WEIGHT FUNCTIONS FOR SOME NEW CLASSES OF ORTHOGONAL POLYNOMIALS

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This paper is dedicated to the founders of *Special* and subsequently *Generalized Functions*, their Theory and Applications.

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This article continue the discussion of finding weight functions for orthogonal polynomials in three situations. The Gegenbauer polynomials are shown to have a distributional weight function. The polynomials of Geronimus [6] which are orthogonal on $[-1, 1]$ are explicitely calculated. An application to Padé approximations is made. Two negative situations are mentioned.

Esta memoria se refiere a la discusión e investigación de funciones de peso de polinomios ortogonales, se aplican las aproximaciones de Padé a los polinomios de Geronimus, que son explícitamente calculados, y se demuestra que los polinomios de Gegenbauer tienen una función de peso distribucional.

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Introduction

In the framework of nowadays researches referred to orthogonal polynomials one finds among others a rather systematic study of an interesting generalization of Gegenbauer, Humbert, and several other polynomial systems, defined by

$$(C - mxt + yt^2)^p = \sum_{n=0}^{\infty} P_n(m, x, y, p, C)\, t^n,$$

where m is a positive integer and other parameters are unrestricted in general [14]; a unified representation of classical polynomials [15], and generation of orthogonal polynomials with respect to some positive measure $d\alpha$ [16], while generalizations to several variables of classical special functions (particularly polynomials) starts as a novelty with nonclassical boundary value problems for polywave or polyvibrating systems of the type

$$M_m^{(1)}[A(x)M_m^{(1)}u(x) + pB(x)u(x)] +$$
$$+ p[B(x)M_m^{(1)}u(x) + C(x)u(x)] = 0,$$

where $x = (x_1, x_2, \ldots, x_m)$ and the operator

$$M_m^{(n)} \equiv \partial^{mn} / \partial^n x_1\, \partial^n x_2 \ldots \partial^n x_m$$

is the n-th order total derivative, $u(x) = 0$ on the boundary or $R\{a_i \leqslant x_i \leqslant b_i, i = 1, 2, \ldots, m\}$ [17], [18], subsequently extended to the non elliptical problems into adequate functional spaces [19], [20], with an extension to polyvibrating special functions of C. Truesdell's «F-Functions» method [21], or with a method of construction of orthonormal system of polynomials in a domain $\Omega \subset R^n$ with a positive weight function [22].

In the authors' previous article [10] a new look at some classic problems in orthogonal polynomials was reached thanks to the introduction of the concept of a distributional weight function for orthogonal polynomials, with the classical orthogonal polynomials used as examples. The Jacobi polynomials were shown to be distributionally orthogonal even when the classical parameters A and B (associated with the weight $(1-x)^A(1+x)^B$) were less than -1. When A, B > -1, the connection to $(1-x)^A (1+x)^B$ was established. Likewise for the Laguerre polynomials, classically orthogonal with respect to $x^\alpha e^{-x}$, a distributional weight function was exhibited even when $\alpha < -1$. Similar results were found concerning Hermite polynomials ([9] also exhibits extensions to the generalized Hermite polynomials), and, finally, a weight function for the Bessel polynomials was exhibited for the first time.

This paper continues the discussion of the use of distributions to three interesting new cases. First the Gegenbauer polynomials [12] are shown to be distributionally orthogonal. Second a connection is made between some polynomials of Geronimus [6] and distributional weight functions.

Finally the application of distributional orthogonality to Padé approximations [3] is examined.

Two negative results are also mentioned. The Brafman polynomials are shown not to be orthogonal. Engels' implicitly defined polynomials do not seem to be appropriate for distributional orthogonality.

The Gegenbauer polynomials

The Gegenbauer polynomials C_n satisfy the ordinary differential equation

$$(1-x^2)\,y'' - (2\lambda+2)\,x\,y' + n\,(2\lambda+1+n)\,y = 0$$

where, here, $\lambda = \nu - 1/2$, ν being the usual parameter [12]. As a result [11], the moments used to define C_n satisfy

$$\mu_n\,(2\lambda+1+n) = (n-1)\,\mu_{n-2},$$

with $\mu_{-1} = 0$, $\mu_0 = 1$. Thus

$$\mu_{2n+1} = 0,$$
$$\mu_{2n} = 1\cdot 3\,\ldots\,(2n-1)/(2\lambda+3)\ldots(2\lambda+2n+1),$$

$n = 0, 1, \ldots$

When $\nu > -1/2$ or $\lambda > -1$, these can be generated by using the standard weight function $(1-x^2)^\lambda$ through the formula

$$\mu_n = \int_{-1}^{1} x^n (1-x^2)^\lambda \, dx$$

as we:. When $\lambda < -1$, however, the integral makes no sense. It is this situation we wish to explore further, since the distribution we shall generate not only generates the moments $\{\mu_n\}_{n=0}^{\infty}$, but also makes the Gegenbauer polynomials orthogonal.

With $\lambda < -1$, the weight function $w = (1-x^2)^\lambda$ is non-integrably singular at both ± 1. This means [11] that a Cauchy regularization is required at both ± 1. We assume $-\alpha - 1 < < \lambda < -\alpha$.

If t^λ is thought of as a distribution acting on a test function f, defined on $[0, 1]$, then when $\lambda < -1$, its Cauchy regularization [5] is given by

$$< f, t^\lambda > = \int_0^1 t^\lambda \left[f(t) - \sum_{j=0}^{\alpha-1} \frac{f^{(j)}(0)}{j!} t^j \right] dt +$$

$$+ \sum_{j=0}^{\alpha-1} \frac{f^{(j)}(0)}{j!(\lambda+j+1} .$$

By splitting up the interval $[-1, 1]$ into two halves and suitably changing variables, we find that

$$< f, w > = \int_0^1 (1-x)^\lambda \Big\{ f(x)(1+x)^\lambda -$$

$$- \sum_{j=0}^{\alpha-1} (-1)^j \frac{[f(x)(1+x)^\lambda]^{(j)}(1)(1-x)^j}{j!} \Big\} dx +$$

$$+ \sum_{j=0}^{\alpha-1} \frac{(-1)^j [f(x)(1+x)^\lambda]^{(j)}(1)}{j!(\lambda+j+1)} +$$

$$+\int_{-1} (1+x)^{\lambda} \left\{ f(x)(1-x)^{\lambda} - \sum_{j=0}^{\alpha-1} \frac{[f(x)(1-x)^{\lambda}]^{(j)}}{j!} \right\} dx +$$

$$+ \sum_{j=0}^{\alpha-1} \frac{[f(x)(1-x)^{\lambda}]^{(j)}(-1)}{j!(\lambda+j+1)} .$$

This can be considerably simplified if we let $\phi(t) = [f(1-t) + f(t-1)](2-t)^{\lambda}$. Then

$$<f, w> = \Gamma(\lambda+1) \left[(-1)^{\alpha} \int_0^1 \frac{t^{\lambda+\alpha}}{\Gamma(\lambda+\alpha+1)} \phi^{(n)}(t)\, dt + \right.$$

$$\left. + \sum_{j=1}^{\alpha} \frac{(-1)^{j+1} \phi^{(j-1)}(1)}{\Gamma(\lambda+j+1)} \right]$$

after integration by parts has been performed α times.

It is tedious to show directly that

THEOREM.—*When* $-\alpha-1 < \lambda < -\alpha$,

$$< x^{2n+1}, = > = 0$$

$$< x^{2n}, w > = 1 \cdot 3 \ldots (2n-1) / (2\lambda+3) \ldots (2\lambda+2n+1) \quad n = 0, 1, \ldots .$$

Consequently it follows [11].

THEOREM.—*When* $-\alpha-1 < \lambda < -\alpha$,

$$< C_n\, C_m, w > = 0 \qquad m \neq n$$

$$= \frac{(2\lambda+1)_n\, \Gamma\left(\frac{1}{2}\right) \Gamma(\lambda+1)}{(\lambda+1/2+n)\, \Gamma(\lambda+1/2)} \qquad m = n,$$

where the symbol $(a)_n = a(a+1) \ldots (a+n-1)$.

In a manner similar to the Laguerre polynomials [8], when $\lambda < -1$ the functional $\langle \cdot, w \rangle$ generates an indefinite inner product and ultimately a Krein or Pontrjagin space [1].

The Geronimus polynomials

In [6] Geronimus defines a set of polynomial $\{\phi_n\}_{n=0}^{\infty}$ through a rather elaborate procedure. By using distributions, when the independent variable is real, the calculation of these polynomials can be considerably simplified.

Let $\psi(x)$ be a distributional weight function on $[-1, 1]$ and let $\{p_n(x)\}_{n=0}^{\infty}$ be the monic polynomials which are orthogonal with respect to ψ. Then the collection $\{p_n\}_{n=0}^{\infty}$ satisfies a three-term recurrence relation [7]

$$p_n(x) = (x - \alpha_n) p_{n-1}(x) - \lambda_n p_{n-2}(x) .$$

If $\mu_n = \langle x^n, \psi \rangle$, then

$$\Delta_n = \begin{vmatrix} \mu_0 & \cdots & \mu_n \\ \vdots & & \vdots \\ \mu_n & \cdots & \mu_{2n} \end{vmatrix} \neq 0,$$

$$p_n(x) = \frac{1}{\Delta_{n-1}} \begin{vmatrix} \mu_0 & \cdots & \mu_n \\ \vdots & & \vdots \\ \mu_{n-1} & \cdots & \mu_{2n-1} \\ 1 & \cdots & x^n \end{vmatrix},$$

$$= x^n - S_n x^{n-1} + \ldots ,$$

and $\alpha_n = S_n - S_{n-3}$, $\lambda_n = \Delta_{n-1} \Delta_{n-3} / \Delta^2_{n-2}$.

Via continued fractions define

$$U_n = 1 - \alpha_{n+1} - \frac{\lambda_{n+1}}{|1 + \alpha_n} - \frac{\lambda_n}{|1 - \alpha_{n-1}} \cdots - \frac{\lambda_2}{|1 - \alpha_1},$$

$$V_n = 1 + \alpha_{n+1} - \frac{\lambda_{n+1}}{|1 + \alpha_n} - \frac{\lambda_n}{|1 + \alpha_{n+1}} \cdots - \frac{\lambda_2}{|1 + \alpha_1},$$

and

$$a_{2n+1} = 1 - U_{n+1} - V_{n+1}$$
$$a_{2n} = (V_n - U_n) / (V_n + U_n).$$

Geronimus' polynomials $\{\phi_n\}_{n=0}^{\infty}$ are then given by the formulas

$$\phi_{n+1}^{*}(x) = \phi_n^{*}(x) - a_n\, x\, \phi_n(x),$$

$$\phi_n^{*}(x) = x^n\, \phi_n\left(\frac{1}{x}\right) \cdot \qquad\qquad n\ 0, 1, \dots .$$

Various growth estimates then follow [6] if ψ is a nondecreasing function of bounded variation with infinitely many points of increase.

Padé approximations

The problem of finding Padé approximations to an analytic function

$$f(t) = \sum_{i=0}^{\infty} \mu_i\, t^i$$

is to find a rational function $p(t)/q(t)$, where the degree of p is $k - 1$, the degree of q is k, for which the error involved is $0(t^k)$ as $t \longrightarrow 0$. Such a rational function is denoted by $\left(\frac{k-1}{k}\right)_f$ (See [3].)

If we denote by w the weight function generated by the moments $\{\mu_i\}_{i=0}^{\infty}$, so

$$w = \sum_{j=0}^{\infty} (-1)^j\, \mu_j\, \delta^{(j)}(x)/j!\,,$$

where $\delta^{(j)}$ is the j-th distributional derivative of the Dirac delta function, then

$$< x^n, w > = \mu_n, \quad n = 0, \dots,$$

and

$$f(t) = < \frac{1}{1 - x t}, w >$$

when expanded.

We assume that Δ_n is never zero (see the previous section), and let $\{P_n(x)\}_{n=0}^{\infty}$ be the normalized polynomials, orthogonal with respect to w:

$$< P_n P_m, w > = 0 \qquad n \neq m$$
$$= 1 \qquad n = m.$$

Then $\{P_n(x)\}_{n=0}^{\infty}$ satisfies a three term recurrence relation [7]

$$P_{k+1} = (A_{k+1} x + B_{k+1}) P_k - C_k P_{k-1}, \; P_{-1} = 0, \; P_0 = P_0.$$

Associated with $\{P_k(x)\}_{n=0}^{\infty}$ is an associated set of polynomials $\{Q_k(x)\}_{k=0}^{\infty}$, defined by

$$Q_k(x) = < \frac{P_k(x) - P_k(t)}{x - t}, w >,$$

which satisfy the same 3 term recurrence relation

$$Q_{k+1} = (A_{k+1} x + B_{k+1}) Q_k - C_k Q_{k-1},$$

but

$$Q_{-1} = -1, \; Q_0 = 0.$$

We then let

$$\tilde{P}_k(x) = x^k P_k(x^{-1})$$

$$\tilde{Q}_k(x) = x^{k-1} Q_k(x^{-1}).$$

Then

$$\left(\frac{k-1}{k} \right)_f = \tilde{Q}_k(t) / \tilde{P}_k(t).$$

That is

$$[\tilde{Q}_k(t) / \tilde{P}_k(t)] - f(t) = 0 \, (t^k)$$

as $t \longrightarrow 0$. We refer to [3] for details.

Some negative results

The question of whether the Brafman polynomials [2], [13] are orthogonal has been asked. Denoted by $\{g_n\}_{n=0}^{\infty}$, they are given by

$$g_n(x) = \sum_{k=0}^{\left[\frac{n}{p}\right]} \frac{(-1)^k (n!)^q \prod_{i=1}^{r} (a_i)_k \, x^{n-pk}}{((n-pk)!)^q \prod_{i=1}^{s} (b_i)_k \, k!}$$

The answer is no. We show this only in the simpler case $p = q = r = s = 1$. Then

$$g_0 = 1,$$

$$g_1 = x - a/b,$$

$$g_2 = x^2 - 2(a/b)x + (a(a+1)/(b(b+1))),$$

$$g_3 = x^3 - 3(a/b)x^2 + 3(a(a+1))/(b(b+1))x - $$
$$- (a(a+1)(a+2))/(b(b+1)(b+2)),$$

etc.

Assuming there is a weight function w which generates moments $\{\mu_n\}_{n=0}^{\infty}$, then since g_1, g_2, g_3 are orthogonal to g_0, we find

$$\mu_0 = 1$$

$$\mu_1 = a/b$$

$$\mu_2 = a(ab + 2a - b)/b^2(b+1)$$

$$\mu_3 = a^2(4a + ab - 3b)/b^3(b+1).$$

Since g_2 is likewise orthogonal to x, we find

$$\mu_3 = \frac{(a^3 b^3 - 3a^2 b^2 + 6a^3 b + 12a^3 - 12a^2 b + 2ab^2}{b^3(b+1)(b+2)}.$$

The expressions for μ_3 are clearly not the same, and so no (distributional) weight function exists.

Finally we comment that H. Engels' implicitly defined orthogonal polynomials, and so our results do not seem to be applicable.

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