\infty} A_{n_j} \right) = \emptyset$$

Let us suppose that there are vectors

$$x_j \in W \cap U_{n_j}, \qquad j = 1, 2, \ldots$$

We write

$$x_j = y_j + z_j, \qquad y_j \in A_{n_j}, \qquad z_j \in V_{n_j}, \qquad j = 1, 2, \ldots$$

Since (z_j) converges to the origin in E and (y_j) is a bounded sequence, we have a cluster point x_0 in $E''[\sigma(E'', E')]$ of the sequence (x_j) which is a cluster point of (y_j) also. It now follows that

$$x_0 \in W \cap \left(\bigcap_{j=1}^{\infty} \tilde{A}_{n_j} \right) = \emptyset$$

Which is a contradiction. So we have now

$$x \in \bigcap_{j=1}^{\infty} \tilde{U}_{n_j} \subset E$$

2) $\Rightarrow$ 1). Let (U_n) be a sequence of neighbourhoods of the origin in E that verifies condition 2). Given any x in E, we find a quasi-bounded subsequence (U_{n_j}) of (U_n) such that

$$x \in \bigcap_{j=1}^{\infty} \tilde{U}_{n_j} \subset E.$$

Then, after the former proposition, we know that

$$x \in \bigcap_{j=1}^{\infty} U^*_{n_j} \subset E$$

from where it follows that (U_n) determines the space $E[\sigma(E, E')]$.

q.e.d.

Proposition 3.– Let (A_n) be a quasi-bounded sequence in a locally convex space E such that

$$\bigcap_{j=1}^{\infty} \tilde{A}_{n_j} \subset E.$$

Let B_n be the absolutely convex cover of A_n, $n = 1, 2, \ldots$ If A is the absolutely convex cover of $\bigcap_{j=1}^{\infty} \tilde{A}_{n_j}$ in $E''[\sigma(E'', E')]$ then

$$A = \bigcap_{n=1}^{\infty} \tilde{B}_n.$$

Proof.– Let us take a point x in E'' that does not belong to A. Since A is $\sigma(E'', E')$-compact we deduce that there is an element u in E' and a number $k < 1$ such that

$$\langle x, u\rangle = 1, \qquad \sup\{|\langle y, u\rangle|:\ y\in A\} < k.$$

Let us write

$$V := \{z\in E'':\ |\langle z, u\rangle| \geqslant k\}$$

and suppose we have a sequence

$$x_n \in A_n \cap V, \qquad n = 1, 2, \ldots$$

From the quasi-boundedness of (A_n) *in* E we know that (x_n) is bounded in E and therefore it has a cluster point x_0 that belongs to A and V, we mean

$$x_0 \in A, \qquad |\langle x_0, u\rangle| \geqslant k$$

which is a contradiction. Consequently, there is a positive integer n_0 such that

$$A_{n_0} \cap V = \emptyset$$

and thus

$$x \notin \tilde{B}_{n_0}$$

from where it follows that

$$A = \bigcap_{n=1}^{\infty} \tilde{B}_n.$$

q.e.d.

Theorem 2.– If E is a Fréchet space, the following conditions are equivalent:

1) E is weakly countably determined.
2) The is a sequence (U_n) of absolutely convex neighbourhoods of the origin in E such that, for every x in E, there is a quasi-bounded subsequence (U_{n_j}) of (U_n) with

$$x \in \bigcap_{n=1}^{\infty} \tilde{U}_{n_j} \subset E.$$

Proof.– It is a straight consequence of Theorem 1, Proposition 3 together with Krein's theorem [6, p. 325].

q.e.d.

Our Theorem 2 suggest the following

Definition.– A metrizable locally convex space E is weakly countably convex-determined if there is a sequence (U_n) of absolutely convex neighbourhoods of the origin such that, for every x in E, there is a quasi-bounded subsequence (U_{n_j}) of (U_n) such that

$$x \in \bigcap_{j=1}^{\infty} \tilde{U}_{n_j} \subset E.$$

Proposition 4.– Let E be a Fréchet space. If there is a metrizable locally convex topology $\mathscr{I}$ on E'' which is coarset than $\mu(E'', E')$ then $E''[\mathscr{I}]$ is weakly countably convex-determined.

Proof.– Let

$$U_1 \supset U_2 \supset \ldots \supset U_n \supset \ldots$$

be a fundamental system of neighbourhoods of the origin in E that we take absolutely convex and closed. Let us consider the family of subsets of E''

$$(m_1 \tilde{U}_1) \cap (m_2 \tilde{U}_2) \cap \ldots \cap (m_p \tilde{U}_p)$$

where $p, m_1, m_2, \ldots, m_p$ are positive integers, and we order it in a sequence (V_r). We set Z_r to denote the closure of V_r in

$$D := (E''[\mathscr{I}])''[\sigma((E''[\mathscr{I}])'', (E''[\mathscr{I}])')]$$

Let us take any point x of E''. For every positive integer n we could find another s_n such that

$$x \in s_n \tilde{U}_n.$$

We extract a subsequence (V_{r_j}) of (V_r) which is also a subsequence of

$$\left(\bigcap_{i=1}^{n} s_i \tilde{U}_i\right)_{n=1}^{\infty}.$$

Obviously, (V_{r_j}) is a decreasing sequence and

$$x \in V_{r_j} \qquad j = 1, 2, \ldots$$

(V_{r_j}) is also a quasi-bounded sequence in $E''[\mathscr{I}]$. Indeed, if W is an absolutely convex closed neighbourhood of the origin in $E''[\mathscr{I}]$, and W_0 is the polar set of W in $(E''[\mathscr{I}])'$, we have a positive integer n such that W_0 is contained in U_n^0 because the boundedness of W_0 in $E'[\sigma(E', E)]$. Consequently $\tilde{U}_n \subset W$. If

$$q_n = \max\{s_1, s_2, \ldots, s_n\}$$

then

$$q_n W \supset \bigcap_{j=1}^{n} s_j \tilde{U}_j \supset V_{r_n}.$$

Now, let us take any point $z \in (E'' [\mathscr{I}])''$ which is not in E''. Obviously

$$V := \bigcap_{j=1}^{\infty} V_{r_j}$$

is a countably compact subset of $E'' (\sigma (E'', E')]$, [3] (see also [6, p. 394]), and therefore V is weakly compact in $E'' [\mathscr{I}]$. It now follows that there is a closed neighbourhood T of z in D that does not meet V. Let us suppose there is a sequence.

$$x_j \in V_{r_j} \cap T, \qquad j = 1, 2, \dots$$

The sequence (x_j) is bounded in $E'' (\sigma (E'', E')]$, hence it has a cluster point x_0 in this space and

$$x_0 \in V \cap T$$

which is a contradiction. It follows from that

$$\bigcap_{j=1}^{\infty} Z_{r_j} \subset E''.$$

it is now obvious, bearing in mind that V_r is absolutely convex, $r = 1, 2, \dots$, that $E'' [\mathscr{I}]$ is weakly countably convex-determined.

q.e.d.

Proposition 5.– Let E be a locally convex space. Let (A_n) be a quasi-bounded sequence of absolutely convex subsets of E. If

$$\bigcap_{n=1}^{\infty} \tilde{A}_{n_j} \subset E,$$

the set $\bigcap_{n=1}^{\infty} A_n^0$ is a neighbourhood of the origin in $E' [\mu (E', E)]$.

Proof.– Given

$$u \in E', \qquad u \notin \bigcap_{n=1}^{\infty} A_n^0$$

we find, for every positive integer n,

$$x_n \in A_n, \qquad |\langle x_n, u\rangle| > 1.$$

The sequence (x_n) is bounded in E and has a cluster point x_0 in $E''\ [\sigma(E'', E')]$ and

$$x_0 \in \bigcap_{n=1}^{\infty} \tilde{A}_n \subset E$$

from where it follows that $|\langle x_0, u\rangle| \geqslant 1$. Then

$$\left\{ v \in E' : |\langle x, v\rangle| < 1,\ x \in \bigcap_{n=1}^{\infty} \tilde{A}_n \right\} \subset \bigcap_{n=1}^{\infty} A_n^{\circ}. \tag{2}$$

Since $\bigcap_{n=1}^{\infty} \tilde{A}_n$ is absolutely convex and weakly compact in E the conclusion follows from (2).

q.e.d.

3. RESOLUTIONS OF IDENTITY IN WEAKLY CONTABLY CONVEX-DETERMINED METRIZABLE SPACES

Teorem 3.– Let E be an infinite dimensional metrizable locally convex space. Let (U_n) be a sequence of absolutely convex and closed neighbourhoods of the origin in E such that, for every x in E, there is a quasi-bounded sequence (U_{n_j}) of (U_n) with

$$x_0 \in \bigcap_{j=1}^{\infty} \tilde{U}_{n_j} \subset E.$$

Let A_0 and B_0 be infinite subsets of E and E', respectively, and let λ be a cardinal number such that $|A_0| \leqslant \lambda$ and $|B_0| \leqslant \lambda$. Then there exists a continuous projection T on E that verifies the following conditions:

1) $T(E) \supset A_0, \quad d(T(E)) \leqslant \lambda, \quad T(U_n) \subset U_n, \quad n = 1, 2, \ldots$
2) $T'(E') \supset B_0, \quad d(T'(E'))\ [\sigma(E', E)] \leqslant \lambda.$

Proof.– Let us write $|.|_m$ for the Minkowski functional of $\hat{U}_m$ and U_m° in $\hat{E}$ and in the linear hull $L(U_m^{\circ})$ of U_m° in E', respectively, $m = 1, 2, \ldots$ For every positive integer r and every x in $\hat{E}$, $u(x, r) \in E'$ is choosed such that

$$|u(x, r)|_r = 1, \qquad |x|_r = \langle x, u(x, r)\rangle$$

Given positive integers r, s and a vector x of E' $u(x, r, s)$ is taken in E such that

$$u(x, r, s) = 0 \quad \text{if} \quad x \notin L(U_r^0),$$

$$|u(x, r, s)|_r = 1, \qquad |\langle u(x, r, s), x\rangle| \geqslant \frac{s-1}{s} |x|_r \quad \text{if} \quad x \in L(U_r^0).$$

We proceed by recurrent. It is supposed that $A_n \subset E$, $B_n \subset E'$, with $|A_n| \leqslant \lambda$, $|B_n| \leqslant \lambda$, has been obtained for a non negative integer n. C_n and D_n are going to be the sets of linear combinations of vectors in A_n and B_n, respectively, with scalar taken in H. We define

$$A_{n+1} = C_n \cup \{u(x, r, s): x \in D_n, \ r, s = 1, 2, \ldots\}$$

$$B_{n+1} = D_n \cup \{u(x, r): x \in C_n, \ r = 1, 2, \ldots\}$$

We set F and G to denote the closures of $\bigcup_{n=0}^{\infty} A_n$ and $\bigcup_{n=1}^{\infty} B_n$ in E and E' $(\sigma(E', E))$], respectively. Since $A_{n+1} \supset C_n$ and $B_{n+1} \supset D_n$ it follows that F and G are vector spaces. Obviously,

$$d(F) \leqslant \lambda, \qquad d(F([\sigma(E', E)]) \leqslant \lambda.$$

Given $x \in \hat{F}$, $z \in (G^{\perp})\hat{}$, $r \in N$ and $\delta > 0$, a positive integer n and $t \in A_n$ is chosen such that $|x - t|_r < \delta$. Then

$$\begin{aligned} |x|_r &\leqslant |x-t|_r + |t|_r < \delta + \langle t, \ u(t, r)\rangle = \delta + \langle t + z, \ u(t, r)\rangle \leqslant \\ &\leqslant \delta + |\langle x + z, \ u(t, r)\rangle| + |\langle t - x, \ u(t, r)\rangle| \leqslant \\ &\leqslant \delta + |x + z|_r + |\ t - z|_r < |x + z|_r + 2\delta \end{aligned}$$

and consequently

$$|x|_r \leqslant |x + z|_r, \qquad r = 1, 2, \ldots \tag{3}$$

Let us suppose that there exists some $v \in G \cap F^{\perp}$, $v \neq 0$. Let x_0 be some point in E with $\langle x_0, v\rangle = 3$. We find a quasi-bounded subsequence (U_{n_j}) of (U_n) such that

$$x_0 \in M := \bigcap_{j=1}^{\infty} \tilde{U}_{n_j} \subset E$$

Obviously,

$$(v + M^0) \cap M^0 = \emptyset. \tag{4}$$

According to Proposition 5, we know that $\bigcap_{j=1}^{\infty} U_{n_j}^0$ is a neighbourhood of the origin in E' $[\mu(E', E)]$, hence we have some U_r in (U_{n_j}) and $x \in E'$ such that

$$x \in (v + U_r^\circ) \cap \left(\bigcap_{n=1}^{\infty} B_n \right), \qquad x \in L(U_r^\circ). \tag{5}$$

It follows from (4) and (5) that

$$|x|_r > 1, \qquad |x - v|_r \leqslant 1.$$

A positive integer s such that

$$\frac{s-1}{s} \; |x|_r > 1$$

can be also determined. Then $u(x, r, s) \in F$ and, consequently,

$$1 \geqslant |x - v|_r \geqslant |\langle u(x, r, s), \; x - v \rangle| = |\langle u(x, r, s), \; x \rangle| \geqslant \frac{s-1}{s} \; |x|_r > 1$$

which is a contradiction. Therefore $G \cap F^{\perp} = \{0\}$, so we have $\hat{F} + (G^{\perp})\hat{} = \hat{E}$. It follows from (3) that $\hat{F}(G^{\perp})\hat{} = \{0\}$ and, if P denotes the projection of $\hat{E}$ onto $\hat{F}$ along $(G^{\perp})\hat{}$ it follows that

$$P(\hat{U}_n) \subset \hat{U}_n, \qquad n = 1, 2, \ldots$$

Let un take a point $z \in E$. We find a quasi-bounded sequence (U_{m_j}) of (U_m) such that

$$z \in \bigcap_{j=1}^{\infty} \tilde{U}_{m_j} \subset E.$$

Then

$$z \in \tilde{U}_{m_j}, \qquad j = 1, 2, \ldots$$

and thus

$$P(z) \in \left(\bigcap_{j=1}^{\infty} \hat{U}_{m_j} \right) \cap \hat{F} \subset E \cap \hat{F} = F$$

and consequently the restriction T of P on E is a continuous projection of E onto F with kernel $G^{\perp}$. It is quite obvious now that T satisfies condition 1) and 2) of the theorem.

q.e.d.

Theorem 4.– Let E be a weakly countably convex-determined metrizable locally convex space of infinite dimension. Let

$$\|\cdot\|_1 \leqslant \|\cdot\|_2 \leqslant \ldots \leqslant \|\cdot\|_n \leqslant \ldots$$

be a fundamental system of continuous seminorms of E. If μ is the first ordinal such that $|\mu| = d(E)$ there is a resolution of identity in E

$$\{P_\alpha: \omega \leqslant \alpha \leqslant \mu\}$$

associated with $\|.\|_n$, $n = 1, 2, \ldots$

Proof.– Let (U_n) be a sequence of absolutely convex closed neighbourhoods of the origin in E such that, given any $x \in E$, there is a quasi-bounded sequence (U_{n_j}) of (U_n) such that

$$x \in \bigcap_{j=1}^{\infty} \tilde{U}_{n_j} \subset E.$$

Without any restriction, it is assumed that the sets

$$\{x \in E: \|x\|_n \leqslant 1\}, \qquad n = 1, 2, \ldots,$$

are members of the sequence (U_n) and that it contains the finite intersections of its elements.

If $d(E) = \aleph_0$, it is enough to take P_ω for the identity mapping and the conclusion is obvious. If $d(E) > \aleph_0$, we take

$$\{x_\upsilon: \upsilon < \mu\}$$

a dense subset of E. We determine a continuous projection T in E that verifies conditions 1) and 2) of the former theorem when

$$A_0 = \{x_\upsilon: \upsilon < \omega\}, \qquad B_0 = \{0\}, \qquad \lambda = \aleph_0.$$

Let us denote by P_ω this projection. Let us take $\omega < \alpha \leqslant \mu$ and suppose se have continuous projections in E,

$$\{P_\beta: \omega \leqslant \beta < \alpha\},$$

that have been defined in such a way that they verifie conditions 1) and 2) of Theorem 3 for $T = P_\beta$, $\lambda = |\beta|$, $A_0 = \{x_\upsilon: \upsilon < \beta\}$ and $B_0 = \{0\}$. It is also assumed that

$$P_\eta \circ P_\zeta = P_\eta = P_\zeta \circ P_\eta, \qquad \omega \leqslant \eta \leqslant \zeta < \alpha.$$

If α is not a limit ordinal, there is γ such that $\gamma + 1 = \alpha$ and we take two dense subsets A_γ and B_γ of $P_\gamma(E)$ and $P'_\gamma(E')[\sigma(E', E)]$, respectively, verifying

$$|A_\gamma(E)| = d(P_\gamma(E)), \quad |B_\gamma| = d(P'_\gamma(E')[\sigma(E', E)]).$$

If we apply Theorem 3 with

$$A_0 = A_\gamma \cup \{x_\upsilon \colon \upsilon < \alpha\}, \qquad \lambda = |\alpha|, \qquad B_0 = B_\gamma,$$

a continuous projection T in E is obtained verifying conditions 1) and 2) of it. Let us denote by P_α the operator T. Obviously, $P_\alpha(E)$ contains $P_\gamma(E)$ and $P'_\alpha(E') \supset P'_\gamma(E')$ because $P'_\alpha(E')$ is $\sigma(E', E)$-closed. Consequently,

$$P_\eta \circ P_\alpha = P_\eta = P_\alpha \circ P_\eta, \qquad \omega \leqslant \eta \leqslant \alpha. \tag{6}$$

If α is a limit ordinal, we set

$$F := \cup \{P_\beta(E) \colon \omega \leqslant \beta < \alpha\}, \qquad G := \cap \{P_\beta^{-1}(0) \colon \omega \leqslant \beta < \alpha\}.$$

Let us take any vector u of E'. We find a positive integer r such that $u \in U_r^0$. Since

$$P_\beta(U_n) \subset U_n, \qquad \omega \leqslant \beta < \alpha, \qquad n = 1, 2, \ldots, \tag{7}$$

the net

$$\{P'_\beta(u) \colon \omega \leqslant \beta < \alpha\}$$

belongs to $U_r^0 \cap G^\perp$ and it has a $\sigma(E', E)$-cluster point v belonging to $G^\perp$. Since

$$u - P'_\beta(u) \in P_\beta(E)^\perp, \qquad \omega \leqslant \beta < \alpha,$$

we have

$$u - v \in \cap \{P_\beta(E)^\perp \colon \omega \leqslant \beta < \alpha\} = F^\perp,$$

from where it follows that $F^\perp + G^\perp = E'$ and consequently $\hat{F} \cap \hat{G} = \{0\}$.

If m is any positive integer and x is a point of E with $\|x\|_m = 1$ we find a quasi-bounded subsequence (U_{n_j}) of (U_n) such that

$$x \in M := \bigcap_{n=1}^{\infty} \tilde{U}_{n_j} \subset E, \qquad U_{n_1} = \{y \in E \colon \|y\|_m \leqslant 1\}.$$

If follows from (7) that the net

$$\{P_\beta(x) \colon \omega \leqslant \beta < \alpha\}$$

is contained in the weakly compact set $M \cap \hat{F}$ and it has a $\sigma(E, E')$-cluster point z in that set. Obviously, $\|\ \ \|_m \leqslant 1$. Since

$$x - P_\beta(x) \in P_\beta^{-1}(0), \qquad \omega \leqslant \beta < \alpha,$$

we have $x - z \in G$. Thus, $E = \hat{F} \cap E + G$ and if we write P_α to denote the projection from E onto $F \cap E$ along G we have $P_\alpha(x) = z$ and so P_α is continuous. It is quite obvious that conditions 1) and 2) of the former theorem for

$$T = P_\alpha, \qquad A_0 = \{x_\upsilon\colon\ \omega \leqslant \upsilon < \alpha\}, \qquad B_0 = \{0\}, \qquad \lambda = |\alpha|,$$

are satisfied. On the other hand, if $\omega \leqslant \beta \leqslant \alpha$, then

$$P_\beta(E) \subset P_\alpha(E), \qquad P'_\beta(E') \supset P'_\alpha(E'),$$

and we have

$$P_\beta \circ P_\alpha = P_\beta = P_\alpha \circ P_\beta.$$

Finally,

$$\{x_\upsilon\colon\ \upsilon < \mu\} \subset P_\mu(E),$$

so P_μ must be the identity operator. obviously,

$$\{P_\alpha\colon\ \omega \leqslant \alpha < \mu\}$$

answer the request of the theorem.

q.e.d.

Note.– For a weakly countably determined Banach space X there is a sequence (A_n) of bounded neighbourhoods of the origin in X, closed and absolutely convex, that determines $X[\sigma(X, X')]$, [11]. So our theorems 3 and 4 can be applied without mention to our part 2. In that way the results of Vasak, [11], extending previous ones of Amir-Lindestrauss, [1], are obtained. Apart from being formulated for weakly countably convex-determined metrizable locally convex spaces, our theorem 3 and 4 use a method to be proved that we have introduced in [10] and that is more simple and direct than the Amir-Lindestrauss-Vasak way, [1] and [11].

4. FRECHET SPACES E WITH $E''[\beta(E'', E)]/E$ SEPARABLE

Proposition 6.– Let P and Q be two closed subspaces of a complete (DF)-space E such that

$$P^\perp + Q^\perp = E', \qquad P^\perp \cap Q^\perp = \{0\}.$$

If P is semireflexive and E/P is ultrabornological then E is the topological direct sum of P and Q. Moreover, $Q^\perp$ is a reflexive subspace of $E'[\beta(E', E)]$.

Proof.– Let

$$A_1 \subset A_2 \subset \ldots \subset A_n \subset \ldots$$

be a fundamental system of bounded subsets of E that we choose absolutely convex and closed. E'' is the topological dual of the Fréchet space $E'[\beta(E', E)]$ and

$$\tilde{A}_1 \subset \tilde{A}_2 \subset \ldots \subset \tilde{A}_n \subset \ldots$$

is a fundamental system of equicontinuous sets of E''. Since P is reflexive it follows that

$$P \cap \tilde{A}_n, \qquad n = 1, 2, \ldots$$

are $\sigma(E'', E')$-compact subsets and, therefore, if we apply the Krein-Smulian theorem, [5, p. 246], P is $\sigma(E'', E')$-closed. The space $E'[\beta(E', E)]$ is the topological direct sum of $P^{\perp}$ and $Q^{\perp}$, consequently,

$$\tilde{P} + \tilde{Q} = E, \qquad \tilde{P} \cap \tilde{Q} = \{0\}.$$

Since $P = \tilde{P}$, it follows that

$$P + Q = E, \qquad P \cap Q = \{0\}.$$

Let φ be the canonical mapping from E onto E/P. Let $\mathscr{U}$ be the topology on Q such that $Q[\mathscr{U}]$ is the ultrabornological space associated with Q. We have that

$$\varphi\colon\ Q[\mathscr{U}] \longrightarrow E/P \tag{8}$$

is a continuous, injective and surjective mapping. We can apply the closed graph theorem in the form given by Grothendieck, [4, p. 17], because $Q[\mathscr{U}]$ is a (LB)-space, and the fact that (8) an isomorphism immediately follows. Then

$$\varphi\colon\ Q \longrightarrow E/P$$

is an isomorphism too, and from that we know as Q is a topological complement of P in E.

E/Q is a semireflexive (DF)-space, and its topological dual with the Mackey topology would be reflexive Frèchet space. On the other hand, $Q^{\perp}$ can be identify, as it is usual, with the topological dual of E/Q, and it is immediate that $\beta(E', E)$ coincides with $\mu(Q^{\perp}, E/Q)$ on $Q^{\perp}$. The conclusion now follows.

q.e.d.

Theorem 5.– Let F be a closed subspace of a Fréchet space E. If $F'[\beta(F', F)]$ is separable and $E + \tilde{F} = E''$, there is a closed suspace G of E with the following properties:

1) $G \supset F$.
2) $G'[\beta(G', G)]$ is separable.
3) The is a reflexive subspace L of $E''[\beta(E'', E')]$ such that

$$L + \tilde{G} = E'', \qquad L \cap \tilde{G} = \{0\}.$$

Proof.– Let

$$U_1 \supset U_2 \supset \ldots \supset U_n \supset \ldots$$

be a fundamental system of neighbourhoods of the origin in E, that we take closed and absolutely convex. If u is an element of E'' then $u = v + w$, $v \in E$ and $w \in F$, so u coincides with v of $F^{\perp}$, from where we see that the topologies $\sigma(E', E'')$ and $\sigma(E', E')$ coincide on $F^{\perp}$. It is now deduced that $F^{\perp} \cap U_n^{\circ}$ is weakly compact in $E'[\beta(E', E')]$, $n = 1, 2, \ldots$

Let φ be the canonical mapping from E' onto $E'/F^{\perp}$. Obviously,

$$\varphi \colon E'[\beta(E_{,}, E)] \longrightarrow (E'/F^{\perp})[\beta(E'/F^{\perp}, F)]$$

is continuous. We have $(E'/F^{\perp})[\beta(E'/F^{\perp}, F)]$ is isomorphic to $F'[\beta(F', F)]$ and it follows that $(E'/F^{\perp})[\beta(E'/F^{\perp}, F)]$ is ultrabornological because the separability of $F'[\beta(F', F)]$, [3] (see also [6, p. 399]). If $\mathscr{U}$ is the topology of E' such that $E'[\mathscr{U}]$ is the ultrabornological space associated with $E'[\beta(E', E)]$, we have an (LB)-space $E'[\mathscr{U}]$. The continuous mapping

$$\varphi \colon E'[\mathscr{U}] \longrightarrow (E'/F^{\perp})[\beta(E'/F^{\perp}, F)]$$

is open after the closed graph theorem given by Grothendieck, [4, p. 17]. It now follows from the separability of $(E'/F^{\perp})[\beta(E'/F^{\perp}, F)]$ the existence of a sequence (u_n) in E' such that the linear hull of

$$\{u_n \colon n = 1, 2, \ldots\} \cup F^{\perp}$$

is dense in $E'[\beta(E', E)]$. Given a positive integer n, we denote by A_n the absolutely convex closed cover of $\{u_1, u_2, \ldots, u_n\}$ in $E'[\beta(E', E)]$. We set

$$W_n = F^{\perp} \cap U_n^{\circ} + nA_n$$

and W_n° for the polar set of W_n in E''. It immediately follows that

$$W_1^{\circ} \supset W_2^{\circ} \supset \ldots \supset W_n^{\circ} \supset \ldots$$

is a fundamental system of neighbourhoods of the origin in E'' for a locally convex metrizable topology T coarser than $\mu(E'', E')$.

In accordance with Proposition 4, $E''[T]$ is a weakly countably convex-determined space. We apply Theorem 3 to obtain a separable closed subspace M of $E''[\mathscr{I}]$ that contains F and has a topologycal complement L in $E''[\mathscr{I}]$. Let M_1 be the orthogonal subspace to M in E'. Obviously, M_1 is $\sigma(E', E'')$-closed and it is contained in $F^{\perp}$. Then $M_1 \cap U_n^{\circ}$ is $\sigma(E', E)$-closed, $n = 1, 2, \ldots,$ and the theorem of Krein-Smulian affirms that M_1 is $\sigma(E', E)$-closed [5, p. 246]. Consequently, if $G := M \cap E$, we have $\tilde{G} = M$ and $G^{\perp} = M_1$. Let Ψ be the canonical mapping from E' onto $E'/G^{\perp}$. Obviously,

$$\Psi \colon E'[\beta(E', E')] \longrightarrow (E'/G^{\perp})[\beta(E'/G^{\perp}, G)] \tag{9}$$

is continuous and therefore $\Psi(W_n)$ is a weakly compact subset of

$(E'/G^{\perp})$ $[\beta\ (E'/G^{\perp},\ G)]$ which is metrizable because the separability of G, from where we deduce that $(E'/G^{\perp})$ $[\beta\ (E'/G^{\perp},\ G)]$ is ultrabornological. Consequently

$$\psi: E'\ [\mathscr{U}] \longrightarrow (E'/G^{\perp})\ [\beta\ (E'/G^{\perp},\ G)]$$

is an homomorphism and (9) too, the fact that E' $[\beta\ (E',\ E]/G^{\perp}$ is ultrabornological is clear now. If Q is the closed subspace of E' which is orthogonal to L, we see that Q and $G^{\perp}$ and two closed subspace of the (DF)-space E' $[\beta\ (E',\ E]$ such that their orthogonal subspace L and $\tilde{G}$ in E'' meets only at the zero vector and $L + \tilde{G} = E''$. Moreover $G^{\perp}$ is a semireflexive subspace of E' $[\beta\ (E',\ E]$. The former proposition is now applied to obtain that L is a reflexive subspace of the Fréchet space E' $[\beta\ (E',\ E']$ and the proof is finished.

q.e.d.

Proposition 7.– If E is a Frèchet space with E'' $[\beta\ (E'',\ E']/E$ separable there is a closed separable subspace F of E'' $[\beta\ (E'',\ E']$ such that $E + F = E''$.

Proof.– Let

$$U_1 \supset U_2 \supset \ldots \supset U_n \supset \ldots$$

be a fundamental system of neighbourhoods of the origin in E. Let φ be the canonical mapping from E'' onto E''/E. For every positive integer n, we take a subset of $\tilde{U}_n$

$$A_n := \{x_{mn}\colon\ m = 1, 2, \ldots\}$$

such that $\varphi\ (A_n)$ is dense in $\varphi\ (\tilde{U}_n)$ in the quotient space E'' $[\beta\ (E'',\ E']/E$. Let F be the closed lineat hull of $\cup\ \{A_n\colon\ n = 1, 2, \ldots\}$ in E'' $[\beta\ (E'',\ E']$ and Ψ the restriction of φ on F. Obviously Ψ is continuous and the closure of $\Psi\ (\tilde{U}_n \cap F)$ in E'' $[\beta\ (E'',\ E']/E$ contains $\varphi\ (\tilde{U}_n)$, $n = 1, 2, \ldots$ So Ψ is an homomorphism from F onto E'' $[\beta\ (E'',\ E']/E$ in accordance with the open mapping theorem, from where it follows that $E + F = E''$.

q.e.d.

Theorem 6.– If E is a Frèchet space with E'' $[\beta\ (E'',\ E']/E$ separable then E is the topological direct sum of two subspaces P and Q such that P'' $[\beta\ (P'',\ P']$ is separable and Q is reflexive.

Proof.– Let φ be the canonical mapping from E'' onto E''/E. Let us determine a sequence (x_n) of E'' such that

$$\{\varphi\ (x_n)\colon\ n = 1, 2, \ldots\}$$

is a dense subset of E'' $[\beta\ (E'',\ E'']/E$. For every positive integer n, we apply result a) together with the former proposition to obtain a sequence $(x_{mn})_{m=1}^{\infty}$

in E that $\sigma(E'', E')$-converges to x_n. Let F be the closed linear hull of

$$\{x_{mn}: m, n = 1, 2, ...\}.$$

$E + \tilde{F}$ is a dense subspace of E'' $[\beta (E'', E']$ and according with result b) we have

$$E'' = E + \tilde{F}.$$

Let Ψ be the restriction of φ on $\tilde{F}$. It is immediate that Ψ is an homomorphism from $\tilde{F}[\beta (E'', E']$ onto E'' $[\beta (E'', E']/E$ with kernel F. Consequently, $\tilde{F}[\beta (E'', E']$ is separable. Every point of F is in the $\sigma(E'', E')$-closure of a sequence in F, thus $\tilde{F}$ is the topological dual of $(E'/F^{\perp})$ $[\beta$ $(E'/F^{\perp}, F]$. If we identify in the usual way F' with $E'/F^{\perp}$ it follows that $\tilde{F}$ $[\beta (E'', E')]$ is isomorphic to F'' $[\beta (F'', F']$ and the separability of F'' $[\beta (F', F')]$ follows from that of F'' $[\beta (F'', F']$. Theorem 5 is now applied to obtain a separable closed subspace G of E that contains F and such that $\tilde{G}$ has a reflexive topological complement L in E'' $[\beta (E'', E']$. Thesam argument used above for $\tilde{F}$ gives us a proof for the separability of $\tilde{G}$ $[\beta (E'', E']$. Then E'' $[\beta (E'', E']$ is isomorphic to the product of two weakly K-analytic spaces $\tilde{G}$ $[\beta (E'', E']$ and L $[\beta (E'', E']$, from where it is deduced that E is a weakly K-analitic. Theorem 3 is applied at this point to obtain a closed separable subspace P of E that contains F and has a topological complement Q in E. It is easy to see now that P'' $[\beta (P'', P')]$ is separable and Q is reflexive.

q.e.d.

Note.– If X is a Banach space with X'' $[\beta (X', X')]/X$ seperable, it is proved in [9] that X is the topological direct sum of two Banach spaces X_1 an X_2 with X_1'' $[\beta (X_1'', X_1')]$ separable and X_2 reflexive. The proof of this result uses a theorem of Amir-Lindenstrauss on weakly compactly generated Banach spaces [1]. Theorem 6 is much more difficult to prove than the corresponding one for Banach spaces and we are based upon Theorem 5 which is even new for Banach spaces up to our knowledge.

BIBLIOGRAPHY

[1] AMIR, D. AND LINDENSTRAUSS, J.: *The structure of weakly compact sets in Banach spaces.* Ann. of Math. *88*, 35-46 (1968).

[2] CHOQUET, G.: *Ensembles K-analytiques et K-sousliniens.* Ann. Inst. Fourier, *9*, 75-89 (1959).

[3] GROTHENDIECK, A.: *Sur les espaces (F) et (DF).* Summa Brasil. Math. *3*, 57-123 (1954).

[4] GROTHENDIECK, A.: *Produits thensoriels topologiques et espaces nucléaires.* Mem. Amer. Math. Soc. No. 16 (1955).

[5] HORVATH, J.: *Topological Vector Spaces and Distributions I.* Massachussets-London. Addison-Wesley, 1966.

[6] KÖTHE, G.: *Topological Vector Spaces I.* Berlin-heidelberg-New York. Springer: 1969.

[7] VALDIVIA, M.: *On metrizable locally convex spaces.* Arch. d. Math. *27*, 79-89 (1976).

[8] VALDIVIA, M.: *Fréchet spaces with strong bidual separable.* Conferenze del Seminario di Matematica dell'Università di Bari, 1978.

[9] VALDIVIA, M.: *On a class of Banach spaces.* Studia Math. *60*, 11-13 (1977).

[10] VALDIVIA, M.: *Espacios de Fréchet de generación débilmente compacta.* Collect. Math. *38*, 17-25 (1987).

[11] VĂSAK, L.: *On one generalization of weakly compactly generated Banach spaces.* Studia Math. *70*, 11-19 (1981).

Prof. Manuel Valdivia
Facultad de Matemáticas
Dr. Moliner, 50
46100-Burjasot (Valencia)
Spain