A nonlinear boundary value problem for a system of hyperbolic partial differential equations

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Recibido: 4 noviembre 1987

Presentado por el académico numerario D. Dario Maravall

Abstract

The paper concerns a boundary value problem for a system of hyperbolic partial differential equations of arbitrary order with two independent variables that is a generalization of the problem examined by Z. Szmydt [7].

1. Boundary value problems for hyperbolic equations or systems of order higher than two with the boundary conditions given on more than two non-characteristic curves have been examined by O. Sjöstrand [6], Z. Szmydt [7], A. Borzymowski [1]-[3] and M. Michalski [4], [5].

The aim of this paper is to study a boundary value problem generalizing that of Z. Szmydt [7] (cp. Remarks 1 and 3 in the sequel). We reduce our problem to a system of nonlinear integro-functional equations and then apply the Schauder fixed point theorem. The uniqueness of the solution is proved by using the Banach fixed point theorem.

2. Let $p, q \in \mathbb{N}$, where $\mathbb{N}$ is the set of all positive integers, and let r_1 and r_2 be positive divisors of p and q, respectively, such that $k := p/r_1 = q/r_2$. Consider the rectangle $\Omega = [0, A] \times [0, B]$ $(0 < A, B < \infty)$ and introduce two systems of curves placed in Ω and given by the equations $y = f_\alpha(x)$ $(\alpha = 1, 2, ..., r_2)$ and $x = h_\beta(y)$ $(\beta = 1, 2, ..., r_1)$, respectively.

For fixed $n \in \mathbb{N}$ we introduce the following notation

$$\begin{aligned} &u = \{u^i\} && (i = 1, 2, ..., n); \\ &V = \{v_{j,\alpha}\} && \text{with} \quad v_{j,\alpha} = D_x^{r_1} D_y^{r_2-\alpha} L^{j-1} u; \\ &W = \{w_{j,\beta}\} && \text{with} \quad w_{j,\beta} = D_x^{r_1-\beta} D_y^{r_2} L^{j-1} u; \\ &Z = \{z_{j,\beta,\alpha}\} && \text{with} \quad z_{j,\beta,\alpha} = D_x^{r_1-\beta} D_y^{r_2-\alpha} L^{j-1} u \end{aligned} \tag{1}$$

$(j = 1, 2, \ldots, k;\ \alpha = 1, 2, \ldots, r_2;\ \beta = 1, 2, \ldots, r_1)$, where $L = D_x^{r_1} D_y^{r_2}$; $L^0 u = u$; $L^\nu u = L(L^{\nu-1}u)$ for $1 \leqslant \nu \leqslant k$.

Consider the following system of partial differential equations

$$L^k u(x,y) = F[x, y, Z(x,y), V(x,y), W(x,y)] \tag{2}$$

$((x,y) \in \Omega)$, where $F = \{F^i\}$ $(i = 1, 2, \ldots, n)$ is a given function.

We denote by $\mathscr{K}$ the class of functions $u\colon \Omega \to \mathbb{R}^n$ such that the derivatives $v_{j,\alpha}$, $w_{j,\beta}$ and $z_{j,\beta,\alpha}$ $(j = 1, 2, \ldots, k;\ \alpha = 1, 2, \ldots, r_2;\ \beta = 1, 2, \ldots, r_1)$ introduced in (1) exist, are continuous and do not depend on the order in which the last mixed differentiation(*) is performed.

By a solution of equation (2) in Ω we mean a function $u \in \mathscr{K}$ possessing continuous derivative $L^k u$ and satisfying system (2) at each point $(x,y) \in \Omega$.

We pose the following problem (P):

Find a solution u of system (2) in Ω satisfying the boundary conditions

$$\begin{gathered} v_{j,\alpha}[x, f_\alpha(x)] = e_{j,\alpha}(x, Z[x, f_\alpha(x)], W[x, f_\alpha(x)]); \\ w_{j,\beta}[h_\beta(y), y] = g_{j,\beta}(y, Z[h_\beta(y), y], V[h_\beta(y), y]) \end{gathered} \tag{3}$$

and

$$z_{j,\beta,\alpha}(\mathring{x}_{j,\beta,\alpha}, \mathring{y}_{j,\beta,\alpha}) = \mathring{u}_{j,\beta,\alpha} \tag{4}$$

$((x,y) \in \Omega;\ j = 1, 2, \ldots, k;\ \alpha = 1, 2, \ldots, r_2;\ \beta = 1, 2, \ldots, r_1)$, where $e_{j,\alpha} = \{e^i_{j,\alpha}\}$ and $g_{j,\beta} = \{g^i_{j,\beta}\}$ $(i = 1, 2, \ldots, n)$ are given functions, $\mathring{u}_{j,\beta,\alpha} = \{\mathring{u}^i_{j,\beta,\alpha}\}$ $(i = 1, 2, \ldots, n)$, where $\mathring{u}^i_{j,\beta,\alpha}$ are given numbers, and the points $(\mathring{x}_{j,\beta,\alpha}, \mathring{y}_{j,\beta,\alpha})$ are arbitrarily fixed in Ω.

We make the following assumptions that will be in force in sections 3 and 4:

I. The functions $f_\alpha\colon [0, A] \to [0, B]$ and $h_\beta\colon [0, B] \to [0, A]$ $(\alpha = 1, 2, \ldots, r_2;\ \beta = 1, 2, \ldots, r_1)$ are continuous.

II. The function $F\colon \Omega \times \mathbb{R}^\kappa \to \mathbb{R}^n$ (where $\kappa = kn(r_1 + r_2 + r_1 r_2)$) is continuous and satisfies the condition(**)

$$|F(x, y, \Xi, H^1, H^2)| \leqslant K_1 + K_2 \mathscr{C}_0 + K_2' \mathscr{C}_0' \tag{5}$$

$(\Xi = \{\xi_{j,\beta,\alpha}\},\ H^1 = \{\eta^1_{j,\alpha}\},\ H^2 = \{\eta^2_{j,\beta}\};\ i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, k;\ \alpha = 1, 2, \ldots, r_2;\ \beta = 1, 2, \ldots, r_1)$, where K_1, K_2 and K_2' are positive constants, and $\mathscr{C}_0$ and $\mathscr{C}_0'$ are given by

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(*) That is the differentiation $D_x^{r_1} D_y^{r_2-\alpha}$, $D_x^{r_1-\beta} D_y^{r_2}$ and $D_x^{r_1-\beta} D_y^{r_2-\alpha}$, respectively.

(**) Above, $|F(x, y, \Xi, H^1, H^2)| = \max\limits_{1 \leqslant i \leqslant n} |F^i(x, y, \Xi, H^1, H^2)|$. The symbols $|e_{j,\alpha}(x, \Xi, H^2)|$ and $|g_{j,\beta}(y, \Xi, H^1)|$ etc. appearing in the sequel will be understood in a similar way.

$$\mathscr{C}_0 = \sum_{j=1}^{k} \sum_{\alpha=1}^{r_2} \sum_{\beta=1}^{r_1} |\xi_{j,\beta,\alpha}|^{r_*} \tag{6}$$

and

$$\mathscr{C}'_0 = \sum_{j=1}^{k} \sum_{\alpha=1}^{r_2} |\eta^1_{j,\alpha}|^{r_*} + \sum_{j=1}^{k} \sum_{\beta=1}^{r_1} |\eta^2_{j,\beta}|^{r_*}, \tag{7}$$

respectively, with $r_* \in (0, 1)$.

III. The functions $e_{j,\alpha}$: $[0, A] \times \mathbb{R}^{\kappa - knr_2} \to \mathbb{R}^n$ and $g_{j,\beta}$: $[0, B] \times \mathbb{R}^{\kappa - knr_1} \to \mathbb{R}^n$ ($j = 1, 2, ..., k$; $\alpha = 1, 2, ..., r_2$; $\beta = 1, 2, ..., r_1$) are continuous and satisfy the conditions

$$\begin{aligned} |e_{j,\alpha}(x, \Xi, H^2)| &\leqslant K_3 + K_4 \mathscr{C}_0 + K'_4 \mathscr{C}'_1 \\ |g_{j,\beta}(y, \Xi, H^1)| &\leqslant K_3 + K_4 \mathscr{C}_0 + K'_4 \mathscr{C}'_2, \end{aligned} \tag{8}$$

respectively, where K_3, K_4 and K'_4 are positive constants, and $\mathscr{C}'_1$ and $\mathscr{C}'_2$ denote the expression $\mathscr{C}'_0$ (cp. (7)) with the first or second term, respectively, being omitted.

Remark 1.– If $r_1 = p$; $r_2 = q$ (as a consequence $k = j = 1$), then our problem (*P*) is identical with the problem (*Q*) examined by Z. Szmydt [7].

3. In this section we will prove some lemmas.

Lemma 1.– If u is of class $\mathscr{K}$ and satisfies condition (4), then

$$z_{j,\beta,\alpha}(x, y) = \int_0^x \frac{(x-\xi)^{\beta-1}}{(\beta-1)!} v_{j,\alpha}(\xi, y)\, d\xi +$$

$$+ \sum_{\nu=1}^{\beta} \frac{x^{\beta-\nu}}{(\beta-\nu)!} \left\{ \int_0^y \frac{(y-\eta)^{\alpha-1}}{(\alpha-1)!} w_{j,\nu}(0, \eta)\, d\eta + \right.$$

$$\left. + \sum_{\mu=1}^{\alpha} C^j_{\nu,\mu} \frac{y^{\alpha-\mu}}{(\alpha-\mu)!} \right\} =$$

$$= \int_0^y \frac{(y-\eta)^{\alpha-1}}{(\alpha-1)!} w_{j,\beta}(x, \eta)\, d\eta + \sum_{\mu=1}^{\alpha} \frac{y^{\alpha-\mu}}{(\alpha-\mu)!} \times$$

$$\times\left\{\int_0^x \frac{(x-\xi)^{\beta-1}}{(\beta-1)!} v_{j,\mu}(\xi,0)\,d\xi + \sum_{\nu=1}^{\beta} C^j_{\nu,\mu} \frac{x^{\beta-\nu}}{(\beta-\nu)!}\right\} \tag{9}$$

$((x,y)\in\Omega;\ j=1,2,\ldots,k;\ \alpha=1,2,\ldots,r_2;\ \beta=1,2,\ldots,r_1)$, where $C^j_{\nu,\mu} = \{C^{j,i}_{\nu,\mu}\}$ are functions of V and W defined recursively by the formula(*)

$$C^j_{\nu,\mu} = \overset{\circ}{u}_{j,\nu,\mu} - \int_0^{\overset{\circ}{x}_{j,\nu,\mu}} \frac{(\overset{\circ}{x}_{j,\nu,\mu}-\xi)^{\nu-1}}{(\nu-1)!} v_{j,\mu}(\xi,\overset{\circ}{y}_{j,\nu,\mu})\,d\xi +$$

$$-\sum_{s=1}^{\nu} \frac{\overset{\circ}{x}{}^{\nu-s}_{j,\nu,\mu}}{(\nu-s)!}\int_0^{\overset{\circ}{y}_{j,\nu,\mu}} \frac{(\overset{\circ}{y}_{j,\nu,\mu}-\eta)^{\mu-1}}{(\mu-1)!} w_{j,s}(0,\eta)\,d\eta +$$

$$+\sum_{\substack{1\leqslant s\leqslant\nu\\ 1\leqslant l\leqslant\mu\\ s+l<\nu+\mu}} C^j_{s,l}\frac{\overset{\circ}{x}{}^{\nu-s}_{j,\nu,\mu}}{(\nu-s)!}\frac{\overset{\circ}{y}{}^{\mu-l}_{j,\nu,\mu}}{(\mu-l)!} \tag{10}$$

$(\nu=1,2,\ldots,\beta;\ \mu=1,2,\ldots,\alpha)$.

Proof. Let $y\in[0,B]$ be fixed. We can write the following Taylor's formulae

$$z_{j,\beta,\alpha}(x,y) = \sum_{\nu=1}^{\beta} \frac{x^{\beta-\nu}}{(\beta-\nu)!} D_x^{\beta-\nu} z_{j,\beta,\alpha}(0,y) +$$

$$+\int_0^x \frac{(x-\xi)^{\beta-1}}{(\beta-1)!} D_x^{\beta} z_{j,\beta,\alpha}(\xi,y)\,d\xi \tag{11}$$

and

$$D_x^{\beta-\nu} z_{j,\beta,\alpha}(0,y) = \sum_{\mu=1}^{\alpha} \frac{y^{\alpha-\mu}}{(\alpha-\mu)!} D_y^{\alpha-\mu} D_x^{\beta-\nu} z_{j,\beta,\alpha}(0,0) +$$

$$+\int_0^y \frac{(y-\eta)^{\alpha-1}}{(\alpha-1)!} D_y^{\alpha} D_x^{\beta-\nu} z_{j,\beta,\alpha}(0,\eta)\,d\eta. \tag{12}$$

Substituting (12) into (11) and using notation (1) and the assumption $u\in\mathscr{K}$, we easily conclude that the first of equalities (9) holds good with $C^j_{\nu,\mu} = z_{j,\nu,\mu}(0,0)$. The proof of the second one is analogous. Finally, relation (10) follows from (9) and (4).

(*) As usual, we set $\sum_{r=\gamma}^{\delta} a_r = 0$ for $\delta<\gamma$.

Remark 2.– If $z_{j,\beta,\alpha}$ $(j = 1, 2, ..., k;\ \beta = 1, 2, ..., r_1;\ \alpha = 1, 2, ..., r_2)$ are given by formula (9), then we shall write (cp. (1))

$$Z(x,y) = R[V(.),\ W(.),\ x, y]. \tag{13}$$

Now, let us consider the following system of integro-functional equations

$$\begin{aligned}
v_{j,\alpha}(x,y) &= e_{j,\alpha}(x,\ R[V(.),\ W(.),\ x,\ f_\alpha(x)],\ W[x,\ f_\alpha(x)]) + \\
&+ \int_{f_\alpha(x)}^{y} v_{j,\alpha-1}(x,\eta)\,d\eta; \qquad (j = 1, 2, ..., k;\ \alpha = 2, 3, ..., r_2) \\
v_{j,1}(x,y) &= e_{j,1}(x,\ R[V(.),\ W(.),\ x,\ f_1(x)],\ W[x,\ f_1(x)]) + \\
&+ \int_{f_1(x)}^{y} d\eta \int_0^x \frac{(x-\xi)^{r_1-1}}{(r_1-1)!} v_{j+1,r_2}(\xi,\eta)\,d\xi + \\
&+ \sum_{\nu=1}^{r_1} \frac{x^{r_1-\nu}}{(r_1-\nu)!} \left\{ \int_{f_1(x)}^{y} d\eta \int_0^\eta \frac{(\eta-\sigma)^{r_2-1}}{(r_2-1)!} w_{j+1,\nu}(0,\sigma)\,d\sigma + \right. \\
&\left. + \sum_{\mu=1}^{r_2} C_{\nu,\mu}^{j+1} \int_{f_1(x)}^{y} \frac{\eta^{r_2-\mu}}{(r_2-\mu)!}\,d\eta \right\}; \qquad (j = 1, 2, ..., k-1) \\
v_{k,1}(x,y) &= e_{k,1}(x,\ R[V(.),\ W(.),\ x,\ f_1(x)],\ W[x,\ f_1(x)]) + \\
&+ \int_{f_1(x)}^{y} F(x,\ \eta,\ R[V(.),\ W(.),\ x,\ \eta],\ V(x,\eta),\ W(x,\eta))\,d\eta; \\
w_{j,\beta}(x,y) &= g_{j,\beta}(y,\ R[V(.),\ W(.),\ h_\beta(y),\ y],\ V[h_\beta(y),\ y]) + \\
&+ \int_{h_\beta(y)}^{x} w_{j,\beta-1}(\xi,y)\,d\xi; \qquad (j = 1, 2, ..., k;\ \beta = 2, 3, ..., r_1) \\
w_{j,1}(x,y) &= g_{j,1}(y,\ R[V(.),\ W(.),\ h_1(y),\ y],\ V[h_1(y),\ y]) + \\
&+ \int_{h_1(y)}^{x} d\xi \int_0^y \frac{(y-\eta)^{r_2-1}}{(r_2-1)!} w_{j+1,r_1}(\xi,\eta)\,d\eta + \\
&+ \sum_{\mu=1}^{r_2} \frac{y^{r_2-\mu}}{(r_2-\mu)!} \left\{ \int_{h_1(y)}^{x} d\xi \int_0^\xi \frac{(\xi-\sigma)^{r_1-1}}{(r_1-1)!} v_{j+1,\mu}(\sigma,0)\,d\sigma + \right.
\end{aligned} \tag{14}$$

$$+ \sum_{\nu=1}^{r_1} C_{\nu,\mu}^{j+1} \int_{h_1(y)}^{x} \frac{\xi^{r_1-\nu}}{(r_1-\nu)!} d\xi \Big\}; \qquad (j=1,2,\dots,k-1)$$

$$w_{k,1}(x,y) = g_{k,1}(y, R[V(.), W(.), h_1(y), y], V[h_1(y), y]) +$$

$$+ \int_{h_1(y)}^{x} F(\xi, y, R[V(.), W(.), \xi, y], V(\xi,y), W(\xi,y))\, d\xi.$$

Lemma 2.— If u is a solution of the (P)-problem, then V and W are continuous and system (14) is satisfied.

Proof. - If follows from the present assumptions, formulae (1) and Lemma 1 that V and W are continuous and relation (13) holds good. Let us observe (cp. (1) and (2)) that the following formula

$$D_y v_{j,\alpha} = \begin{cases} v_{j,\alpha-1} & \text{for } j=1,2,\dots,k;\ \alpha=2,3,\dots,r_2 \\ F & \text{for } j=k;\ \alpha=1 \\ z_{j+1,r_1,r_2} & \text{for } j=1,2,\dots,k-1;\ \alpha=1 \end{cases} \tag{15}$$

is valid. Integrating (15) over $[f_\alpha(x), y]$ and using conditions (3), equation (2) or formula (9), respectively, we get the first three of equations (14). The derivation of the remaining three of these equations is analogous.

Lemma 3.— If relation (13) holds true, where the system (V, W) is a continuous solution of system (14), then z_{1,r_1,r_2} is a solution of the problem (P).

Proof.— It easily follows from our assumptions that

$$Lz_{j,r_1,r_2} = \begin{cases} z_{j+1,r_1,r_2} & \text{for } j=1,2,\dots,k-1 \\ F & \text{for } j=k. \end{cases} \tag{16}$$

As a consequence of (16) we get

$$L^{j-1} z_{1,r_1,r_2} = z_{j,r_1,r_2} \tag{17}$$

$(j=1,2,\dots,k)$. Moreover, using (9), (14) and (17), we have

$$D_x^{r_1} D_y^{r_2-\alpha} L^{j-1} z_{1,r_1,r_2} = D_x^{r_1} D_y^{r_2-\alpha} z_{j,r_1,r_2} =$$
$$= D_y^{r_2-\alpha} v_{j,r_2} = v_{j,\alpha} \tag{18}$$

$(j = 1, 2, ..., k;\ \alpha = 1, 2, ..., r_2)$.

In a similar way we get the relations

$$D_x^{r_1-\beta} D_y^{r_2} L^{j-1} z_{1,r_1,r_2} = w_{j,\beta} \tag{19}$$

and

$$D_x^{r_1-\beta} D_y^{r_2-\alpha} L^{j-1} z_{1,r_1,r_2} = z_{j,\beta,\alpha} \tag{20}$$

$(j = 1, 2, ..., k;\ \alpha = 1, 2, ..., r_2;\ \beta = 1, 2, ..., r_1)$.

It follows from equalities (18)-(20) that the function z_{1,r_1,r_2} belongs to $\mathscr{K}$. Evidently, by (16) and (17) the said function satisfies system (2) in Ω. Finally, one can easily deduce from (9), (10), (14), (19) and (20) that conditions (3) and (4) are satisfied. Thus, the proof of Lemma 3 is completed.

As a result of Lemmas 2 and 3 we can formulate the following

Proposition 1.— There is one-to-one correspondence between the solutions u of the problem (P) and continuous solutions (V, W) of system (14).

4. In this section we will prove the existence of a continuous solution of system (14) and hence (cp. Proposition 1) of a solution of problem (P). To this end we will apply the well known Schauder fixed point theorem.

Let Λ be the Banach space of all systems $\phi = (V, W)$ of continuous functions(*) with the norm

$$\|\phi\| = \max\Big[\max_{\substack{1 \leq j \leq k \\ 1 \leq \alpha \leq r_2}} \sup_{\Omega} |v_{j,\alpha}(x,y)|, \max_{\substack{1 \leq j \leq k \\ 1 \leq \beta \leq r_1}} \sup_{\Omega} |w_{j,\beta}(x,y)|\Big]. \tag{21}$$

We consider the set $\mathscr{K}$ of all points $\phi \in \Lambda$ that are equicontinuous and satisfy the condition

$$\|\phi\| \leqslant \rho \tag{22}$$

where ρ is a parameter to be suitably chosen in the sequel (cp. p. 9).

Evidently, $\mathscr{K}$ is a closed convex and compact set.

In view of system (14), we map the set $\mathscr{K}$ by the following transformation

$$\tilde{\phi} = T\phi \tag{23}$$

$(\phi = (V, W) \in \mathscr{K};\ \tilde{\phi} = (\tilde{V}, \tilde{W})$, where $\tilde{V} = \{\tilde{v}_{j,\alpha}\};\ \tilde{W} = \{\tilde{w}_{j,\beta}\})$ with

$$\tilde{v}_{j,\alpha}(x,y) = e_{j,\alpha}(x, R[V(.), W(.), x, f_\alpha(x)], W[x, f_\alpha(x)]) +$$

(*) $V = \{v_{j,\alpha}\};\ W = \{w_{j,\beta}\}$, where $j = 1, 2, ..., k;\ \alpha = 1, 2, ..., r_2;\ \beta = 1, 2, ..., r_1$.

$$+\int_{f_\alpha(x)}^{y} \tilde{v}_{j,\alpha-1}(x,\eta)\,d\eta; \qquad (j=1,2,\dots,k;\ \alpha=2,3,\dots,r_2)$$

$$\tilde{v}_{j,1}(x,y)=e_{j,1}(x,\ R[V(.),\ W(.),\ x,\ f_1(x)],\ W[x,\ f_1(x)])+$$

$$+\int_{f_1(x)}^{y} d\eta \int_0^x \frac{(x-\xi)^{r_1-1}}{(r_1-1)!}\,\tilde{v}_{j+1,r_2}(\xi,\eta)\,d\xi+$$

$$+\sum_{\nu=1}^{r_1} \frac{x^{r_1-\nu}}{(r_1-\nu)!}\left\{\int_{f_1(x)}^{y} d\eta \int_0^{\eta} \frac{(\eta-\sigma)^{r_2-1}}{(r_2-1)!}\,\tilde{w}_{j+1,\nu}(0,\sigma)\,d\sigma+\right.$$

$$\left.+\sum_{\mu=1}^{r_2} \tilde{C}^{j+1}_{\nu,\mu}\int_{f_1(x)}^{y} \frac{\eta^{r_2-\mu}}{(r_2-\mu)!}\,d\eta\right\}; \qquad (j=1,2,\dots,k-1)$$

$$\tilde{v}_{k,1}(x,y)=e_{k,1}(x,\ R[V(.),\ W(.),\ x,\ f_1(x)],\ W[x,\ f_1(x)])+$$

$$+\int_{f_1(x)}^{y} F(x,\ \eta,\ R[V(.),\ W(.),\ x,\ \eta],\ V(x,\eta),\ W(x,\eta))\,d\eta;$$

(24)

$$\tilde{w}_{j,\beta}(x,y)=g_{j,\beta}(y,\ R[V(.),\ W(.),\ h_\beta(y),\ y],\ V[h_\beta(y),\ y])+$$

$$+\int_{h_\beta(y)}^{x} \tilde{w}_{j,\beta-1}(\xi,y)\,d\xi; \qquad (j=1,2,\dots,k;\ \beta=2,3,\dots,r_1)$$

$$\tilde{w}_{j,1}(x,y)=g_{j,1}(y,\ R[V(.),\ W(.),\ h_1(y),\ y],\ V[h_1(y),\ y])+$$

$$+\int_{h_1(y)}^{x} d\xi \int_0^y \frac{(y-\eta)^{r_2-1}}{(r_2-1)!}\,\tilde{w}_{j+1,r_1}(\xi,\eta)\,d\eta+$$

$$+\sum_{\mu=1}^{r_2} \frac{y^{r_2-\mu}}{(r_2-\mu)!}\left\{\int_{h_1(y)}^{x} d\xi \int_0^{\xi} \frac{(\eta-\sigma)^{r_1-1}}{(r_1-1)!}\,\tilde{v}_{j+1,\mu}(\sigma,0)\,d\sigma+\right.$$

$$\left.+\sum_{\nu=1}^{r_1} \tilde{C}^{j+1}_{\nu,\mu}\int_{h_1(y)}^{x} \frac{\xi^{r_1-\mu}}{(r_1-\mu)!}\,d\xi\right\}; \qquad (j=1,2,\dots,k-1)$$

$$\tilde{w}_{k,1}(x,y)=g_{k,1}(y,\ R[V(.),\ W(.),\ h_1(y),\ y],\ V[h_1(y),\ y])+$$

$$+\int_{h_1(y)}^{x} F(\xi,\ y,\ R[V(.),\ W(.),\ \xi,\ y],\ V(\xi,y),\ W(\xi,y))\,d\xi,$$

where $\tilde{C}^{j+1}_{\nu,\mu}$ are given by formula (10) with $v_{j,\mu}$ and $w_{j,s}$ replaced by $\tilde{v}_{j,\mu}$ and $\tilde{w}_{j,s}$, respectively (and j by $j+1$).

We will find sufficient conditions for the inclusion $T(\mathscr{K}) \subset \mathscr{K}$.

Let us observe that by formulae (9) and (22), and Assumptions I-III, the following inequalities

$$|F(x,\ y,\ R[V(.),\ W(.),\ x,\ y],\ V(x,y),\ W(x,y))| \leqslant M_1 \tag{25}$$

$((x,y) \in \Omega)$ and

$$\begin{aligned} &|e_{j,\alpha}(x,\ R[V(.),\ W(.),\ x,\ f_\alpha(x)],\ W[x,\ f_\alpha(x)])| \leqslant M_2 \\ &|g_{j,\beta}(y,\ R[V(.),\ W(.),\ h_\beta(y),\ y],\ V[h_\beta(y),\ y])| \leqslant M_2 \end{aligned} \tag{26}$$

$((x,y) \in \Omega;\ j=1,2,\dots,k;\ \alpha=1,2,\dots,r_2;\ \beta=1,2,\dots,r_1)$ are valid, where

$$\begin{aligned} M_1 &= C_1\,[K_1 + K_2\,(1+(\tilde{A}\rho)^{r_*}) + K_2'\rho^{r_*}] \\ M_2 &= C_1\,[K_3 + K_4\,(1+(\tilde{A}\rho)^{r_*}) + K_4'\rho^{r_*}] \end{aligned} \tag{27}$$

$(\tilde{A} = \max(A,B))$ with C_1 being a positive constant independent of ρ.

By using (24)-(26) one can prove that

$$\begin{aligned} |\tilde{v}_{j,\alpha}(x,y)| &\leqslant C_2\,(\tilde{A}+M_2+\tilde{A}M_1) \\ |\tilde{w}_{j,\beta}(x,y)| &\leqslant C_2\,(\tilde{A}+M_2+\tilde{A}M_1), \end{aligned} \tag{28}$$

where C_2 is a constant of the same type as C_1 above.

It is easily seen that by Assumptions I-III, the construction of the set $\mathscr{K}$ and formula (24), the functions $\tilde{\phi} = T\phi$ $(\phi \in \mathscr{K})$ are equicontinuous.

Thus, bearing in mind afore-obtained results, we can conclude that $T(\mathscr{K}) \subset \mathscr{K}$ if the following inequality

$$C\{1+\tilde{A}K_1+K_3+(\tilde{A}K_2+K_4)[1+(\tilde{A}\rho)^{r_*}]+(\tilde{A}K_2'+K_4')\rho^{r_*}\} \leqslant \rho \tag{29}$$

is satisfied, where C is a positive constant independent of ρ.

It is clear that (29) holds good if the parameter ρ in (22) is chosen sufficiently large.

Onē can also prove that the following lemma is valid

Lemma 4.– The transformation (23) is continuous.

Thus, all assumptions of Schauder's fixed point theorem are satisfied and using this theorem we can conclude that there is a fixed point $\phi^* = (V^*, W^*) \in \mathscr{K}$ of transformation (23), whence (V^*, W^*) is a continuous solution of system (14). As a consequence (cp. Lemma 3), we can assert that the corresponding function z^*_{1,r_1,r_2} (cp. formula (9)) is a solution of the problem (P) and so the following theorem is established.

Theorem 1.— If Assumptions I-III are satisfied, then problem (P) has a solution.

Remark 3.— Let us note that we have proved the global existence of a solution of problem (P), and hence also of problem (Q) of paper [7], for arbitrary values of K_ν, K'_2, K'_4 $(\nu = 1, 2, 3, 4)$ and without any additional conditions on the curves considered (cp. [7], Theorems 1-3 and Remark 4).

5. This section is devoted to the existence of a unique solution of problem (P).

We retain Assumption I and replace Assumptions II and III by the following ones:

II'. The function $F\colon \Omega \times \mathbb{R}^\kappa \to \mathbb{R}^n$ is continuous and satisfies the Lipschitz condition

$$|F(x,y,\Xi,H^1,H^2) - F(x,y,\tilde{\Xi},\tilde{H}^1,\tilde{H}^2)| \leqslant \leqslant K_5 \hat{\mathscr{C}}_0 + K'_5 \hat{\mathscr{C}}'_0, \tag{30}$$

where K_5 and K'_5 are positive constants, and $\hat{\mathscr{C}}_0$ and $\hat{\mathscr{C}}'_0$ are given by

$$\hat{\mathscr{C}}_0 = \sum_{j=1}^{k} \sum_{\alpha=1}^{r_2} \sum_{\beta=1}^{r_1} |\xi_{j,\beta,\alpha} - \tilde{\xi}_{j,\beta,\alpha}| \tag{31}$$

and

$$\hat{\mathscr{C}}'_0 = \sum_{j=1}^{k} \sum_{\alpha=1}^{r_2} |\eta^1_{j,\alpha} - \tilde{\eta}^1_{j,\alpha}| + \sum_{j=1}^{k} \sum_{\beta=1}^{r_1} |\eta^2_{j,\beta} - \tilde{\eta}^2_{j,\beta}|, \tag{32}$$

respectively (particular symbols are understood analogously as the corresponding symbols in (6), (7)).

III'. The functions $e_{j,\alpha}\colon [0,A] \times \mathbb{R}^{\kappa - knr_2} \to \mathbb{R}^n$ and $g_{j,\beta}\colon [0,B] \times \times \mathbb{R}^{\kappa - knr_1} \to \mathbb{R}^n$ $(j = 1, 2, ..., k;\ \alpha = 1, 2, ..., r_2;\ \beta = 1, 2, ..., r_1)$ are continuous and satisfy the Lipschitz conditions

$$\begin{aligned} &|e_{j,\alpha}(x,\Xi,H^2) - e_{j,\alpha}(x,\tilde{\Xi},\tilde{H}^2)| \leqslant K_6 \hat{\mathscr{C}}_0 + K'_6 \hat{\mathscr{C}}'_1;\\ &|g_{j,\beta}(y,\Xi,H^1) - g_{j,\beta}(y,\tilde{\Xi},\tilde{H}^1)| \leqslant K_6 \hat{\mathscr{C}}_0 + K'_6 \hat{\mathscr{C}}'_2, \end{aligned} \tag{33}$$

where K_6 and K'_6 are positive constants and $\hat{\mathscr{C}}'_\nu$ $(\nu = 1, 2)$ denote the expression $\hat{\mathscr{C}}'_0$ with the first or second term, respectively, being omitted.

Remark 4.– Let us denote

$$K_* = \max \Big\{ \sup_{\Omega} |F(x,y,0,0,0)|,$$

$$\max_{1 \leqslant j \leqslant k} \max_{1 \leqslant \alpha \leqslant r_2} \sup_{[0,\tilde{A}]} |e_{j,\alpha}(x,0,0)|,$$

$$\max_{1 \leqslant j \leqslant k} \max_{1 \leqslant \beta \leqslant r_1} \sup_{[0,\tilde{B}]} |g_{j,\beta}(y,0,0)| \Big\}. \tag{34}$$

It follows from Assumptions II' and III' that

$$\begin{aligned} |F(x,y,\Xi,H^1,H^2)| &\leqslant K_* + K_5 \mathscr{C}_{0*} + K_5' \mathscr{C}_{0*}' \\ |e_{j,\alpha}(x,\Xi,H^2)| &\leqslant K_* + K_6 \mathscr{C}_{0*} + K_6' \mathscr{C}_{1*}' \cdot \\ |g_{j,\beta}(y,\Xi,H^1)| &\leqslant K_* + K_6 \mathscr{C}_{0*} + K_6' \mathscr{C}_{2*}' \end{aligned} \tag{35}$$

where $\mathscr{C}_{0*}$ and $\mathscr{C}_{\nu *}'$ ($\nu = 0, 1, 2$) denote the expressions $\mathscr{C}_0$ and $\mathscr{C}_\nu'$ ($\nu = 0, 1, 2$), respectively, with $r_* = 1$.

Now we will apply the Banach fixed point theorem. Let us consider the set $\mathscr{X}$ (cp. p. 7) and the transformation T (cp. (23)). Evidently, $\mathscr{X}$ can be treated as a complete metric space with the distance $d(\phi_1, \phi) = \|\phi_1 - \phi_2\|$. Moreover, it follows from the results obtained in Section 4 and from Remark 4 that, under Assumptions I, II' and III', the inclusion $T(\mathscr{X}) \subset \mathscr{X}$ holds good if the following inequality.

$$C' [(1+\tilde{A}) K_* + (K_5 + K_5' + K_6)(1 + \tilde{A}\rho) + K_6' \rho] \leqslant \rho \tag{36}$$

is valid, where C' is a positive constant independent of ρ.

It is clear that (36) holds good if ρ is properly chosen and if the Lipschitz coefficients K_5, K_5', K_6, or the Lipschitz coefficient K_6' and the value of $\tilde{A}$ (cp. p. 9), are sufficiently small.

Thus, in order to apply the Banach fixed point theorem we have only to prove that the transformation T is a contraction.

Let us observe that by formulae (9), (10), (30) and (33) we have

$$\begin{aligned} &|F(x, y, R[V_1(.), W_1(.), x, y], V_1(x,y), W_1(x,y)) - \\ &- F(x, y, R[V_2(.), W_2(.), x, y], V_2(x,y), W_2(x,y))| \leqslant \\ &\leqslant C_3 (\tilde{A} K_5 + K_5') d(\phi_1, \phi_2), \end{aligned} \tag{37}$$

$$\begin{aligned} &|e_{j,\alpha}(x, R[V_1(.), W_1(.), x, f_\alpha(x)], W_1[x, f_\alpha(x)]) - \\ &- e_{j,\alpha}(x, R[V_2(.), W_2(.), x, f_\alpha(x)], W_2[x, f_\alpha(x)])| \leqslant \\ &\leqslant C_4 (\tilde{A} K_6 + K_6') d(\phi_1, \phi_2) \end{aligned} \tag{38}$$

and

$$\begin{aligned}|g_{j,\beta}(y, R[V_1(.), W_1(.), h_\beta(y), y], V_1[h_\beta(y), y]) - \\ - g_{j,\beta}(y, R[V_2(.), W_2(.), h_\beta(y), y], V_2[h_\beta(y), y])| \leqslant \\ \leqslant C_4(\tilde{A}K_6 + K'_6)\, d(\phi_1, \phi_2)\end{aligned} \tag{39}$$

($\phi_\nu = (V_\nu, W_\nu) \in \mathscr{X}$; $\nu = 1, 2$), where C_ν ($\nu = 3, 4$) are positive constants independent of ϕ_1 and ϕ_2.

Using (37)-(39) and (24), one can prove that

$$\begin{aligned}|v^1_{j,\alpha}(x,y) - v^2_{j,\alpha}(x,y)| \leqslant C_5[\tilde{A}(K_5 + K_6 + K'_5) + K_6]\, d(\phi_1, \phi_2) \\ |w^1_{j,\beta}(x,y) - w^2_{j,\beta}(x,y)| \leqslant C_5[\tilde{A}(K_5 + K_6 + K'_5) + K_6]\, d(\phi_1, \phi_2)\end{aligned} \tag{40}$$

with C_5 being a constant of the same type as C_3 and C_4 above, and as a consequence of (40) we can assert that the transformation T is a contraction provided that

$$C''[\tilde{A}(K_5 + K_6 + K'_5) + K'_6] < 1, \tag{41}$$

where C'' is a positive constant independent of ϕ_1 and ϕ_2.

Evidently, (41) is satisfied if the Lipschitz coefficients K_5, K'_5, K_6 and K'_6, or the Lipschitz coefficient K'_6 and the value of $\tilde{A}$, are sufficiently small.

Let us assume that inequalities (36) and (41) are satisfied. It follows from the Banach fixed point theorem that the transformation T (cp. (23)) has a unique fixed point $\phi^0 \in \mathscr{X}$, whence, and by (14) and Lemma 3, we can assert that there is a unique solution of problem (P).

As a consequence, we can formulate the following

Theorem 2.– If Assumptions I, II' and III' are satisfied and if the inequalities (36) and (41) hold good, then there exists a unique solution of problem (P).

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