On a littlewood–paley theorem and connections between some non-isotropic distributions spaces

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Abstract

A Littlewood–Paley type theorem for L^P spaces with mixed norms is established. As an application connections between spaces of Besov–Nikol'skii (non–isotropic Besov–spaces) and spaces of Lizorkin–Nikol'skii (non–isotropic potential spaces) are given.

INTRODUCTION

The purpose of this note is twofold. The first objetive is to answer a question raised by J. Peetre in his "New Thoughts on Besov Spaces" ([7]), which is to give a proof of the Littlewood–Paley inequalities that uses only scalar versions of Mihlin's multiplier theorem. We shall state non–isotropic Littlewood–Paley inequalities and we shall give a proof that uses the Lizorkin scalar version of Mihlin's theorem ([4]). The multipliers will be used to prove only one side of the inequality. The other side will be obtained from the former, by a modification of an idea of E. M. Stein ([9, p. 105]). It is clear that the idea of our proof can be easily adapted to the usual isotropic inequality. The second goal is to give connections between some non–isotropic distribution spaces namely the Besov–Nikol'skii spaces (Nikol'skii [6], Fernandez [3]) and the Lizorkin–Nikol'skii spaces (Lizorkin–Nikol'skii [5]). These connections will be a direct consequence of the non–isotropic Littlewood–Paley inequality.

We shall work in the context of the L^P spaces with mixed norm of Benedek–Panzone [1].

1. THE LITTLEWOOD–PALEY INEQUALITIES

1.1.—Let ψ be a rapidly decreasing function on $\mathbb{R}$ such that

(1) $\quad \text{supp}\ \psi \subset [t|2^{-1} \leqslant |t| \leqslant 2]$;

(2) $\quad \psi(t) > 0 \ \text{ if } 2^{-1} < |t| < 2$

(3) $\quad \sum_{n=-\infty}^{\infty} \psi(2^{-n} t) = 1;$

and let us define φ_n, $n \in \mathbb{N}$, by $\hat{\varphi}_n(t) = \psi(2^{-n}t)$ if $n = 1, 2, \ldots$, and

$$\hat{\varphi}_0(t) = 1 - \sum_{n=1}^{\infty} \psi(2^{-n}t).$$

Let us consider d functions $\psi^1, \ldots, \psi^d$ from $S(\mathbb{R})$ which satisfy (1)–(3) and the corresponding sequences $(\varphi^1_{n_1}), \ldots, (\varphi^d_{n_d})$. Now, let us set

$$\varphi_N = \varphi_{n_1 \ldots n_d} = \varphi^1_{n_1} \otimes \ldots \otimes \varphi^d_{n_d}, \quad N = (n_1, \ldots, n_d) \in \mathbb{N}^d.$$

We call the multiple sequence $(\varphi_N)_{N \in \mathbb{N}^d}$ a system of test functions on $\mathbb{R}^d$.

1.2.–*Theorem (of Littlewood–Paley):* Let f be a function in $L^P(\mathbb{R}^d)$, where $P = (p_1, \ldots, p_d)$ and $1 < p_1, \ldots, p_d < \infty$, and let $(\varphi_N)_N$ be a system of test functions. Then, there exist positive constants A and B, independent of f, such that

$$(1) \qquad A\|f\|_{L^P} \leqslant \|(\sum_{N \in \mathbb{N}^d} |\varphi_N * f|^2)^{1/2}\|_{L^P} \leqslant B\|f\|_{L^P}.$$

Proof: Let $(r_n)_{n \geqslant 0}$ be the sequence of Rademacher functions on $[0,1]$. We consider the sequence $(r_N)_{N \geqslant 0}$ given by $r_N = r_{n_1} \ldots r_{n_d}$, $N = (n_1, \ldots, n_d) \geqslant 0$. For each $N \geqslant 0$, the functions r_N are defined on Q, the unit cube in $\mathbb{R}^d$, $Q = \{t = (t_1, \ldots, t_d) / 0 \leqslant t_j \leqslant 1\}$. Next we consider, for each $t \in Q$, the operator

$$T_t f = \sum_{N \geqslant 0} r_N(t)\, \varphi_N * f.$$

The condition 1.1(3) assures that Lizorkin's criterion is satisfied and we see that T_t is a Fourier multiplier on L^P. Whence there is a constant $C > 0$, independent of f, such that

$$\|\sum_{N \geqslant 0} r_n(t)\, \varphi_N * f\|_{L^P} \leqslant C\|f\|_{L^P}.$$

Now, by Minkowski's inequality and a property of the Rademacher functions, we have

$$(2) \qquad \|f\|_{L^P} \geqslant C \int_Q \|\sum_{N \geqslant 0} r_N(t)\, \varphi_N * f\|_{L^P} dt$$
$$\geqslant C \| \int_Q |\sum_{N \geqslant 0} r_N(t)\, \varphi_N * f| dt \|_{L^P} \geqslant A \|[\sum_{N \geqslant 0} |\varphi_N * f|^2]^{1/2}\|_{L^P},$$

which is one half of (1).

The second part of (1) follows from the first. Suppose $f \in L^P$ and $g \in L^{P'}$, where $P' = (p'_1, \ldots, p'_d)$ and $1/p'_j + 1/p_j = 1, j = 1, 2, \ldots, d$. Now, the Schwarz inequality, the Hölder inequality and the inequality (2) imply that

$$|\int_{\mathbb{R}^d} f(x)\overline{g(x)}dx| = |\Sigma_N \Sigma_N \langle \varphi_N * \overline{g}, \varphi_M * f \rangle|$$
$$= \Sigma_N \sum_{\substack{l_j = -2 \\ j = 1, \ldots, d}}^{2} \langle \varphi_N * \overline{g}, \varphi_{N+L} * f \rangle| \quad (L = (l_1, \ldots, l_d))$$

$$\leqslant C \int_{\mathbb{R}^d} \| (\varphi_N * f)_N \|_{l^2} \ \|(\varphi_N * \bar{g})_N \|_{l^2} \ dx$$

$$\leqslant C \,\|(\varphi_N * f)_N \|_{L^P(l^2)} \ \|(\varphi_N * \bar{g})_N \|_{L^{P'}(l^2)} \leqslant CA^{-1} \|(\varphi_N * f)_N \|_{L^P(l^2)} \ \|g\|_{L^{P'}}$$

Taking the supremun over all $g \in L^{P'}$ with the restriction $\|g\|_{L^{P'}} \leqslant 1$, we get the second half of (1) with $B = CA^{-1}$.

The proof is complete.

2. BESOV–NIKOL'SKII SPACES AND LIZORKIN–NIKOL'SKII SPACES

Let us recall the definition of the Besov–Nikol'skii spaces $B_P^{S,Q}$ and the Lizorkin–Nikol'skii spaces $H^{S,P}$. For the properties of these spaces we refer to Nikol'skii [6], Lizorkin–Nikol'skii [5] and Fernandez [3].

2.1.– *Definition:* Suppose $S = (s_1, \ldots, s_d) \in \mathbb{R}^d$ and $1 \leqslant P,\ Q \leqslant \infty$. Let $(\varphi_N)_{N \in \mathbb{N}^d}$ be a sequence of test functions. The linear space

$$B_P^{S,Q} = B_P^{S,Q}(\mathbb{R}^d) = \left\{ f \in S'(\mathbb{R}^d) \,|\, (2^{S \cdot N} \varphi_N * f)_{N \in \mathbb{N}^d} \in l^Q(L^P) \right\}$$

is called a Besov–Nikol'kii space. We equip the space $B_P^{S,Q}$ with the norm

$$(1) \qquad \|f\|_{B_P^{S,Q}} = \|(2^{S\cdot N} \varphi_N * f)_{N \in \mathbb{N}^d} \|_{l^Q(L^P)} = \|(2^{S\cdot N} \|\varphi_N * f\|_{L^P})_{N \in \mathbb{N}^d}\|_{l^Q}.$$

The linear space

$$H^{S,P} = H^{S,P}(\mathbb{R}^d) = \left\{ f \in S'(\mathbb{R}^d) \,|\, \bar{F}\ \Pi_{j=1}^d (1 + |t_j|^2)^{s_j/2} Ff \in L^P \right\}$$

is called a Lizorkin–Nikol'skii space or non–isotropic potential space. We equip $H^{S,P}$ with the norm

$$(2) \qquad \|f\|_{H^{S,P}} = \|\bar{F}\, \Pi_{j=1}^d (1 + |t_j|^2)^{s_j/2} Ff\|_{L^P}.$$

We need the following lemma whose proof is analogous to the isotropic case, with some natural modifications, and we omit it.

2.2.–*Lemma:* Suppose $R = (r_1, \ldots, r_d) \in \mathbb{R}^d$ and for $f \in S'(\mathbb{R}^d)$ let us set

$$(1) \qquad J^R f = \bar{F}\, \Pi_{j=1}^d (1 + |t_j|^2)^{s_j/2} Ff.$$

Then, the operator J^R is an isomorphism from $B_P^{S,Q}$ onto $B_P^{S-R,Q}$ and from $H^{S,Q}$ onto $H^{S-P,P}$. Moreover $H^{S,P} = J^{-S} L^P$.

Now, we are ready to state and prove the promised connections between the $B^{S,Q}$ and $H^{S,P}$ spaces.

2.3.–Theorem: Suppose $S = (s_1, \dots, s_d) \in \mathbb{R}^d$ and let $p = (p_1, \dots, p_d)$ be such that $1 < p_1 \leqslant p_2 \leqslant \dots \leqslant p_d \leqslant 2$. We have

$$(1) \qquad B_{P}^{S,P} \subset H^{S,P} \subset B_{P'}^{S,2},$$

and

$$(2) \qquad B_{P'}^{S,2} \subset H^{S,P'} \subset B_{P'}^{S,P'}, \quad (P+P' = PP'),$$

where the embedding mappings are continuous. In particular

$$(3) \qquad B_{2}^{S,2} = H^{S,2}$$

Proof: Due to the above lemma, it is enough to prove (1) and (2) for $S = 0$. Then, due to the Minkowski inequality, the Littlewood–Paley inequalities, the fact that $l^P \subset l^2$ which holds under our hypothesis and Minkowski inequality once again we get

$$\begin{aligned} A\,\|f\|_{B_P^{0,2}} &= A\,\|(\varphi_N * f)_N\|_{l^2(L^P)} \leqslant A\,\|(\varphi_N * f)_N\|_{L^P(l^2)} \\ &\leqslant \|f\|_{L^P} \leqslant B\,\|(\varphi_N * f)_N\|_{L^P(l^2)} \\ &\leqslant B\,\|(\varphi_N * f)_N\|_{L^P(l^P)} \leqslant B\,\|(\varphi_N * f)_N\|_{l^P(L^P)} \\ &= B\,\|f\|_{B_P^{O,P}} \end{aligned}$$

These inequalities allow us to infer (1). The inclusions (2) follow by duality and (3) follows by taking $P = 2$ in (1) or (2).

The proof is complete.

The identity 2.3(3) follows also from the connections between the real and the complex interpolation spaces of several Banach spaces (see Bertolo–Fernández [2]).

REMARKS

(A) The proof of theorem 1.2 can obviously be adapted to the usual eliptic or isotropic case. In this way, we have answered Peetre's questions raised in the Introduction. Thus, our proof seems to be of independent interest.

(B) Closely related non–isotropic Littlewood–Paley inequalities were obtained by N. M. Rivière [5, Ch. II, Th. 1.8–9]. However, Rivière's proof is a consequence of *vectorial* singular integrals theorems, and then it does not answer Peetre's question. On the other hand, it is not clear our results can be derived from those of Rivière's. Thus, Theorem 2.3 is not a (clear, at least) consequence of the results in [5].

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