Torres de espectros asociadas a cadenas de complejos

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Resumen

Se presenta el concepto de una torre de espectros asociada a una cadena de complejos sobre un A–módulo, donde A es el álgebra de Steenrod mód. 2, dentro de la categoría de espectros localizados en $p = 2$ y clase de homotopía de funciones entre tales espectros.

Se dan tres ejemplos (los dos primeros bien conocidos):

1) el espectro *bu* (que representa la K–teoría conectiva compleja)
2) una torre de espectros con periodicidad, donde el espectro de la base es *bo* (que representa la K–teoría conectiva real) y
3) un espectro Y_5 asociado a una cadena de complejos finita, cuyos grupos de homotopía son:

$$\pi_i(Y_5) = \begin{cases} Z \text{ si } i = 2^k - 2,\ 1 \leqslant k \leqslant 6 \\ 0 \text{ en todo otro caso} \end{cases}$$

Este trabajo fue motivado por un problema propuesto por F. Peterson en 1970 (ver [6] en la Bibliografía) sobre el cual también han trabajado D. S. Kahn, D. Kraines y R. Steiner, entre otros.

I. INTRODUCTION

This paper presents the concept of tower of spectra associated to chain complexes over a Steenrod algebra module and gives three examples of such towers.

The Steenrod algebra modulo 2 will be denoted by A. We will work in the category of spectra localized at two and homotopy classes of maps between such spectra. $KG(m)$ wil represent the m–fold suspension of the Eilenberg–MacLane spectrum KG (or $KG(0)$) for a group G. Thus, $KG(m) = \Sigma^m KG(0)$ and $(KG(m))_n = K(G, m+n)$ for all nonnegative integers m, n. Other spectra which appear in this paper are *bo* and *bu* (the Ω–spectra representing connective real and complex K–theory, respectively), $\Sigma^m bo[t]$ (the m–fold suspension of the (t–1)–connected covering spectrum over *bo*) and $\Sigma^4 bsp = bo[3]$.

Section II gives the necessary algebraic and topological background, as well as the definition of a realization of a chain complex by a tower of spectra. Sections III, IV and V present three examples of such towers, with the corresponding constructions and proofs.

A problem related to our present task was posed by F. Peterson in 1970

"Does there exist an Ω–spectrum $X(n)$ with homotopy groups

$$\pi_k(X(n)) = \begin{cases} \mathbb{Z}_2 \text{ if } k = 2^r n,\ r \geqslant 0 \\ 0 \text{ otherwise} \end{cases}$$

and so that the rth stage Postnikov system has k–invariants coming from the relations

$$Sq^{n+1} = 0$$

$$Sq^{2n+1}\ Sq^{n+1} = 0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$Sq^{2^{r-1}\ n+1}\ (\ldots Sq^{n+1}) = 0?$$

One would expect the answer to be Yes. Also, the 0-th term of the spectrum should be a product of $K(\mathbb{Z}_2,n)$x...x$K(\mathbb{Z}_2,2^i)$x... with twisted Hopf algebra structure, so the homology generator in dimension n generates a polynomial algebra. D. S. Kahn and D. Kraines have partial results." [6]

The three examples presented here are partial contributions to achieving a better understanding of Peterson's problem, which has been solved by Steiner [14], using an algebraic method of Segal's [13], and to extend the problem to cases not covered by them.

The first example consists of an infinite tower of fibrations of spectra with bottom spectrum $KZ(0)$, the Eilenberg–Mac Lane spectrum for additive group of integers Z. In the end, we obtain a spectrum S equivalent to bu. The tower constructed is associated with a chain complex $\mathscr{C}_1$ over $H^*(KZ(0);Z_2) = A/ASq^1$, with differentials $d_k = Sq^3$.

In the second example an infinite periodic tower of fibrations of spectra is constructed, starting with the real connective spectrum bo. At each of the intermediate stages a well–known spectrum appears. The tower itself is associated to a chain complex $\mathscr{C}_2$with periodic differentials ($d_k = d_{k+4}$ for $k = 0, 1,2, \ldots$) over $H^*(bo;Z_2)$.

The third example es a finite Postnikov tower with bottom spectrum KZ(0) that realizes a finite chain complex $\mathscr{C}_3$ over $H^*(KZ(0);Z_2)$. The final spectum obtained has integral homotopy groups in dimensions 2^k-2 ($1 \leqslant k \leqslant 6$) and zero otherwise. Some relations involving X, the Thom antiautomorphism of the Steenrod algebra modulo 2, are used in the construction of the tower.

The techniques used in this paper can also be found in works by Davis [2], Mahowald ([4], [5]), Milgram ([5], [7]), Mosher and Tangora [8] and others.

II. ALGEBRAIC AND TOPOLOGICAL BACKGROUND

Let A_k be the subset of A generated by the elements Sq^{2^i}, $i = 1, 2, ..., k$. A_k is a subalgebra of A over the field Z_2. The following Z_2–cohomology groups are well known in the literature ([5], [7], [8], [9], [10]):

$$H^*(KZ_2) \cong A(i_0)$$

$$H^*(KZ) \cong (A/ASq^1)(i_0)$$

$$H^*(bo) \cong (A/A(Sq^1, Sq^2))(\nu_0)$$

$$H^*(bu) \cong (A/A(Sq^1, Sq^3))(x_0)$$

$$H^*(bo[1]) \cong (A/ASq^2)(\nu_1)$$

$$H^*(bo[2]) \cong (A/Sq^3)(\nu_2)$$

$$H^*(bo[3]) \cong H^*(\Sigma^4 bsp) = (A/A(Sq^1, Sq^5))(\nu_4)$$

where i_j, x_k and ν_t are generators in the lowest dimension.

A chain complex is a sequence of groups and homomorphisms

$$\mathfrak{G} : G_0 \longleftarrow G_1 \longleftarrow G_2 \longleftarrow G_3 \longleftarrow G_4 \longleftarrow \ldots$$

such that $d_t d_{t+1} = 0$. If $G_k = 0$ for all k greater than some positive integer m we say that $\mathfrak{G}$ is a finite chain complex of length m.

Consider a spectrum X and a map of spectra $f: X \to KG(m+1)$. The path fibration over $KG(m+1)$ has total space $PKG(m+1)$ and fiber $\Omega KG(m+1) \cong KG(m)$ [8]. The fibration over X induced by f has the same fiber and total space X' consisting of all pairs (x, t) such that $f(x) = \pi(t)$ in $KG(m+1)$. So the following is a commutative diagram of spectra and maps of spectra:

$$\begin{array}{ccccc} KG(m) & \longrightarrow & X' & \longrightarrow & PKG(m+1) \\ & & \downarrow p & & \downarrow \pi \\ & & X & \xrightarrow[f]{} & KG(m+1) \end{array}$$

One can repeat the process of obtaining induced fibrations, obtaining a tower of fibrations (or stalactite of fibrations) of spectra.

$$
\begin{array}{ccccc}
 & & \downarrow & & \\
KG(m_1) & \xrightarrow{j_2} & X_2 & \xrightarrow{f_2} & KG(m_2+1) \\
 & & \downarrow & & \\
KG(m_0) & \xrightarrow{j_1} & X_1 & \xrightarrow{f_1} & KG(m_1+1) \\
 & & \downarrow & & \\
 & & X_0 & \xrightarrow{f_0} & KG(m_0+1)
\end{array}
$$

where each $f_k \in [X_k, KG(m_k+1)] \cong H^{m_k+1}$ $(X_k;G)$. The f_k's are called k–invariants. The maps j_k and p_k are inclusion of the fiber and proyection onto the base spectrum respectively.

Suppose we have a tower of fibrations of spectra whoose s-th, $(s+1)$-th and $(s+2)$-th levels look as follows:

$$
\begin{array}{ccccc}
 & & \downarrow & & \\
KG(m_2) & \xrightarrow[j_{s+2}]{} & X_{s\ 2} & & \\
 & & \downarrow & & \\
KG(m_1) & \xrightarrow[j_{s+1}]{} & X_{s\ 1} & \xrightarrow[\nu_{s+1}]{} & KG(m_2+1) \\
 & & \downarrow & & \\
 & & X_s & \xrightarrow[\nu_s]{} & KG(m_1+1) \\
 & & \downarrow & &
\end{array}
$$

Diagram A

with $KG(s)$ = the Eilenberg–Mac Lane spectrum $\Sigma^s KG$, and $G = Z$ or Z_2; ν_s is the s-th k–invariant, and each level is induced over the previous one by the path fibration over the corresponding $KG(t)$. At each level we obtain long exact sequences of cohomology groups with Z_2 for coefficients:

$$\begin{array}{lllll} ..H^*(X_{s+2}) \xrightarrow{j^*} H^*(KG(m_2)) \xrightarrow{\tau} & H^{*+1}(X_{s+1}) \xrightarrow{p^*} & (*) \\ & \| & \\ ...H^*(KG(m_1)) \longrightarrow H^{*+1}(X_s) \longrightarrow & H^{*+1}(X_{s+1}) \longrightarrow & (**) \end{array}$$

$$\begin{array}{ll} (*) \quad H^{*+1}(X_{s+2}) \xrightarrow{j^*} H^{*+1}(KG(m_2)) \xrightarrow{\tau} & H^{*+2}(X_{s+1}) \xrightarrow{p^*} ... \\ & \| \\ (**) \quad H^{*+1}(KG(m_1)) \longrightarrow H^{*+2}(X_s) \longrightarrow & H^{*+2}(X_{s+1})) \longrightarrow ... \end{array}$$

Diagram B

Suppose now that $\mathfrak{G} = \{G_k, d_k\}$ is a chain complex, with $d_k = \beta = Sq^I$ for some admissible sequence $I = (i_1, ..., i_r)$ and $d_{k+1} = \alpha = Sq^J$ for some admissible sequence $J = (j_1, ..., j_q)$. Then $d_k d_{k+1} = \beta\alpha = 0$. We say that the tower is associated to a chain complex $\mathfrak{G}$ or that the tower "realizes" $\mathfrak{G}$ if

$$\beta\alpha(i_{m_1}) = \beta j^*_{s+1}(\nu_{s+1}) = j^*_{s+1}(\beta\, \nu_{s+1}) = 0 \quad (***)$$

One observes that $\tau(i_{m_1}) = \nu_s$. Then ν_{s+1} is the class in $H^*(X_{s+1})$ such that $j^*_{s+1}(\nu_{s+1}) = \alpha(i_{m_1})$ and it arises by exactness of the long cohomology sequence when $\tau(\alpha(i_{m_1})) = \alpha(\tau(i_{m_1})) = \alpha(\nu_s) = 0$. Then $\tau(i_{m_2}) = \nu_{s+1}$ and $\tau(\beta(i_{m_2})) = \beta(\nu_{s+1})$. If this last term is zero, then we obtain the desired chain of equalities in (***).

Also by exactness one notices that there exists a class ν_{s+2} in $H^*(X_{s+2})$ such that

$$j^*_{s+2}(\nu_{s+2}) = \beta(i_{m_2})$$

and this ν_{s+2} becomes the next k-invariant.

The explanation given above can best be described by the following diagrams, which must be read together with diagram B in mind:

$$\begin{array}{ccccc} ...\longrightarrow H^*(X_{s+2}) & \xrightarrow[j^*_{s+2}]{} & H^*(KG(m_2)) & \xrightarrow[\tau]{} & H^*l1(X_{s+1}) \longrightarrow ... \\ & & i_{m_2} & \longmapsto & \nu_{s+1} \\ \nu_{s+2} & \longmapsto & \beta(i_{m_2}) & \longmapsto & \beta(\nu_{s+1}) = 0 \end{array}$$

Diagram C

$$
\begin{array}{ccccccc}
\cdots \longrightarrow H^*(X_{s+1}) & \xrightarrow[j^*_{s+1}]{} & H^*(KG(m_1)) & \xrightarrow[\tau]{} & H^*(X_s) & \longrightarrow \cdots \\
 & & i_{m_1} & \longmapsto & \nu_s & \\
\nu_{s+1} & \longmapsto & \alpha(i_{m_1}) & \longmapsto & \alpha(\nu_s) = 0 &
\end{array}
$$

Diagram D

The rest of the paper consists of three examples in which a given chain complex is realized by a tower of fibrations of spectra in the way described above.

III. Example 1

Let $A' = A/ASq^1 = H^*(KZ;Z_2)$. Then, the following is an infinite chain complex:

$$\mathscr{C}_1 : B_0 \xleftarrow{d_0} B_1 \xleftarrow{d_1} B_2 \longleftarrow \cdots \longleftarrow B_{n-1} \xleftarrow{d_{n-1}} B_n \longleftarrow \cdots$$

where $B_k = A'$ and $d_k = Sq^3$ for every $k = 0, 1, 2, \ldots$.
We observe that $d_k d_{k+1} = Sq^3 Sq^3 = Sq^5 Sq^1 = 0 \bmod A'$.

Theorem 3.1. There is a tower of spectra and a spectrum S that realize the chain complex $\mathscr{C}_1$. Furthermore, S satisfies the following:

a) $H^*(S;Z_2) \cong A/A(\mathrm{Sq}^1, Sq^3)$

b) $\pi_i(S) = \begin{cases} Z & \text{if } i \text{ is even} \\ 0 & \text{otherwise} \end{cases}$

(Thus $S \approx bu$.)

Proof: Consider the fibration of spectra

$$
\begin{array}{ccccc}
KZ(2) & \xrightarrow{j_1} & X_1 & \longrightarrow & PKZ(3) \\
 & & \downarrow & & \downarrow \\
X_0 = & & KZ(0) & \xrightarrow[\nu_3]{} & KZ(3)
\end{array}
$$

where $\nu_3 \in [KZ(0),KZ(3)]$ corresponds to the class $bSq^2(i_0) \in H^3(KZ(0);Z)$, i_0 is the generator of $H^0(K\mathbb{Z}(0);\mathbb{Z})$ corresponding to the identity in $[K\mathbb{Z}(0), K\mathbb{Z}(0)]$, b is the integral Bockstein and X_1 is the induced spectrum over the path fibration $\Omega KZ(3) \approx KZ(2) \longrightarrow PKZ(3) \longrightarrow KZ(3)$.

Consider the long exact Z_2–cohomology sequence of the induced fibration:

$$\ldots \longrightarrow H^n(KZ(2)) \xrightarrow{\tau} H^{n+1}(KZ(0)) \xrightarrow{p_1^*} H^{n+1}(X_1) \xrightarrow{j_1^*} H^{n+1}(KZ(2)) \rightarrow$$

where τ is the cohomology transgression homomorphism. If i_s is the nonzero generator of $H^s(KZ(s);Z_2)$ $(s = 1, 2)$ we have

$$\tau(i_2) = Sq^3(i_0)$$

and

$$\tau(Sq^2(i_2)) = Sq^2Sq^3(i_0)$$

Furthermore,

$$\tau\, Sq^3(i_2) = Sq^3Sq^3(i_0) = 0 \bmod A/ASq^1$$

By exactness of the long cohomology sequence we obtain the following

a) There exist classes $\nu'_5, \nu_7 \in H^*(X_1;Z_2)$ with $\nu_7 = Sq^2(\nu'_5)$ such that

$$j_1{}^*(\nu'_5) = Sq^1Sq^2(i_2)$$

$$j_1{}^*(\nu_7) = Sq^2Sq^1Sq^2(i_2)$$

b) ν_5 is an integral class in $H^5(X_1;Z)$ which reduces modulo 2 to ν'_5. This is easy to see, since $Sq^1Sq^2(i_2)$ = reduction modulo 2 of $bSq^2(i_2) \in H^5(KZ(2);Z)$. Hence,

$$j_1{}^*(\nu_5) = bSq^2(i_2)$$

and

$$j_1{}^*(r_2(\nu_5)) = r_2(bSq^2(i_2)) = r_2j_1{}^*(\nu_5) = Sq^3(i_2).$$

c) The two nonzero classes in dimension 0 and 2 in $H^*(KZ(0);Z_2)$ are mapped under $p_1{}^*$ into $H^*(X_1;Z_2)$ by exactness.

Taking

$$\nu_5 \in H^5(X_1;Z) \cong [X_1, KZ(5)]$$

as our k–invariant, and proceeding as before, we obtain the following tower of fibrations, where each X^{n+1} is obtained from the induced fibration over the previous k–invariant $\nu_{2n+3} \in [X^n, KZ(2n+3)]$ with fiber $\Omega KZ(2n+3) \cong KZ(2n+2)$:

$$
\begin{array}{ccccc}
 & & \downarrow & & \\
KZ(2n+2) & \xrightarrow{j_{n+1}} & X_{n+1} & \xrightarrow{\nu_{2n+5}} & KZ(2n+5) \\
 & & \downarrow & & \\
KZ(2n) & \xrightarrow{j_n} & X_n & \xrightarrow{\nu_{2n+3}} & KZ(2n+3) \\
 & & \downarrow & & \\
 & & \vdots & & \\
 & & \downarrow & & \\
KZ(4) & \xrightarrow{j_2} & X_2 & \xrightarrow{\nu_7} & KZ(7) \\
 & & \downarrow & & \\
KZ(2) & \xrightarrow{j_1} & X_1 & \xrightarrow{\nu_5} & KZ(5) \\
 & & \downarrow & & \\
 & KZ(0) = & X_0 & \xrightarrow{\nu_3} & KZ(3)
\end{array}
$$

Diagram E

Remark: Recall that $H^*(bo; Z_2) \cong A \otimes_{A_1} Z_2$. We need the following

Lemma 3.2 A Z_2–basis for $A \otimes_{A_1} \mathbb{Z}_2$ is given by $\{ XSq^I / I$ an admissible sequence $(i_1, ..., i_n)$, $i_s \geqslant 2i_{s+1}$ and $i_1 \equiv 0 \pmod 4$, $i_2 \equiv 0 \pmod 2) \}$. (See [4], p. 367).

Proof: It is well–nown that $\{ Sq^J / J = (j_1, ... j_t), j_s \geqslant 2j_{s+1}\}$ is a Z_2–basis for A (Serre–Cartan basis). Also $\{ XSq^J \}$ is a basis, where X is Thom anti-automorphism of the Steenrod algebra. One can verfy that

$$Sq^{2n+1} = Sq^1 Sq^{2n}$$

$$Sq^{4n+2} = Sq^2 Sq^{4n} + Sq^1 Sq^{4n} Sq^1$$

by the Adem relations

$$Sq^a Sq^b = \sum_{c=0}^{[a/2]} \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c$$

for $a < 2b$, where $[a/2]$ is the greatest integer less than or equal to $a/2$ and the binominal coefficient is taken modulo 2. Thus,

$$XSq^{2n+1} = X(Sq^1 Sq^{2n}) = (XSq^{2n}) Sq^1$$

and

$$XSq^{4n+2} = X(Sq^2 Sq^{4n}) = (XSq^{4n}) Sq^2$$

and both equations are equal to zero in A_1. So, for every n, a nonnegative integer, we obtain that the only possible nonzero elements of the form XSq^I are those whose first term i_1 is of the form $4n$.

Also, we have $X(Sq^{4n}Sq^2) = Sq^2(XSq^{4n})$ and, by the Adem relations:

$$X(Sq^{4n}Sq^1) = X(Sq^1Sq^{4n} + Sq^2Sq^{4n-1}) =$$

$$= (XSq^{4n})Sq^1 + (XSq^{4n-1})Sq^2.$$

But this last term is zero in A_1, which also shows that $i_2 \equiv 0 \pmod 2$ for $XSq_I \neq 0$ in the basis of $A \otimes_{A_1} \mathbb{Z}_2$.

We now continue with the proof of Theorem 3.1.

We observe that

$$H^*(X^1; Z_2) = (A/A(Sq^1, Sq^3))p_1{}^*(i_0) \oplus (A/A(Sq^1, Sq^3))\nu_5'$$

As we proceed to the n-th level we obtain an exact $\mathbb{Z}_2$–cohomology sequence

$$\ldots \longrightarrow H^m(KZ(2n)) \underset{\tau}{\longrightarrow} H^{m+1}(X_{n-1}) \underset{p_n^*}{\longrightarrow} H^{m+1}(X_n) \underset{j_n^*}{\longrightarrow} H^{m+1}(KZ(2n)) \longrightarrow \ldots$$

corresponding to the fibration

$$\begin{array}{ccccc} KZ(2n) & \xrightarrow{\;j_n\;} & X_n & \longrightarrow & PKZ(2n+1) \\ & & \Big\downarrow p_n & & \Big\downarrow \\ & & X_{n-1} & \xrightarrow[\nu_{2n+1}]{} & KZ(2n+1) \end{array}$$

The classes in dimensions $2n+1$ and $2n+3$ in $H^*(X_{n-1}; Z_2)$ are "killed" by transgressions from the cohomology of the fiber. The classes in dimensions 0 and 2, which come from the cohomology of the base by exactness, keep "surviving" and are mapped by p_n^* into $H^*(X_n; Z_2)$, also by exactness. Two new classes in dimensions 2n+3 and $2n+5$ arise in the last cohomology group, so that

$$j_n^*(\nu_{2n+3}) = Sq^1Sq^2(i_{2n})$$

and

$$j_n^*(\nu_{2n+5}) = j_n^*(Sq^2\nu_{2n+3}) = Sq^2Sq^1Sq^2(i_{2n})$$

where ν_{2n+3} is the reduction modulo 2 of an integer class, by the same argument used for ν_3 above. This ν_{2n+3} is chosen as our next k-invariant.

Taking $S = \lim X_n$ (inverse limit in the category of spectra and homotopy maps of spectra) we obtain the desired spectrum.

It is imediate that

$$H^*(S; Z_2) \cong A/A(Sq^1, Sq^3) \cong H^*(bu; Z_2)$$

by the previous two paragraphs. Also from the homotopy exact sequence of each fibration one obtains

$$\pi_i(X_n) = \begin{cases} Z \text{ if } i \text{ is even, } i \leqslant 2n \\ 0 \text{ otherwise} \end{cases}$$

In the limit, one obtains

$$\pi_i(S) = \begin{cases} Z \text{ if } i \text{ is even} \\ 0 \text{ otherwise} \end{cases}$$

In order to check that the tower constructed realizes the chain complex $\mathscr{C}_1$ in the sense of section II, consider the portion of diagram E

$$\begin{array}{ccccc}
 & & \downarrow & & \\
KZ(2n+2) & \xrightarrow{j_{n+1}} & X_{n+1} & \xrightarrow{\nu_{2n+5}} & KZ(2n+5) \\
 & & \downarrow & & \\
KZ(2n) & \xrightarrow{j_n} & X_n & \xrightarrow{\nu_{2n+3}} & KZ(2n+3) \\
 & & \downarrow & & \\
KZ(2n-2) & \xrightarrow{j_{n-1}} & X_{n-1} & \xrightarrow{\nu_{2n+1}} & KZ(2n+1) \\
 & & \downarrow & &
\end{array}$$

We observe that

$$Sq^3(i_{2n}) = j_n^*(\nu_{2n+3})$$

and

$$0 = Sq^3 Sq^3(i_{2n}) = Sq^3 j_n^*(\nu_{2n+3}) = j_n^*(Sq^3 \nu_{2n+3}).$$

This completes the proof of Theorem 3.1.

IV Example 2

Let $C_0 = A/A(Sq^1, Sq^2)$. For $j \neq 0$ define

$$C_j = \begin{cases} A/ASq^1 \text{ if } j \equiv 0,1 \pmod 4 \\ A \text{ if } j \equiv 2,3 \pmod 4 \end{cases}$$

Also, define the following homomorphisms:

$$d_j = \begin{cases} Sq^2 \text{ if } j \equiv 0,1 \pmod 4 \\ Sq^1 Sq^2 \text{ if } j \equiv 2 \pmod 4 \\ Sq^2 Sq^1 Sq^2 \text{ if } j \equiv 3 \pmod 4 \end{cases}$$

We have the next

Proposition 4.1: Let $\mathscr{C}_2$ be the sequence of groups and homomophisms

$$\mathscr{C}_2 : C_0 \xleftarrow[d_0]{} C_1 \xleftarrow[d_1]{} C_2 \xleftarrow[d_2]{} C_3 \xleftarrow[d_3]{} C_4 \xleftarrow[d_4]{} \cdots$$

Then $\mathscr{C}_2$ is a chain complex.

Proof. It is easy to check that for every nonnegative integer k we obtain $d_k d_{k+1} = 0$ (modulo C_k).

Now, consider the fibration induced over the path fibration

$$KZ(0) \longrightarrow PKZ(0) \longrightarrow KZ(0)$$

by the map $\nu_0 \in [bo, KZ(0)] \cong H^0(bo; Z)$ which corresponds to the generator 1. We obtain the following diagram:

$$\begin{array}{ccccc} KZ(-1) \approx \Omega KZ(0) & \xrightarrow{j_1} & X_1 & \longrightarrow & PKZ(0) \\ & & \downarrow{p_1} & & \downarrow \\ & & bo & \longrightarrow & KZ(0) \end{array}$$

with X_1 = induced total space.

In the Z_2–cohomology exact sequence of the induced fibration we have:

$$\tau\ (i_{-1}) = \nu_0$$

$$\tau\ (Sq^2 i_{-1}) = Sq^2 \tau\ (i_{-1}) = Sq^2(\nu_0)$$

$$\tau\ (Sq^1 Sq^2 i_{-1}) = Sq^1 Sq^2 (\nu_0).$$

But the last two equations are equal to zero en $H^*(bo; Z_2)$. Thus. by exacness, a class ν_1 in dimension one exists in $H^*(X_1 : Z_2)$ such that $j_1^*(\nu_1) = Sq^2(i_{-1})$. Similarly,

$$j_1^*(Sq^1 \nu_1) = Sq^1 Sq^2 (i_{-1}).$$

Analizing the long exact sequence in Z_2–cohomology

$$\ldots \longrightarrow H^*(KZ(-1)) \xrightarrow{\tau} H^{*+1}(bo) \xrightarrow{p_1^*} H^{*+1}(X_1) \xrightarrow{j_1^*} H^{*+1}(KZ(-1)) \longrightarrow \ldots$$

we observe that

$$H^{*+1}(X_1) = \text{Ker}\,(\tau) = (A/A(Sq^1, Sq^2)) \otimes_Z (A_1/A_1Sq^2)(\nu_1)$$

and the latter is isomorphic to $A/ASq^2(\nu_1)$. Thus, $X_1 \approx bo[1]$.

For the second k–invariant of the tower of spectra we wish to construct we take $\nu_1 \in [bo[1], KZ_2(1)] = H^1(bo[1], Z_2)$. Consider now the induced fibration with total space X_2 over the path fibration of $KZ_2(1)$ induced by ν_1:

$$\begin{array}{ccccc}
KZ_2\,(0) \approx \Omega K\mathbb{Z}_2(1) & \xrightarrow{j_2} & X_2 & \longrightarrow & PKZ_2(1) \\
 & & \downarrow{\scriptstyle p_2} & & \downarrow \\
 & & bo[1] & \xrightarrow{\nu_1} & KZ_2(1)
\end{array}$$

We conclude that:

$$\tau(i_0) = \nu_1$$

$$\tau(Sq^1(i_0)) = Sq^1(\nu_1)$$

$$\tau(Sq^2Sq^1(i_0)) = Sq^2Sq^1(\nu_1)$$

$$\tau(Sq^2(i_0)) = Sq^2(\nu_1) = 0 \text{ in } H^*(bo[1];Z_2)$$

$$\tau(Sq^I Sq^2(i_0)) = Sq^I Sq^2(\nu_1) = 0 \text{ in } H^*(bo[1])$$

for any admissible sequence I.

By exactness, $\text{Ker}(\tau) = Im\, j_2^*$ and so, $X_2 \approx bo[2]$. Notice that $H^*(X_2;Z_2) = (A/ASq^3)(\nu_2)$, where ν_2 is the nonzero class in $H^2(X_2;Z_2)$ that arises by exactness so that $j_2^*(\nu_2) = Sq^2(i_0)$.

Thus far, we have obtained a two–story tower of fibrations of spectra

$$\begin{array}{ccccc}
KZ_2(0) & \xrightarrow{j_2} & bo[2] & & \\
 & & \downarrow{\scriptstyle p_2} & & \\
KZ(-1) & \xrightarrow{j_1} & bo[1] & \xrightarrow{\nu_1} & KZ_2(1) \\
 & & \downarrow{\scriptstyle p_1} & & \\
 & & bo & \xrightarrow{\nu_2} & KZ(0)
\end{array}$$

Now, taking $\nu_2 \in [bo[2], KZ_2(2)] \cong H^2(bo[2]; Z_2)$ as our next k–invariant, consider the induced fibration over the path fibration and ν_2, with total space X_3:

$$\begin{array}{ccccc} KZ_2(1) \approx \Omega K_2(2) & \xrightarrow{j_3} & X_3 & \longrightarrow & PKZ_2(2) \\ & & \downarrow p_3 & & \\ & & bo[2] = X_2 & \xrightarrow{\nu_2} & KZ_2(2) \end{array}$$

and let us look again at the long exact Z_2–cohomology sequence of the induced fibration:

$$\ldots \longrightarrow H^*(X_3) \underset{j_3^*}{\longrightarrow} H^*(KZ_2(1)) \underset{\tau}{\longrightarrow} H^{*+1}(bo[2]) \underset{p_3^*}{\longrightarrow} H^{*+1}(X_3) \longrightarrow \ldots$$

Then

$$\tau \ (i_1) = \nu_2$$

$$\tau \ (Sq^1(i_1)) = Sq^1 \nu_2$$

$$\tau \ (Sq^2(i_1)) = Sq^2 \nu_2$$

$$\tau \ (Sq^2 Sq^1(i_1)) = Sq^2 Sq^1 \nu_2$$

All of the above are nonzero calsses in $H^*(bo[2]; Z_2)$. But

$$\tau \ (Sq^1 Sq^2(i_1)) = Sq^3 \nu_2 = 0 \ \text{ in } H^*(bo[2]; Z_2).$$

By exactness of the long cohomology sequence there exists a class $\nu_4 \in H^*(X_3; Z_2)$ such that $j_3^*(\nu_4) = Sq^3(i_1)$. Furthermore, ν_4 is the reduction mod 2 of an integer class ν_4. We take this integer class ν_4 as our next k–invariant.

One can easily see that $H^*(X_3; Z_2) \cong (A/A(Sq^1, Sq^5))(\nu_4)$. Thus, we conclude that $X_3 \approx bo[3] = \Sigma^4 bsp$.

Consider now the induced fibration over the path fibration and ν_4:

$$\begin{array}{ccccc} KZ(3) & \longrightarrow & X_4 & \longrightarrow & PKZ(4) \\ & & \downarrow p_4 & & \downarrow \\ & & bo\,[3] = X_3 & \longrightarrow & KZ(4) \end{array}$$

From the Z_2–cohomology exact sequence

$$\ldots \longrightarrow H^*(X_4) \xrightarrow{j_4^*} H^*(KZ(3)) \xrightarrow{\tau} H^{*+1}(bo[3]) \xrightarrow{p_4^*} H^{*+1}(X_4) \longrightarrow \ldots$$

we verify that

$$\tau\ (i_3) = \nu_4$$

$$\tau\ (Sq^2(i_3)) = Sq^2\nu_4$$

$$\tau\ (Sq^1Sq^2(i_3)) = Sq^1Sq^2(\nu_4)$$

$$\tau\ (Sq^2Sq^1Sq^2(i_3)) = Sq^2Sq^1Sq^2\nu_4 = 0 \text{ in } H^*(bo[3])$$

Thus, by exactness, there exists a class $\nu_8 \in H^8(X_4; Z_2)$ such that $j_4^*(\nu_8) = = Sq^5(i_3)$.

It can be verified that $H^*(X_4; Z_2) = (A/A(Sq^1 . Sq^2))(\nu_8)$. Hence, $X_4 \approx \Sigma^8 bo$. Thus, the construction process starts all over again, but eight dimensions higher.

We have seen that an infinite tower can be constructed. It remains to prove that it realizes the infinite chain complex $\mathscr{C}_2$ of proposition 4.1. We summarize what we are set out to prove in:

Theorem 4.2.—The following is an infinite tower of spectra that realizes $\mathscr{C}_2$.

$$
\begin{array}{ccccc}
 & & \downarrow & & \\
KZ_2(8) & \xrightarrow{j_6} & X_6 = \Sigma^8 bo[2] & \xrightarrow{\nu_{10}} & KZ(10) \\
 & & \downarrow & & \\
KZ(7) & \xrightarrow{j_5} & X_5 = \Sigma^8 bo[1] & \xrightarrow{\nu_9} & KZ_2(9) \\
 & & \downarrow & & \\
KZ(3) & \xrightarrow{j_4} & X_4 = \Sigma^8 bo & \xrightarrow{\nu_8} & KZ(8) \\
 & & \downarrow & & \\
KZ_2(1) & \xrightarrow{j_3} & bo[3] = \Sigma^4 bsp & \xrightarrow{\nu_4} & KZ(4) \\
 & & \downarrow & & \\
KZ_2(0) & \xrightarrow{j_2} & X_2 = bo[2] & \xrightarrow{\nu_2} & KZ_2(2) \\
 & & \downarrow & & \\
KZ(-1) & \xrightarrow{j_1} & X_1 = bo[1] & \xrightarrow{\nu_1} & KZ_2(1) \\
 & & \downarrow & & \\
 & & X_0 = bo & \xrightarrow{\nu_0} & KZ(0)
\end{array}
$$

Diagram F

Proof: The construction of the first four stages has been indicated above. In general, one can check that for $k = 0, 1, 2, 4$ and for any nonnegative n, $\nu_{8n+k} = \Sigma^{8n}\nu_k$, with ν_k as before.

It remains to show that this tower realizes the chain complex $\mathscr{C}_2$ of Proposition 4.1. With diagram F in mind, we see that the following diagram illustrates the form in which Z_2–cohomology classes are mapped:

$$\begin{array}{llllllllll}
i_7 & \xrightarrow[\tau]{} & \nu_8 & & & & & & & \\
 & & \downarrow j_4^* & & & & & & & \\
 & & Sq^2Sq^1Sq^2(i_3) & & & & & & & \\
 & & i_3 & \xrightarrow[\tau]{} & \nu_4 & & & & & \\
 & & & & \downarrow j_3^* & & & & & \\
 & & & & Sq^1Sq^2(i_1) & & & & & \\
 & & & & i_1 & \xrightarrow[\tau]{} & \nu_2 & & & \\
 & & & & & & \downarrow j_2^* & & & \\
 & & & & & & Sq^2(i_0) & & & \\
 & & & & & & i_0 & \xrightarrow[\tau]{} & \nu_1 & \\
 & & & & & & & & \downarrow j_1^* & \\
 & & & & & & & & Sq^2(i_{-1}) & \\
 & & & & & & & & i_{-1} \xrightarrow[\tau]{} & \nu_0
\end{array}$$

Therefore,

$Sq^2Sq^2(i_{-1}) = Sq^2(j_1^*(\nu_1)) = j_1^*(Sq^2\nu_1) = 0$, since $Sq^2\nu_1 = 0$ in $H^*(bo[1];Z_2 \cong A/ASq^2(\nu_1)$.

$Sq^1Sq^2Sq^2(i_0) = \mathrm{Sq}^1Sq^2(j_2^*(\nu_2)) = j_2^*(Sq^3(\nu_2)) = 0$, since $Sq^3(\nu_2) = 0$ in $H^*(bo[2];Z_2) \cong A/ASq^3(\nu_2)$. Notice that also $Sq^1Sq^2Sq^2 = 0$ by the Adem relations.

$Sq^2Sq^1Sq^2(Sq^1Sq^2(i_1)) = Sq^2Sq^1Sq^2(j_3^*(\nu_4)) = j_3^*(Sq^2Sq^1Sq^2\nu_4) = 0$, since $Sq^2Sq^1Sq^2\nu_4 = (Sq^5 + Sq^4Sq^1)(\nu_4) = 0$ in $H^*(bo[3];\mathbb{Z}_2) \cong (A/A(Sq^1,Sq^5))(\nu_4)$.

$Sq^2(Sq^2Sq^1Sq^2(i_3)) = Sq^2(j_4^*(\nu_8)) = j_4^*Sq^2(\nu_8) = 0$, since $Sq^2(\nu_8) = 0$ in $H^*(\Sigma^8 bo; Z_2)$.

The proof of this relation relies on computations due to D. Davis and others (Cfr. [1], [11]). We list here some relevant formulas involving the Thom antiautomorphism X used in the proof. Part c) is just Proposition 5.1.

Lemma 5.2.—The following relations hold in A/ASq^1.

$$\text{a)}\quad Sq^1(XSq^{2^k}) = Sq^{2^{k-1}}Sq^1(XSq^{2^{k-1}})$$

$$\text{b)}\quad (XSq^{2^k})Sq^1(XSq^{2^{k+1}}) = 0$$

$$\text{c)}\quad Sq^1(XSq^{2^{k+1}})Sq^1(XSq^{2^k}) = 0$$

Now, we come to the main theorem of this example:

Theorem 5.3.—There is a finite tower of fibrations of spectra, with bottom spectrum $KZ(0)$, that realizes the chain complex $\mathscr{C}_3$. Furthermore, this is an "induced tower" over another tower of fibrations of spectra with bottom spectrum $\Sigma^3 bu$, and the two towers can be represented by the following picture:

$$\begin{array}{ccccccc}
KZ(62) & \xrightarrow{\;j_5\;} & Y_5 & \xrightarrow{\;a_5\;} & T_5 & & \\
 & & \downarrow & & \downarrow & & \\
KZ(30) & \xrightarrow{\;j_4\;} & Y_4 & \xrightarrow{\;a_4\;} & T_4 & \xrightarrow{\;f_4\;} & KZ(63) \\
 & & \downarrow & & \downarrow & & \\
KZ(14) & \xrightarrow{\;j_3\;} & Y_3 & \xrightarrow{\;a_2\;} & T_3 & \xrightarrow{\;f_3\;} & KZ(31) \\
 & & \downarrow & & \downarrow & & \\
KZ(6) & \xrightarrow{\;j_2\;} & Y_2 & \xrightarrow{\;a_2\;} & T_2 & \xrightarrow{\;f_2\;} & KZ(15) \\
 & & \downarrow & & \downarrow & & \\
KZ(2) & \xrightarrow{\;j_1\;} & Y_1 & \xrightarrow{\;a_1\;} & T_1 & \xrightarrow{\;f_1\;} & KZ(7) \\
 & & \downarrow & & \downarrow & & \\
KZ(0) & \underset{j_0 = Id}{=\!=\!=} & Y_0 & \underset{a_0}{\longrightarrow} \Sigma^3 bu = & T_0 & \underset{f_0}{\longrightarrow} & KZ(3)
\end{array}$$

Corollary 5.4.—The homotopy groups of the spectrum Y_5 are as follows:

$$\pi_i(Y_5) = \begin{cases} Z \text{ if } i = 2^k-2,\ 1 \leqslant k \leqslant 6 \\ 0 \text{ otherwise} \end{cases}$$

The proofs appear in [11] (see also [15]). Let it suffice to say that the T—tower looks as follows:

By the periodicity of both the tower and the chain complex one sees that these are all the possible cases to be considered. The proof of Theorem 4.2 is complete.

Finally, one sees the following immediately:

Corollary 4.3. The tower of Theorem 4.2 is periodic and each piece looks as follows:

$$
\begin{array}{ccccc}
 & & \downarrow & & \\
KZ(8n+3) & \longrightarrow & \Sigma^{8(n+1)}bo & \xrightarrow[\nu_{8(n+1)}]{} & KZ(8(n+1)) \\
 & & \downarrow & & \\
KZ_2(8n+1) & \longrightarrow & \Sigma^{8n}bo[3] & \xrightarrow[\nu_{8n+4}]{} & KZ(8n+4) \\
 & & \downarrow & & \\
KZ_2(8n) & \longrightarrow & \Sigma^{8n}bo[2] & \xrightarrow[\nu_{8n+2}]{} & KZ_2(8n+2) \\
 & & \downarrow & & \\
KZ(8n-1) & \longrightarrow & \Sigma^{8n}bo[1] & \xrightarrow[\nu_{8n+1}]{} & KZ_2(8n+1) \\
 & & \downarrow & & \\
KZ(8(n-1)+3) & \longrightarrow & \Sigma^{8n}bo & \xrightarrow[\nu_{8n}]{} & KZ(8n) \\
 & & \downarrow & &
\end{array}
$$

V. Example 3

Let $A' = A/ASq^1$, as in example 1. Let $D_k = A'$ and $d_k = Sq^1(XSq^{2^k})$ for $1 \leqslant k \leqslant 6$. Then:

Proposition 5.1.—Let $\mathscr{C}_3$ be the finite chain of groups and homomorphisms

$$\mathscr{C}_3 : D_1 \xleftarrow{d_1} D_2 \xleftarrow{d_2} D_3 \xleftarrow{d_3} D_4 \xleftarrow{d_4} D_5 \xleftarrow{d_5} D_6$$

Then $\mathscr{C}_3$ is a chain complex.

Proof: It suffices to show that $d_k d_{k+1} = 0 (\mathrm{mod}\ Sq^1)$ for all nonnegative integers. This is equivalent to showing that

$$Sq^1(XSq^{2^{k+1}})Sq^1(XSq^{2^k}) \equiv 0 \ (\mathrm{mod}\ Sq')$$

$$
\begin{array}{ccccc}
KZ(62) & \xrightarrow{i_5} & T_5 & & \\
 & & \downarrow & & \\
KZ(30) & \xrightarrow{i_4} & T_4 & \xrightarrow{f_4} & KZ(63) \\
 & & \downarrow & & \\
KZ(14) & \xrightarrow{i_3} & T_3 & \xrightarrow{f_3} & KZ(31) \\
 & & \downarrow & & \\
KZ(6) & \xrightarrow{i_2} & T_2 & \xrightarrow{f_2} & KZ(15) \\
 & & \downarrow & & \\
KZ(2) & \xrightarrow{i_1} & T_1 = \Sigma^5 bu & \xrightarrow{f_1} & KZ(7) \\
 & & \downarrow & & \\
 & & T_0 = \Sigma^3 bu & \xrightarrow{f_0} & KZ(3)
\end{array}
$$

Furthermore, each f_i satisties the following.

f_0 = nonzero generator of $H^3(\Sigma^3 bu; Z)$

$f_1 = Sq^2(v_5) \in H^7(\Sigma^5 bu; Z)$, where v_5 = nonzero generator of $H^5(\Sigma^5 bu; Z)$.

$f_2 = Sq^4 Sq^2 \; Sq^1(v_8) \in H^{15}(T_2; \mathbb{Z})$

$f_3 = Sq^8 Sq^4 Sq^1(v_{18}) \in H^{31}(T_3 \cdot Z)$

$f^4 = Sq^{16} Sq^8 Sq^1(v_{38}) \in H^{63}(T_4; Z)$

where each v_m appears by exactness of the long exact sequence in cohomology of each fibration ($m = 2^{k+1} + 2^{k-1} - 2$, $k = 2, 3, 4$) and each f_k is an integer class ($0 \leqslant k \leqslant 4$).

The Y—tower ia induced over the T—tower in the following way. At the first stage we have

$$
\begin{array}{ccccc}
 & \swarrow & KZ(2) & \searrow & \\
Y_1 & \xrightarrow{a_1} & \Sigma^5 bu & \longrightarrow & PKZ(3) \\
\downarrow & & \downarrow & & \downarrow \\
KZ(0) & \xrightarrow{a_0} & \Sigma^3 bu & \xrightarrow{f_0} & KZ(3)
\end{array}
$$

where $a_0 \in [KZ(0), \Sigma^3 bu]$ such that $f_0 a_0 = bSq^2(i_0) \in [KZ(0), KZ(3)] \cong H^3(KZ(0); Z)$. Then Y_1 is the total spectrum over the path fibration $K\mathbb{Z}(2) \longrightarrow PKZ(3) \longrightarrow KZ(3)$ induced by $f_0 a_0$. By the functorial properties of induced fibrations Y_1 is equivalent to the total spectrum induced by a_0 over the fibration $KZ(2) \longrightarrow \Sigma^5 bu \longrightarrow \Sigma^3\, bu$, so there exists a map $a_1 : Y_1 \longrightarrow \Sigma^5\, bu$.

Taking the composition of a_1 with $f_1 \in [T_1, KZ(7)]$ we obtain a map $f_1 a_1 \in [Y_1, K\mathbb{Z}(7)]$ which induces the spectrum Y_2 over the path fibration $KZ(6) \longrightarrow PKZ(7) \longrightarrow KZ(7)$, and the process can be repeated again.

Using this constructive method, the process can only go as far as the picture of the complete tower indicates. It is not clear that $v_s \in H^t(Y_k; Z_2)$ exists ($s = 2^{k+1} + 2^{k-1} - 2$; $t = 2^{k+2} - 1$) such that $Sq^{2^k} Sq^{2^{k-1}} Sq^1(v_s)$ is an integer class for $k \geqslant 5$.

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