Polycircular and semi-tubular domains in normed spaces

By IRINEU SALLES BATARCE

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Presentado por el académico numerario D. Darío Maravall Casesnoves

Resumen

Se investigan las propiedades y caracterizaciones equivalentes de los conjuntos abiertos pseudo-convexos en espacios normados complejos de infinitas dimensiones, apoyándose en teoremas previos de Ph. Noverraz.

Se obtiene entre otros resultados el siguiente: si f es una función holomorfa definida en un dominio de Hartog, que corta a su plano de simetría, entonces f puede ser desarrollada uniformemente, sobre subconjuntos compactos, en una serie uniforme de Hartog.

1. INTRODUCTION

In finite dimension one can study certain special domains which have interesting applications in quantum field theory. They also have connections with the theory of partial differential equations with constant coefficients. (see Vladimirov [7, ch. V]0. So it seems natural to study those domains in spaces of infinite dimension. For this purpose we try to characterize those concepts in infinite dimension; then we analyse the pseudoconvexity property of such domains looking for necessary and sufficient conditions. The chosen space could be a seminormed space, for by a theorem of Noverraz [6, th. 2.1.1] we have several equivalents characterization of pseudoconvex open sets, which would be sufficient to prove the results in this paper. But another theorem by Noverraz ([6 th. 2.1.7]) shows that in order to study pseudoconvexity one just have to look at normed spaces. Thence the space E considered in this paper will be a complex normed space. (dim $E \geqslant 2$)

Another result obtained in this paper is the following: if f is a holomorphic function, defined on a Hartog's domain which meets its symmetry plane, then f can be uniformly expanded on compacts subsets in a series of Hartogs.

2. PLURISUBHARMONIC FUNCTIONS AND PSEUDOCONVEX OPEN SETS

2.1 Definitions; Let U be an open set in E. The map

$$u: U \longrightarrow [-\infty \,.\, +\infty\,)$$

$u \not\equiv -\infty$ is said to be p.s.h. (plurisubharmonic) if:

i) u is u.s.c. (upper semi–continuous), that is, the set $\{x \in U; u(x) < \alpha, \alpha \in R\}$ is open.

ii) For every $a \in U$ and $b \in E - \{0\}$ the map $\lambda \longrightarrow u(a+\lambda b)$ is subharmonic or $-\infty$ in $V = \{\lambda \in C; a+\lambda b \in U\}$.

2.2.–Definition: If U is an open in E we put

$$\delta: Ux(E - \{0\}) \longrightarrow (0, +\infty]$$

given by $\delta(x,y) = \sup r$ *if* $\{x + \lambda y; |\lambda| < r\} \in U$. Note that for each fixed y in $E - \{0\}$, $\delta(x,y)$ is the distance of x to the boundary of U in the direction of y.

2.3.–Definition: An open set U in E is said to be pseudoconvex if $-\ln \delta$ is p.s.h. in $U \times (E - \{0\})$.

3. POLYCIRCULAR DOMAINS

The following remark can be easily proved:

3.1.–Remark: Let $H \subset E$ be a hyperplane and $y \in E - H$, then every $z \in E$ can be writen in a unique way as $z = \tilde{z} + z_1 y$ where $\tilde{z} \in H$ and $z_1 \in C$. Furthermore, the function

$$\varphi_H : E \longrightarrow C \times H$$

given by $\varphi_H(z) = (z_1, \tilde{z})$ is an isomorphism.

Using the notation introduced above we now consider $y \in E - H$ fixed such that $\|y\| = 1$. Then we have the following:

3.2.–Definition: A complete polycircular domain G_y with plane of symmetry H is a domain G_y with the property:

$$\text{if } (z_1, \tilde{z}) \in G_y \subset E \Rightarrow (\lambda z_1, \tilde{z}) \in G_y \text{ with } |\lambda| \leqslant 1.$$

3.3.–Lemma: A complete polycircular domain G_y with plane of symmetry H can be characterized as:

$$G_y = \{(z_1 \tilde{z}); \tilde{z} \in B = G_y \cap H \text{ and } |z_1| < \mathfrak{R}(\tilde{z})\}$$

where $\mathfrak{R}: B \longrightarrow (0, +\infty]$ is given by $\mathfrak{R}(\tilde{z}) = \sup r$ if $\{(z_1, \tilde{z}); |z_1| < r\} \subset G_y$ is l.s.c. (lower semi–continuous).

Proof: By the definition of $\mathfrak{R}(\tilde{z})$ and the fact that G_y is polycircular domain we have that $G_y = \{(z_1, \tilde{z})$ such that $\tilde{z} \in B$ *and* $|z_1| < \mathfrak{R}(\tilde{z})\}$.

To show that $\mathfrak{R}$ is a l.s.c. in B, we have to verfy that for all $\beta \in R$ the subset $\{\tilde{z} \in B;\ \mathfrak{R}(\tilde{z}) > \beta\}$ is open in B. Let $\tilde{w} \in B$ be such that $\mathfrak{R}(\tilde{z}) > \beta$ and $\beta < \alpha < \mathfrak{R}(\tilde{w})$. Consider the set $K = \{ w = (z_1, \tilde{w});\ |z_1| \leqslant \alpha \}$ then $K \subset G_y$ and K is compact in G_y, hence there exists a neighborhood of the origen W such that $K + W \subset G_y$. Thus if $\tilde{z} \in \tilde{w} + W$, then $(z_1, \tilde{z}) = (0, \tilde{z} - \tilde{w}) + (z_1, \tilde{w}) \in G_y$ if $|z_1| \leqslant \alpha$, for what we have is the sum of an element of W with an element of K, therefore $\mathfrak{R}(\tilde{z}) \geqslant \alpha > \beta$.

3.4.–Theorem: A complete polycircular domain G_y is pseudoconvex if and only if B is pseudoconvex in H and the function $-\ln \mathfrak{R}(\tilde{z})$ is p.s.h. in B.

Proof: The function $p: E \longrightarrow H$ given by $p(z_1, \tilde{z}) = \tilde{z}$ is open linear continuous and onto. Besides $p(G_y) = B$, then by Noverraz ([7], th. 2.1.7), we have that B is pseudoconvex in H. Since G_y is pseudoconvex $-\ln \delta$ is p.s.h. in $G_y \times (E - \{0\})$. In particular $-\ln (\tilde{z}, y)$ is p.s.h., where $\tilde{z} \in B$. But note that $\delta(\tilde{z}, y) = \mathfrak{R}(\tilde{z})$, then $-\ln \delta(\tilde{z}, y) = -\ln \mathfrak{R}(\tilde{z})$ is p.s.h. in B.

Conversely, if $B = G_y \cap H$ is pseudoconvex and $-\ln \mathfrak{R}(\tilde{z})$ is p.s.h. then $C \times B$ is pseudoconvex and the function $\ln |z_1| - \ln \mathfrak{R}(\tilde{z})$ is p.s.h. in $C \times B$. Recall now that $G_y = \{ z = (z_1, \tilde{z});\ |z_1| < \mathfrak{R}(\tilde{z}) \}$, that is, $G_y = \{ z; \ln |z_1| - -\ln \mathfrak{R}(\tilde{z}) < 0 \}$ and by the result of Noverraz ([6] th. 2.2.1), G_y is pseudoconvex.

3.5.–Definition: A polycircular domain G_y that contains no points of its plane of symmetry H, is a domain of the type:

$$G_y = \left\{ z = (z_1, \tilde{z}); r(\tilde{z}) < |z_1| < \mathfrak{R}(\tilde{z}), \tilde{z} \in B \right\}$$

where B is a domain contained in H, $r(\tilde{z})$ is u.s.c. function and $\mathfrak{R}(\tilde{z})$ is a l.s.c. function.

3.6.–Theorem: If $B \subset H$ is a pseudoconvex domain and the functions $r(\tilde{z})$ and $-\ln \mathfrak{R}(\tilde{z})$ are p.s.h. in B, then the polycircular domain G_y is pseudoconvex.

Proof: If B is pseudoconvex the $(C - \{0\}) \times B$ is pseudoconvex. Let $u(z) = \sup \{\ln |z_1| - \ln \mathfrak{R}(\tilde{z}), \ln 1/z_1 + \ln r(\tilde{z}) \}$, u is p.s.h. since it is the supremo of p.s.h. functions and it is also u.s.c. (see Noverraz [6]). Thus $G_y = \{z; u(z) < 0 \}$ is pseudoconvex.

4. SEMI–TUBULAR DOMAINS

We shall be using the same notation previously introduced, that is, let E be a normed space, $H \subset E$ a hyperplane, $y \in E - H$ and $\|y\| = 1$.

4.1.–Definition: A semi-tubular domain is a domain of the type:

$$G_y = \{ z = (x_1 + iy_1, \tilde{z}); (x_1, \tilde{z}) \in D,\ y_1 \text{ arbitrary}\}$$

where D is a domain contained in $R \times H$.

We now consider semi-tubular domain of the type:

$$G_y = \{z = (x_1 + iy_1 \bar{z});\ \nu(\bar{z}) < x_1 < V(\bar{z}), \bar{z} \in B \text{ and } |y| < \infty\}$$

where $B \subset H$ is a domain and the functions $\nu(\bar{z})$ and $V(\bar{z})$ are upper semi–continuos and lower semicontinuous, respectively.

We the have the following result:

4.2.–Theorem: A semi–tubular domain G_y is pseudoconvex if B is pseudoconvex and $\nu(\bar{z})$ and $-\nu(\bar{z})$ are p.s.h. in B.

Proof: We just consider the p.s.h. function $\nu^*(\bar{z}) = \sup\{x_1 - V(\bar{z}), \nu(\bar{z}) - x_1\}$ and recall that, $G_y = \{z \text{ such that } \nu^*(\bar{z}) < 0, z \in C \times B\}$.

5. HARTOG'S DOMAIN AND SERIES

We will consider non complete polycircular domains given by the definition:

5.1.–Definition: A non complete polycircular domains with plane of symmetry H is a domain G_y satisfyng:

$$\text{if } z(z_1, \bar{z}) \in G_y \subset E \Rightarrow (\lambda z_1, \bar{z}) \in G_y \quad \text{with } |\lambda| = 1$$

where $z = (z_1, \bar{z})$ and such that $z = z_1 y + \bar{z}$, $z_1 \in C$ and $\bar{z} \in H$. Such domains will be called Hartog's domains.

If E and F are normed spaces and $U \subset E$ is a non–empty open subset of E, we have (see [4]) the following results.

5.2.–Proposition: (Cauchy integral). Suppose $f \in \mathcal{H}(U,F)$ (space of all holomorphic functions). Let $\xi \in U$, $x \in U$ and $\rho > 1$ be such that $(1-\lambda)\xi + \lambda x \in U$ for every $\lambda \in C$, $|\lambda| \leqslant \rho$. Then

$$f(x) = \frac{1}{2\pi i} \int_{|\lambda| = \rho} \frac{f[(1-\lambda)\xi + \lambda x]}{\lambda - 1}$$

5.3.–Proposition: Let $f \in \mathcal{H}(U,F)$, $\xi \in U$, $y \in E$ and $\rho > 0$ be such that $\xi + \lambda y \in U$ for every $\lambda \in C$, $|\lambda| \leqslant \rho$ Then

$$\frac{1}{m!} d^m f(\xi)(y) = \frac{1}{2\pi i} \int_{|\lambda| = \rho} \frac{f(\xi + \lambda y)}{\lambda^{m+1}}$$

for $m = 0, 1, \ldots$

Let G_y be a Hartog's domain containing points of its plane of symmetry. Suppose that a function $f(z)$ is holomorphic in G_y. Let $K \subset G_y$ be a compact such that $K \cap H \neq \phi$. Let $z = (z_1, \bar{z})$, $z \in K$ and $\bar{z} \in K \cap H$. We put:

$$R = \sup_{z \in K} |z_1|, \quad d(K, \complement G_y) = \eta, \quad r = 1 + \frac{\eta}{2R}.$$

Let us construct the function

$$\varphi(z) = \frac{1}{2\pi i} \int_{|\lambda_1| = r} \frac{f(\lambda_1 z_1, \tilde{z})}{\lambda_1 - 1} \, d\lambda_1$$

When $z \in K$, the points $\lambda_1 z_1 + \tilde{z} \in G_y$ if $|\lambda_1| = r$. *Indeed*

$$\|(\lambda_1 z_1, \tilde{z}) - (e^{i \arg \lambda_1} z_1, \tilde{z})\| \leqslant |\lambda_1 z_1 - e^{i \arg \lambda_1} z_1| =$$

$$= |z_1| |\lambda_1 - e^{i \arg \lambda_1}| \leqslant R(|\lambda_1| - 1) = \frac{\eta}{2}.$$

Note that

$$(1 - \lambda_1)(0, \tilde{z}) + \lambda_1 (z_1, \tilde{z}) = (\lambda_1 z_1, \tilde{z}) \in G_y \text{ if } |\lambda_1| = r,$$

therefore by proposition 5.2, $\varphi(z) \equiv f(z)$ since $f[(1 - \lambda_1)(0, \tilde{z}) + \lambda_1(z_1, \tilde{z})] = f(\lambda_1 z_1, \tilde{z})$ when $|\lambda_1| = r$, that is, for all $z \in K$. Let us show, now, that

$$f(z) = \sum_{m=0}^{\infty} z_1^m \frac{\hat{d}^m f(\tilde{z})}{m!}$$

uniformly for $z \in K$. We have that

$$\frac{1}{\lambda - 1} = \sum_{m=0}^{\infty} \frac{1}{\lambda^m}$$

uniformly for $|\lambda| = r$, hence

$$\frac{f(\lambda z_1, \tilde{z})}{\lambda - 1} = \sum_{m=0}^{\infty} \frac{f(\lambda z_1, \tilde{z})}{\lambda - 1}$$

uniformly for $z \in K$ and $|\lambda| = r$. Therefore we have

$$f(z) = \frac{1}{2\pi i} \int_{|\lambda| = r} \frac{f(\lambda z_1, \tilde{z})}{\lambda - 1} d\lambda = \sum_{m=0}^{\infty} \frac{1}{2\pi i} \int_{|\lambda| = r} \frac{f(\lambda z_1, \tilde{z})}{\lambda^{m+1}} d\lambda = \sum_{m=0}^{\infty} \frac{z_1^m}{2\pi i} \int \frac{f(z'_1, \tilde{z})}{z_1'^{m+1}} dz'_1$$

Then by 5.3 we have $|z'_1| = r|z_1|$

$$f(z) = \sum_{m=0}^{\infty} z_1^m \frac{\hat{d} f(\tilde{z})}{m!} y$$

uniformly for $z \in K$, where $z = z_1 y + \tilde{z} \in K$.

We have just proved the following.

5.4.–Theorem: If G_y is a Hartog s domain containing points of its plane of symmetry and $K \subset G_y$ is a compact such that $K \cap H \neq \phi$, then

$$f(z) = \sum_{m=0}^{\infty} z_1^m \frac{\hat{d} f(\tilde{z})}{m!} y$$

uniformly for every $z = z_1 y + \tilde{z} \in K$, $\tilde{z} \in K \cap H$.

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Instituto de Matemática
Universidade Estadual de Campinas
Caixa Postâl – 6065.
13081.– CAMPINAS SP
Brasil