On the Radon-Nikodym property in locally convex spaces and the completeness of L_E^1

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Abstract

The purpose of this paper is to characterize the class of locally convex spaces E for which the space of integrable E-valued functions L_E^1 is quasi-complete and to connect this problem with the Radon-Nikodym Property, martingale convergence, geometric properties of E and nuclearity of operators in $L(L^\infty, E)$. Also permanence properties and examples of locally convex spaces for which L_E^1 is quasi-complete are given.

0. INTRODUCTION

Throughout this paper (X, Σ, μ) will stand for a complete finite measure space, $m: \Sigma \to E$ will be a vector measure while (E, P) will denote a locally convex space E with system of seminorms P.

In case E is a Banach space the Radon-Nikodym Theorem for the Bochner integral can be formulated as follows (see [15], Theorem 8, p. 244).

Radom-Nikodym Theorem. If (X, Σ, μ) is a finite positive measure space and if $m: \Sigma \to E$, E a Banach space, is a measure, then there exists a Bochner integrable function f such that for each $A \in \Sigma$, $m(A) = \int_A f d\mu$ iff:

i) $m \ll \mu$.
ii) $|m|(X) < +\infty$.

and one of the following equivalent conditions is satisfied:

RN1: m has locally relatively compact average range.
RN2: m has locally relatively weakly compact average range.
RN3: m has locally dentable average range.
RN4: m has locally σ-dentable average range.
RN5: m has locally small average range.

In a foregoing paper [2] we have proved a Radon-Nikodym Theorem in case E is a general locally convex space whereby the integral we have used generalizes the Bochner integral. In order to get both necessary and sufficient conditions for the existence of a density, it turns out that one has to impose conditions on the average range which ly between *RN*1 and *RN*5. Although in this general setting the most natural condition seems to be *RN*5, by the lack of countability of the system of seminorms P, *RN*5 does not imply the existence of a density.

On the other hand, it is well known that if E is a Banach space the existence of a density is closely related to the nuclearity of operators in $L(L^\infty, E)$ (see [19]) and the convergence of vector valued martingales (see [4]).

In sectión two these relationships are extended to locally convex spaces. It is shown that $RN5$ can be replaced by the nuclearity by seminorm of the correspondig operator of $L(L^{\infty}, E)$ or by the fact the corresponding martingale is L_E^1-Cauchy.

In the third section we characterize those locally convex spaces for which $RN5$ or one of its equivalent formulations is both necessary and sufficient for m to be indefinite integral of an E-valued function. It appears that this holds exactly for the class of locally convex spaces for which the space L_E^1 of integrable E-valued functions is quasi-complete. We also investigate the relationship between the quasi-completeness of L_E^1 and the Radon-Nikodym Property (RNP) of E. In section four a relationship between the Radon-Nikodym Property and s-dentability for locally convex spaces is established. Hare again it appears that the completeness of L_E^1 plays a fundamental role.

Let us remark that in [11], Egghe has already noticed that RNP and σ-dentability are equivalente notions under certain completeness conditions for L_E^1.

Finally, in the fifth section we give permanence properties and examples of locally convex spaces for which L_E^1 is quasi-complete.

1. DEFINITIONS AND PRELIMINARIES

Let (X, S, m), (E, P) and m be as before. For the terminology used in the sequel concerning locally convex spaces we refer to [13].

For every seminorm $p \in P$ the p-variation of m over $A \in S$ is defined by:

$$|m|_p(A) = \sup_{(A_i) \in P(A)} \sum_{(i)} p(m(A_i))$$

where $P(A)$ denotes the set of all finite partitions of A by means of measurable sets A_i. If for each $p \in P$, $|m|_p(X) < +\infty$, then m is said to be of bounded variation (notation $|m|(X) < \infty$). The vector measure m is called absolutely continous with respect to μ iff for each $p \in P$, $|m|_p(A) \to 0$ whenever $\mu(A) \to 0$ (notation $m \ll \mu$).

The set $A_A(m) = \left\{\frac{m(B)}{\mu(B)} : B \in \Sigma^+ \text{ and } B \subset A\right\}$, where $\Sigma^+ = \{B \in \Sigma : \mu(B) > 0\}$, is called the average range of m on $A \in \Sigma$.

$A_X(m)$ is said to be locally small if for each $p \in P$, $\varepsilon > 0$ and $A \in \Sigma^+$ there exists $B \in \Sigma^+$, with $B \subset A$, such that $A_B(m)$ has p-diameter less than 2ε.

Let us now define a type of integrability of E-valued functions which generalizes the classical Bochner integral. Let us first recall that a function $f: X \to E$ is said to be simple if range f is finite and for every $y \in$ range f, $f_{-1}(y) \in \Sigma$.

The integral of a simple funcion $f = \sum_{(i)} y_i \delta_{A_i}$ is defined by $\int_A f d\mu = \sum_{(i)} y_i \mu(A \cap A_i)$.

Definition 1.1. A function $f: X \to E$ is said to be *integrable by seminorm* if for each $p \in P$ there exist a set $X_0^p \subset X$, with $\mu(X_0^p) = 0$, and a sequence $(f_n^p)_{n \in N}$ of simple functions such that:

(i) $\lim_{n \to \infty} p(f(x) - f_n^p(x)) = 0$, for each $x \in X \setminus X_0^p$, i.e. f is *measurable by seminorm;*

(ii) $p(f(x) - f_n^p(x)) \in L^1(X, \Sigma, \mu)$, for every $n \in \mathbf{N}$ and $\lim_{n\to\infty} \int_X p(f(x) - f_n^p(x))\, d\mu = 0$, for all $p \in P$;

(iii) for each $A \in \Sigma$ there exists $y_A \in E$ such that for every seminorm $p \in P$:

$$\lim_{n\to\infty} p\left(\int_A f_n^p\, d\mu - y_A\right) = 0.$$

We then define $y_A = \int_A f d\mu$.

For further properties on measurability and integrability by seminorm we refer to [1] and [6].

Definition 1.2. Call $\mathrm{L}_E^1(X, \Sigma, \mu)$ the set of all functions $f: X \to E$ which are integrable by seminorm and η the subset of $\mathrm{L}_E^1(X, \Sigma, \mu)$ consisting of those elements f such that for each $p \in P$:

$$\int_X p(f(x))\, d\mu = 0.$$

Define $\mathrm{L}_E^1 = L_E^1(X, \Sigma, \mu) = \mathrm{L}_E^1(X, \Sigma, \mu)/\eta$ and $p^*(f^*) = \int_X p(f)\, d\mu$, for each $f^* \in \mathrm{L}_E^1$ and $p \in P$. Then onviously $P^* = \{p^* : p \in P\}$ is a system of seminorms on L_E^1.

In the sequel, when we shall speak of the completeness of L_E^1, convergence in L_E^1, etc., then these notions are always linked to the system of seminorms P^*.

Definition 1.3. A locally convex space E is said to hace the *Radon-Nikodym Property* (*RNP*) if for every complete finite measure space (X, Σ, μ) and every vector measure $m: \Sigma \to E$ with $m \ll \mu$ and $|m|(X) < +\infty$ there exists a function $f: X \to E$ which is integrable by seminorm such that for each $A \in \Sigma$, $m(A) = \int_A f d\mu$.

Many authors (see e.g. [11], [16], [18], ...) impose a supplementary condition on the vector measure m in the definition of *RNP*, namely m has (locally) bounded average range. Locally convex spaces which satisfy this form of Radom-Nikodym Property are called [*RNP*]-spaces.

In the sequel we shall use the following notations:

(i) For each $p \in P$ calle $\hat{E}_p$ the completion of the quotient space $E/p_{-1}(0)$ provided with the norm $\hat{p}$ deduced from the seminorm p and denote by j_p the projection $E \to \hat{E}_p$.

(ii) Π is the set of all finite partitions π of X by means of measurable sets.

For each $\pi_1, \pi_2 \in \Pi$ we define $\pi_1 \leqslant \pi_2$ iff each member of π_1 is μ-almost the union of members of π_2.

2. VECTOR MEASURES, ABSOLUTELY SUMMING OPERATORS IN $L(L^\infty, E)$ AND MARTINGALES

In the sequel we assime E to be a quasi-complete locally convex space. Let us first recall some definitions about operators (see [7]).

Definition 2.1. Let (E, P) and (F, Q) be locally convex spaces.

(i) An operator $T \in L(E, F)$ is said to be *nuclear by seminorm* if for each $q \in Q$ there exist $p \in P$, $(t_n)_{n \in N}$ in $\mathbf{R}$, $(y'_n)_{n \in N}$ in E' and $(x_n)_{n \in N}$ in F such that:

$$T(y) \overset{q}{=} \sum_{n=1}^{\infty} t_n \langle y'_n, y \rangle x_n$$

whereby $\sum_{n=1}^{\infty} |t_n| < +\infty$, $y'_n \in b_p^{\Delta}(1)$ and $q(x_n) \leqslant 1$ (notice that $\overset{q}{=}$ means convergence for the seminorm q).

(ii) An operator $T \in L(E, F)$ is said to be *absolutely summing* if it maps unconditional convergent series in E into absolutely convergent series in E.

In this section we shall give an extension of the relationship established in [19] between absolutely summing operators in $L(L^\infty, E)$ and E-valued measures, E being a Banach space, to the case where E is a general locally convex space.

Let $m : \Sigma \to E$ be a vector measure with $m \ll \mu$ and $|m|(X) < +\infty$.

Define for each scalar valued simple function $h' = \sum_{(i)} c_i \delta_{A_i}$:

$$T(h') = \int_X h' \, dm = \sum_{(i)} c_i m(A_i).$$

Since $p(\int_X h' \, dm) \leqslant \|h'\|_\infty |m|_p(X)$, the operator $T : L^\infty \to E$ is well defined and $T \in L(L^\infty, E)$.

Conversely, let $T \in L(L^\infty, E)$; the $m(A) = T(\delta_A)$ defines a finitely additive vector measure.

The operator T is absolutely summing if and only if the corresponding (finitely additive) vector measure has bounded variation.

In case E is a Banach space or a Fréchet space, m has a density if an only if the induced operator T is nuclear (see [19]). It is possible to give a partial generalization of this result for locally convex spaces.

Proposition 2.2. *A μ-continous E-evalued masure m of bounded variation has locally small average range if and only if the corresponding absolutely summing operator $T : L^\infty \to E$ is nuclear by seminorm. Moreover m possesses a density f which is integrable by seminorm if and only if T is of the following form:*

$$T(h') = \int_X h' f d\mu, \text{ for each } h' \in L^\infty.$$

Now let us show vector measures are realted to vector valued martingales.

Definition 2.3. Let $(I, \ll)$ be a directed set and $(\Sigma_\alpha)_{\alpha \in I}$ a family of sub σ-algebras of Σ such that $\Sigma_{\alpha_1} \subset \Sigma_{\alpha_2}$, whenever $\alpha_1 \ll \alpha_2$.

Let furthermore $f_\alpha : X \to E$ be a family of functions which are Σ_α-measurable by seminorm and integrable by seminorm.

The family $(f_\alpha, \Sigma_\alpha)_{\alpha \in I}$ is called *a martingale* iff:

$$\int_A f_\alpha d\mu = \int_A f_\beta d\mu, \text{ for all } \beta \gg \alpha \text{ and } A \in \Sigma_\alpha.$$

We always assume that $\sigma\left(\bigcup_{\alpha \in I} \Sigma_\alpha\right) = \Sigma$. A martingale $(f_\alpha, \Sigma_\alpha)$ is said to be uniformly integrable if for each $p \in P$:

$$\lim_{\mu(A) \to 0} \sup_{\alpha \in I} \int_A p(f_\alpha)\, d\mu = 0$$

Now let $m: \Sigma \to E$ be a vector measure with $m \ll \mu$ and $|m|(X) < +\infty$. Define:

$$f_\pi = \sum_{A \in \pi} \frac{m(A)}{\mu(A)} \delta_A$$

where $\sigma(\pi)$ is the σ-algebra generated by π.

Then $(f_\pi, \sigma(\pi))_{\pi \in \Pi}$ is a bounded and uniformly integrable martingale in L^1_E.

Moreover $\int_A f_\pi\, d\mu \to m(A)$, for each $A \in \Sigma$.

Conversely, let $(f_\alpha, \Sigma_\alpha)_{\alpha \in I}$ be a bounded and uniformly integrable martingale and define for each $A \in \bigcup_{\alpha \in I} \Sigma_\alpha$:

$$m(A) = \lim_{\alpha \in I} \int_A f_\alpha d\mu$$

Then, just as in the Banach space setting, m may be extended to a σ-additive $\Sigma \to E$ vector measure with $m \ll \mu$ and $|m|(X) < +\infty$.

Now we formulate the results of Proposition 2.2 in terms of uniformly integrable martingales. By doing this we generalize [4], Theorem 6.

Proposition 2.4. *A vector measure $m: \Sigma \to E$ with $m \ll \mu$ and $|m|(X) < +\infty$ has locally small average range if and only if the corresponding bounded and uniformly integrable martingale is L^1_E-Cauchy. Moreover this vector measure possesses a density which is integrable by seminorm if and only if the corresponding martingale is convergent in L^1_E.*

Proof. Follows immediately from the Radon-Nikodym Theorem in Banach spaces applied to each measure $j_p \circ m: \Sigma \to \hat{E}_p$, and [12], Theorem 6.

Combining the previus results, we obtain:

Theorem 2.5. *Let E be a quasi-complete locally convex space. Then the following are equivalent:*

(i) *Every vector measure $m: \Sigma \to E$ with $m \ll \mu$ and $|m|(X) < +\infty$ has locally small average range;*

(ii) *Every absolutely summing operator $T \in L(L^\infty, E)$ which induces a σ-additive vector measure is nuclear by seminorm;*

(iii) *Every uniformly integrable martingale is L^1_E-Cauchy.*

Teorem 2.6. *Let E be a quasi-complete locally convex space. Then the following are equivalent:*

(i) *E has RNP;*
(ii) *Every absolutely summing operator $T \in L(L^\infty, E)$ which induces a σ-additive vector measure can be written as $T(h') = \int_X h' f d\mu$, whereby $f \in L_E^1$;*
(iii) *Every uniformly integrable martingale is L_E^1-convergent.*

Remark that if E is either a Banach space or a Fréchet space the conditions in Theorem 2.5 are equivalent to those in Theorem 2.6. In the following section we show that this equivalence fails if E is a general locally convex space. It will appear that this is due essentially to the lack of quasi-completeness of L_E^1.

3. THE COMPLETENESS OF L_E^1

In [2] we have proved a general Radon-Nikodym Theorem for E-valued functions by associating a density to a lifting ρ for (X, Σ, μ) in the following way.

Ley for every $x \in X$:

$$S(x) = \{\rho(A) : x \in \rho(A) \text{ and } A \in \Sigma^+\}$$

and $\rho(A) \ll \rho(B)$ iff $\rho(B) \subset \rho(A)$.

Theorem 3.1. (Radon-Nikodym). *Let E be a locally convex space, let (X, Σ, μ) be a complete finite measure space and let ρ be a lifting on Σ.*

If m is an E-valued vector measure, then there exists a function $f: X \to E$ which is integrable by seminorm such that $m(A) = \int_A f d\mu$, for each $A \in \Sigma$, if $X_0 \subset X$ may be found, with $\mu(X_0) = 0$, such that:

(i) *$m \ll \mu$;*
(ii) *m has finite variation;*
(iii) *m has locally small average range;*
(iv) *The net $\left(\dfrac{m(A)}{\mu(A)}\right)_{A \in S(x)}$ converges for every $x \in X \setminus X_0$.*

If one analyses carefully the proof of Theorem 3.1 (see [2]) then one may observe that condition (iv) is necessary for the construction of the density f while conditions (i), (ii) and (iii) yield the integrability by seminorm of f. But when the system of seminorms is countable, condition (iv) becomes superfluous. It should be interesting to characterize those spaces for which condition (iv) of Theorem 3.1 is unnecessary. From what follows it apperars that this question if closely related to a fundamental problem in vector valued integration theory, namely the completeness of L_E^1.

First we associate to a bounded Cauchy net in L_E^1 a vector measure in the following way.

Let (E, P) be a quasi-complete locally convex space and let $(f_\alpha)_{\alpha \in I}$ be a bounded Cauchy net in L_E^1.

Put for each $A \in \Sigma$, $m(A) = \lim_{\alpha \in I} \int_A f_\alpha d\mu$. Since for each $p \in P$, $(j_p \circ f_\alpha)_{\alpha \in I}$ is a Cauchy net in $L_{E_p}^1$, there exists a Bochner integrable density for $j_p \circ m$. Hence m is a μ-continuous measure of bounded variation with a locally small average range. We call m limit measure of the Cauchy net $(f_\alpha)_{\alpha \in I}$. Let us now show how the limit of a bounded Cauchy net in L^1_E is related to the density of the corresponding limit measure.

Lemma 3.2. *Let $(f_\alpha)_{\alpha \in I}$ be a Cauchy net in L^1_E and let $m: \Sigma \to E$ be its limit measure. If there exists a function $f: X \to E$ which is integrable by seminorm and such that for each $A \in \Sigma$, $m(A) = \int_A f d\mu$, then $f_\alpha \underset{L^1_E}{\to} f$.*

Proof. Define for each $\pi \in \Pi$ the simple function:

$$f_\pi = \sum_{A \in \pi} \frac{m(A)}{\mu(A)} \delta_A.$$

The clearly $(f_\pi, \sigma(\pi)) \cup \{f, \Sigma\}$ is a martingale and $f_\pi \underset{L^1_E}{\to} f$.

Let $p \in P$ and $\varepsilon > 0$ be given. Since for each $\alpha \in I, f_\alpha$ is integrable by seminorm, there exist simple functions $(g^p_\alpha)_{\alpha \in I}$ such that:

$$\int_X p(f_\alpha - g^p_\alpha)\, d\mu \leqslant \frac{\varepsilon}{6}.$$

Choose $\alpha_0 \in I$ such that for $\alpha, \beta \gg \alpha_0$:

$$\int_X p(f_\alpha - f_\beta)\, d\mu \leqslant \frac{\varepsilon}{6}.$$

For each $\alpha \in I$, $\pi_\alpha \in \Pi$ can be found such that for the corresponding g^p_α:

$$\sum_{A \in \pi} \frac{\int_A g^p_\alpha d\mu}{\mu(A)} \delta_A = g^p_\alpha$$

and this for every $\pi \geqslant \pi_\alpha$.

Finally choose $\pi_0 \in \Pi$ in such a way that, if $\pi \geqslant \pi_0$,

$$\int_X p(f - f_\pi)\, d\mu \leqslant \frac{\varepsilon}{2}$$

Whenever $\alpha \gg \alpha_0$, take $\pi \geqslant \{\pi_0, \pi_\alpha\}$. Then

$$\int_X p(f - f_\alpha)\, d\mu \leqslant \int_X p(f - f_\pi)\, d\mu + \int_X p(f_\pi - f_\alpha)\, d\mu \leqslant$$

$$\leqslant \frac{\varepsilon}{2} + \int_X p(f_\alpha - g^p_\alpha)\, d\mu + \int_X p\left[\sum_{A \in \pi} \frac{\left(\int_A g^p_\alpha d\mu - \int_A f_\alpha d\mu\right)}{\mu(A)} \delta_A\right] d\mu +$$

$$+ \int_X p\left[\sum_{A\in\pi} \frac{\left(\int_A f_\alpha d\mu - m(A)\right)}{\mu(A)} \delta_A\right] d\mu \leqslant$$

$$\leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{6} + \sum_{A\in\pi} \int_A p(g_\alpha^p - f_\alpha)\, d\mu + \sum_{A\in\pi} p\left(\int_A f_\alpha d\mu - m(A)\right) \leqslant$$

$$\leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \lim_{\beta\in I} \int_X p(f_\alpha - f_\beta)\, d\mu \leqslant \varepsilon$$

Hence $f_\alpha \to f$ in L_E^1. ∎

Theorem 3.3. *If E is a quasi-complete locally convex space, then the following conditions are equivalent:*

(i) L_E^1 *is quasi-complete;*

(ii) *For every vector measure* $m : \Sigma \to E$ *whith* $m \ll \mu$, $|m|(X) < +\infty$, *and having locally small average range, there exists a density which is integrable by seminorm;*

(iii) *For every absolutely summing operator* $T \in L(L^\infty, E)$ *which induces a σ-additive vector measure and which is nuclear by seminorm there exists a function f which is integrable by seminorm such that*

$$T(h') = \int_X h' f d\mu, \quad \textit{for each } h' \in L^\infty.$$

(iv) *Every bounded and uniformly integrable martingale which is* L_E^1*-Cauchy is* L_E^1*-convergent.*

Proof. It is sufficient to prove (i) ⇔ (ii).

The implication (ii) ⇒ (i) follows immediately from Lemma 3.2. Conversely, let L_E^1 be quasi-complete and let m be a vector measure with $m \ll \mu$, $|m|(X) < +\infty$ and having locally small average range. By similar arguments as in the Banach setting (see [10]) one can easily prove that $(f_\pi, \sigma(\pi))_{\pi\in\Pi}$, with

$$f_\pi = \sum_{A\in\pi} \frac{m(A)}{\mu(A)} \delta_A,$$

is a bounded Cauchy net for which the limit f satisfies $m(A) = \int_A f d\mu$, for each $A \in \Sigma$. ∎

Remark 3.4. Since un general the space L_E^1 is not quasi-complete, condition (iv) in Theorem 3.1 can not be omitted. Combining Theorem 2.5 and Theorem 3.3, we are now able to give a relationship between the Radon-Nikodym Property for a locally convex space E and the quasi-completeness of L_E^1.

Theorem 3.5. *A quasi-complete locally convex space E has RNP if and only if* L_E^1 *is quasi-complete and one of the equivalent statements in Theorem* 2.5 *is satisfied.*

4. RNP AND σ-DENTABILITY

In case E is a Banach space or a Fréchet space the problem of giving geometrical characterizations of *RNP*-spaces has been the subject of intensive research (see [14], [17], etc.).

These results have been extended to locally convex spaces in which every bounded subset it metrizable (see [18]). For a treatment in general locally convex spaces we refer to [12] and [16].

Let us recall that a subset B of a locally convex space (E, P) is said to be *σ-dentable* if for every $p \in P$ and $\varepsilon > 0$ there exists $y \in B$ such that $y \notin \sigma(B \setminus b_p(y, \varepsilon))$, whereby:

$$\sigma(A) = \left\{ \sum_{i=1}^{\infty} \alpha_i y_i \mid y_i \in B, \alpha_i \geqslant 0 \text{ and } \sum_{i=1}^{\infty} \alpha_i = 1 \right\}$$

The space E is called *σ-dentable* if every non-empty bounded subset of E is σ-dentable.

The following theorem is due to Egghe (see [12], Theorem 3).

Theorem 4.1. *Let E be a quasi-complete locally convex space. Then E possesses [RNP] if and only if E is σ-dentable and every uniformly bounded Cauchy net in* L_E^1 *is convergent.*

In [16], Rodríguez-Salinas proves the equivalence between [*RNP*] and σ-dentability for general locally convex spaces. Yet remark that the density in his definition of [*RNP*] takes values in the bidual E''.

5. PERMANENCE PROPERTIES AND EXAMPLES

From sections three and four it follows that many results concerning the representation of Banach space valued measures by means of an integrable density can be extended to locally convex spaces if and only if L_E^1 is quasi-complete or complete with respect to uniformly bounded nets. In this section we intend to give permanence properties and examples of locally convex spaces for which L_E^1 satisfies this completeness conditions. For that purpose, Theorem 3.3 will be very usefull.

Obviously for every *RNP*-space (resp. [*RNP*]-space) E, L_E^1 is quasi-complete (resp. complete w.r.t. uniformly bounded nets). For examples of such spaces we refer to [3]. Since for every Banach space E, L_E^1 is complete, but on the other hand not every Banach space possesses *RNP*, it makes sense to search for examples of locally convex spaces E for which L_E^1 is quasi-complete (resp. complete w.r.t. uniformly bounded nets) without assuming that E possesses *RNP* (resp. [*RNP*]).

We start with some permanence properties.

Proposition 5.1. *Let $(E_i, P_i)_{i \in N}$ be an increasing sequence of quasi-complete locally convex spaces for which L_E^1 is quasi-complete (or complete w.r.t. uniformly bounded nests). If the inductive limit E of the spaces $(E_i, P_i)_{i \in N}$ is strict and moreover localizes convergent sequences, then L_E^1 is quasi-complete (or complete w.r.t. uniformly bounded nets).*

Proof. Let $m\colon \Sigma \to E$ be a μ-continuous measure of bounded variation with locally small average range. Then there exists a partition $(X_i)_{i \in N}$ of X μ.a.e. with $X_i \in \Sigma^+$, such that $A_{X_i}(m) \subset E_i$.

If this was not the cse, one could choose a pairwise disjoint sequence $(A_i)_{i \in N}$ in Σ^+ for which $m(A_i) \in E_i \setminus E_{i-1}$. But since $m(A_i) \to 0$, there exists an integer n such that for each $i \in N$, $m(A_i) \in E_n$, a contradiction.

Now let for each $i \in N$, $\Sigma_i = \Sigma \cap X_i$; then $m_i : \Sigma_i \to E_i$ is a μ-continuous vector measure of bounded variation with locally small average range. In view of Theorem 3.3 there exists $f_i \in \mathrm{L}^1_{E_i}(X_i, \Sigma_i, \mu)$ such that $m_i(A) = \int_A f_i d\mu$, $A \in \Sigma_i$.

Then the function $f = \sum_{i=1}^{\infty} f_i \delta_{X_i}$ is a density for m, and by Theorem 3.3 the space L^1_E is quasi-complete. ■

Let (E, P) and (F, Q) be locally convex spaces. Let $\mathscr{F}$ be a family of bounded subsets of E such that $\bigcup_{B \in \mathscr{F}} B = E$ and for all $B, B' \in \mathscr{F}$, there exist $B'' \in \mathscr{F}$ and $c > 0$ such that $B, B' \subset cB''$. The space $L_{\mathscr{F}}(E, F)$ is the space $L(E, F)$ equiped with the topology defined by the seminorms $q_B(T) = \sup_{y \in B} q(Ty)$, $B \in \mathscr{F}$ and $q \in Q$.

Proposition 5.2. *Let $(E_i, P_i)_{i \in N}$ be an increasing sequence of separable locally convex spaces. Let furthermore (F, Q) be locally convex space and suppose that for each $i \in N$, $\mathrm{L}^1_{L_F(E_i, F)}$ is complete w.r.t. uniformly bounded nets. If the inductive limit (E, P) of the spaces $(E_i, P_i)_{i \in N}$ is hyperstrict then $\mathrm{L}^1_{L_F(E, F)}$ is complete w.r.t.-uniformly bounded nets.*

Proof. Let $m : \Sigma \to L_{\mathscr{F}}(E, F)$ be a μ-continuous measure of bounded variation with an average range which is bounded and locally small.

Define for each $i \in N$, $m_i : \Sigma \to L_{\mathscr{F}}(E_i, F)$, with $m_i(A) = m(A) \mid E_i$. Then m_i has the same properties as m.

Since $L^1_{L_F(E_i, F)}$ is complete w.r.t. uniformly bounded nets, there exists an $L_{\mathscr{F}}(E_i, F)$-valued density f_i for m_i, which is integrable by seminorm.

Now let $y \in E$ and let $i \in N$ be the smalles integer for which $y \in E_i$.

Since for each $j \geqslant i$, $\int_A f_i(x)(y)\, d\mu = m_i(A)(y) = m_j(A)(y) = \int_A f_j(x)(y)\, d\mu$, there exists a set $X^y_{i,j}$, with $\mu(X^y_{i,j}) = 0$ such that $f_i(x)(y) = f_j(x)(y)$, for all $x \in X \setminus X^y_{i,j}$. Let $\{y_{i,n} \parallel n \in N\}$ be a countable dense subset of E_i. Call $X_0 = \bigcup_{i=1}^{\infty} \bigcup_{j>i} \bigcup_{n=1}^{\infty} X^{y_{i,n}}_{i,j}$; then $\mu(X_0) = 0$.

Define $f\colon X \to L_{\mathscr{F}}(E, F)$ as follows:

$$f(x)(y) = f_i(x)(y), \quad \text{if} \quad y \in E_i \quad \text{and} \quad x \in X \setminus X_0$$
$$f(x) = 0, \quad \text{if} \quad x \in X_0.$$

Then f is the desired density for m. ■

Corolario 5.3. *Let $(E_i, P_i)_{i \in N}$ be an increasing sequence os separable locally convex spaces such that for each $n \in N$, $\mathrm{L}^1_{(E_i)'_F}$ is complete w.r.t. uniformly bounded nets. If the inductive limit (E, P) of the space $(E_i, P_i)_{i \in N}$ is hyperstrict, then $\mathrm{L}^1_{E'_F}$ is complete w.r.t. uniformly bounded nets.*

Proposition 5.4. *Let $(E_i, P_i)_{i \in N}$ be a sequence of locally convex spaces for which $L^1_{E_i}$ is quasi-complete (resp. complete w.r.t. uniformly bounded nets). If E is the product of the spaces $(E_i, P_i)_{i \in N}$, then L^1_E is quasi-complete (resp. complete w.r.t. uniformly bounded nets).*

Proof. Apply Theorem 3.3 to each projection $m_i : \Sigma \to E_i$ of m. ■

Proposition 5.5. *Let E be a closed linear subspace of a Suslin space F and suppose that L^1_F is quasi-complete (resp. complete w.r.t. uniformly bounded nets). Then L^1_E is also quasi-complete (resp. complete w.r.t. uniformly bounded nets).*

Proof. Follows immediately from [20], Property VIII.

Let us give examples of locally convex spaces for which L^1_E is quasi-complete (resp. complete w.r.t. uniformly bounded nets).

Examples 5.6. *Let E be a quasi-complete locally convex space in which bounded subset is metrizable. Then L^1_E is complete w.r.t. uniformly bounded nets.*

Proof. Apply [18], Theorem 2.5 and Theorem 4.1. ■

Example 5.7. *Let E be a quasi-complete locally convex space with property (BM)* (see [5]). *Then L^1_E is quasi-complete.*

Proof. Let $m : \Sigma \to E$ a μ-continuous measure of bounded variation. According to [5], Theorem 3.1, it is possible to find a partition $(X)_{i \in N}$ of X μ*.a.e.*, such that $A_{X_i}(m)$ is contained in a bounded metrizable subset of E. The conclusion follows as in Example 5.6. ■

Consequently if E is a Fréchet space or a strict (LF)-space, L^1_E es quasi-complete locally convex spaces with the property (BM) we refer to [5].

Example 5.8. Let E be a separable evaluable space and let F be a quasi-complete Suslin space for which L^1_F is complete w.r.t. uniformly bounded nets. Then the space $L^1_{L_{pc}(E, F)}$ is complete w.r.t. uniformly bounded nets.

Proof. Let $m : \Sigma \to L_{pc}(E, F)$ be a μ-continuous measure of bounded variation and with an average range which is bounded and locally small. Since E is evaluable, $A_X(m)$ is equicontinuous. Let $\{y_i \| i \in N\}$ be a countable dense subset of E. Define for each

$$i \in N, \quad m_i : \Sigma \to F \text{ by } m_i(A) = m(A)(y_i), \quad A \in \Sigma.$$

Then m_i is a μ-continuous F-valued measure of bounded variation with an average range which is bounded and locally small. Hence there exists a function $f_i : X \to F$ which is integrable by seminorm such that $m_i(A) = \int_A f_i d\mu$, $A \in \Sigma$. Since F is a Suslin space, a set $X_{0,i} \subset X$, with $\mu(X_{0,i}) = 0$, may be found such that for every $x \in X \setminus$

$X_{0,i}$, the net $\left(\frac{m(A)}{\mu(A)}\right)_{A\in S(x)}$ converges (see [2], Proposition 3.4). Let $X_0 = \bigcup_{i=1}^{\infty} X_{0,i}$; then $\mu(X_0) = 0$.

Considerer for each $x \in X \setminus X_0$, the net $\left(\frac{m(A)}{\mu(A)}\right)_{A\in S(x)}$

In view of the Banach-Steinhaus Theorem this net is convergent in $L_{pc}(E, F)$. Consequently the space $L^1_{L_{pc}(E,F)}$ is complete w.r.t. uniformly bounded nets. ■

Example 5.9. *Let E be a Fréchet-Schwartz space and let F be a quasi-complete Suslin space for which* L^1_F *is complete w.r.t. uniformly bounded nets. Then the space* $L^1_{E\hat{\oplus}F}$ *is complete w.r.t. uniformly bounded nets. The same conclusion holds in case E is a strict inductive limit of Fréchet-Schwartz spaces and F is quasi-complete Suslin space of type Q* (see [9], p. 84).

Proof. In both cases, the tensor product $E \mathbin{\hat{\oplus}_\varepsilon} F$ is a closed subspace of the Suslin space $L_{pc}(E'_b, F)$ (see [8], [9]). In view of Proposition 5.3 we may restrict ourselves to the case where E is a Fréchet-Schwarts space. Then E'_b is a separable evaluable space, and so the conclusion follows by means of Example 5.8 and Proposition 5.5.

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