INVERSION THEOREM FOR THE WHITTAKER-TRANSFORM OF CERTAIN DISTRIBUTIONS

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1. Introduction

In this paper we propose to extend a Complex-inversion formula given by R. S. Varma [3] to a certain class of distributions.

R. S. Varma [3] proved the following Complex-inversion formula for the Whittaker-transform:

$$\frac{1}{2}\{f(t+0)+f(t-0)\} =$$

$$\frac{1}{\pi i} \lim_{\tau\to\infty} \int_{c-i\tau}^{c+i\tau} \frac{(2t)^{-l}\,\Gamma\left(-l-k+\frac{7}{4}\right)}{\Gamma\left(-l+m+\frac{5}{4}\right)\Gamma\left(-l-m+\frac{5}{4}\right)} \psi \quad) \, d\, l$$

where

$${}_2F_1\left(-l+m+\frac{5}{4},\ -l-m+\frac{5}{4},\ -l-k+\frac{7}{4},\frac{1}{2}\right)\psi(l) =$$

$$= \int_0^\infty s^{-l}\,\varphi^k{}_m(s)\,d\,s,$$

where

$$\varphi^k{}_m(s) = \int_0^\infty (2\,s\,t)^{-\frac{1}{4}}\, W_{k,\,m}(2\,s\,t) f(t)\, d\,t$$

provided that

(i) $x^{c-1} f(x)$ belongs to L $(0, \infty)$

ii) $x^{-l} \varphi^{k}_{m}(x)$ also belongs to L $(0, \infty)$ for

$$l = c \pm i\tau, \; -\infty < \tau < \infty$$

(iii) $f(x)$ is of bounded variation in the neighbourbood of the point $x = t$

A classical complex-inversión theorem for Erdélyi's generalization of Laplace transform whose kernal contains a confluent hypergeometric function, was recently extended to distributions by Rao [1]. The problem of Complex inversion for certain distributional generalized Laplace transforms was solved by a different method by Rao [2] recently.

In this paper we will extend the above classical result to distributions by transferring the classical Complex inversion formula into the testing function space for the distribution under consideration and then proving that the limiting process in the resulting formula converges with respect to the topology of the testing function space. This was thé method used by Zemanian [[5], section 3.5] and also Rao [1].

2. Some preliminary notation and definitions

Testing function space L_a and its dual Zem. [[5], section 3.10] and Rao [2].

Let a be a real number. A function $\varphi(t)$ is said to belong to L_a if and only if $\varphi(t)$ is smooth on $-\infty < t < \infty$ and for each non-negative integer k it satisfies

$$\nu_{a, T, k}(\varphi) = \sup_{T < t < \infty} | e^{at} D_t^k \varphi(t) | < \infty .$$

The following facts are straight forward to prove:

(i) $L_{a, b}$ is a subspace of L_a for every b. Since $\mathcal{D} \subset L_{a, b}$, $\mathcal{D} \subset L_a$. also Zemanian [[5], p.p. 85]

(ii) $(2sy)^{-\frac{1}{4}} W_{k,m}(2sy) \in L_{a,b} \Rightarrow (2sy)^{-\frac{1}{4}} W_{k,m}(2sy) \in L_a$. provided $a <$ Re s and y > 0 and $k < \frac{1}{4}$ with Re $s = a$.

For the spaces L'_a, $L(w)$, $L'(w)$, see Zem [[5] p. 85 Sec. 3.10]. There exists a real number c_1 such that $f \in L'(c_1)$ and $f \notin L'(w)$ if $w < c_1$.

We can now define the right-sided Whittaker-transform of a distribution f by the equation

$$F(s) = \langle f(y), \quad (2sy)^{-\frac{1}{4}} W_{k,m}(2sy) \rangle \tag{1}$$

where $s \in \Gamma f$ which is defined by

$$\Gamma f = \{ \mathrm{Re}\, s, c_1 < \mathrm{Re}\, s < \infty \}.$$

The right-hand side of (1) has a sense as the application of $f \in L'(c_1)$ to $(2sy)^{-\frac{1}{4}} W_{k,m}(2sy) \in L(c_1)$ provided $c_1 < s$ and $y > 0$. It is clear from Zem. [[5], 51 p] that $L(a) \subset L_a$ and $L'_a \subset L'(a)$.

3. Complex inversion theorem for a distributional Whittaker-transform

We shall prove the Complex inversion theorem by the help of the following three Lemmas;

LEMMA 3.1.—Let

$$\varphi = \varphi(st) = (2st)^{-\frac{1}{4}} W_{k,m}(2st) \tag{2}$$

be a testing function in $\mathcal{D}$ and let $f(t)$ be a distribution in L'_a. Then

$$\int_0^\infty s^{-l} K \langle f(t), \varphi(st) \rangle \, ds = \left\langle f(t), \int_0^\infty K \varphi s^{-l} \, ds \right\rangle \tag{3}$$

where

$$K = \frac{\Gamma\left(-l - k + \frac{7}{4}\right)}{\Gamma\left(-l + m + \frac{5}{4}\right) \Gamma\left(-l - m + \frac{5}{4}\right)}$$

provided $a < \mathrm{Re}\, s$, $t > 0$, $k < \frac{1}{4}$ with $\mathrm{Re}\, s = a$.

PROOF.—We shall first prove that

$$s^{-l}\,K\,\langle f(t), \varphi\rangle = \langle f(t), K\,\varphi\, s^{-l}\rangle. \tag{4}$$

Since $\varphi \in \mathcal{D}$ and $\mathcal{D} \subset L_a$, it follows that $\varphi \in L_a$.
We shall prove that $\varphi\, K\, s^{-l}$ also $\in L_a$.
It is sufficient if we show that

$$|\, e^{at}\, D_t^{\nu} \{(2\,s\,t)^{-\frac{1}{4}}\, s^{-l}\, K\, W_{k,m}(2\,s\,t)\}\, | < \varepsilon, (\varepsilon > 0) \quad \text{as} \quad t \to \infty\,.$$

Left-hand side of this inequality which is equal to

$$|\, K\, s^{-\frac{1}{4}-l}\, e^{at}\, D_t^{\nu} \{(2\,t)^{-\frac{1}{4}}\, W_{k,m}(2\,s\,t)\}\, |\,,$$

Obviously converges to zero as $t \to \infty$ when $a < \operatorname{Re} s$, $t > 0$ and $k < \frac{1}{4}$ with $\operatorname{Re} s = a$ as it was shown in section 2. Hence (4) is proved. So we have

$$\int_0^{\infty} s^{-l}\, K\, \langle f(t), \varphi\rangle\, d\,s = \int_0^{\infty} \langle f(t), K\,\varphi\, s^{-l}\, d\,s\rangle. \tag{5}$$

Since $\varphi \in \mathcal{D}$ and $f \in \mathcal{D}_t'$ it follows from Zem. [[4]; Corolary 5.3.2 b]] that

$$\int_0^{\infty} \langle f(t), \varphi\, K\, s^{-l}\rangle\, d\,s = \left\langle f(t), \int_0^{\infty} K\,\varphi\, s^{-l}\, d\,s \right\rangle.$$

This, with (5), proves (3).

LEMMA 3.2.—Let $\varphi \in \mathcal{D}$ and r be a fixed positive real number $(0 < r < \infty)$.
Let

$$\psi(l) = \int_0^{\infty} \varphi(y)\, y^{-l}\, d\,y$$

where $l = c + i\,\tau$ $(-\infty < \tau < \infty)$ and c is fixed such that $c_1 < c < \infty$ then

$$\frac{1}{2\pi} \int_{-r}^{r} \langle f(t), t^{l-1}\rangle\, \psi(l)\, d\,\tau = \left\langle f(t), \frac{1}{2\pi} \int_{-r}^{r} t^{l-1}\, \psi(l)\, d\,\tau \right\rangle \tag{6}$$

PROOF.—The proof of this theorem can be constructed as in Rao [1] and Zem. [5].

LEMMA 3.3.—Let $\varphi \in \mathfrak{D}$. Let a, c and r be real numbers such that $c > 1$ and $0 < r < \infty$. Then

$$\frac{1}{\pi} \int_0^\infty \frac{\varphi(y)}{t \log \frac{t}{y}} \left(\frac{t}{y}\right)^c \operatorname{Sin}\left(r \log \frac{t}{y}\right) dy$$

converges in L_a to $\varphi(t)$ as $r \to \infty$.

PROOF.—The proof is the same as in Rao [1].

THEOREM 3.1.—*The complex inversion theorem*: If

(i) $F(s) = \langle f(y), (2sy)^{-\frac{1}{4}} W_{k,m}(2sy) \rangle$ and c_1 is a real number such that $c_1 < s < \infty$,

(ii) $\varphi(l) = \int_0^\infty s^{-l} F(s)\, ds$,

(iii) r be a real number,

(iv) c (real part of l) be any fixed real number such that $c_1 < c < \infty$,

(v) $0 < \operatorname{Re}\left(-l - m + \frac{5}{4}\right) < \operatorname{Re}\left(-l + m + \frac{5}{4}\right)$, $m > 0$, $l - k + \frac{7}{4} < 0$.

Then, in the sense of convergence in $\mathfrak{D}'$

$$f(y) = \frac{1}{2\pi i} \lim_{r \to \infty} \int_{c-ir}^{c+ir} \frac{\Gamma\left(-l - k + \frac{7}{4}\right) y^{-l}}{\Gamma\left(-l + m + \frac{5}{4}\right) \Gamma\left(-l - m + \frac{5}{4}\right)} \psi(l)\, dl \tag{7}$$

PROOF.—Here we transfer the inversion formula into a transform of $\varphi \in \mathfrak{D}$ and show that the resulting expression converges to φ in L_a.

Let $\varphi \in \mathfrak{D}$ and let $c_1 < a < c < \infty$.

We have to prove that

$$\lim_{r \to \infty} \left\langle \frac{1}{2\pi i} \int_{c-ir}^{c+ir} K y^{-l} \psi(l)\, dl, \varphi(y) \right\rangle = \langle f, \varphi \rangle \tag{8}$$

where

$$K = \frac{\Gamma\left(-l-k+\frac{7}{4}\right)}{\Gamma\left(-l+m+\frac{5}{4}\right)\Gamma\left(-l-m+\frac{5}{4}\right)}.$$

We notice first that F (s) is continuous in s in $c_1 < s < \infty$ and hence bounded and integrable in that interval.

Therefone $\varphi\,(l)$ is a continuous function of l which proves that the integral on l on the left-hand side of (8) is a continuous function of y.

Thus the left-hand side of (8) without the limit notation hecomes

$$\frac{1}{2\pi}\int_0^\infty \varphi(y) \int_{-r}^{r} K y^{-l} \psi(l)\, d\tau\, dy$$

where $l = c + i\tau$ and $r > 0$. Since the integrand in (9) is a continuous function of y and τ and the support of $\varphi\,(y)$ is bounded, we can change the order of integration in (9).

Then we have

$$\frac{1}{2\pi}\int_{-r}^{r} \psi(l) K \int_0^\infty \varphi(y) y^{-l}\, dy\, d\tau =$$

$$= \frac{1}{2\pi}\int_{-r}^{r} \left\{ \int^{\infty} K s^{-l} \left\langle f(t), (2st)^{-\frac{1}{4}} W_{k,m}(2st) \right\rangle ds \right\} \times$$

$$\times \int_0^\infty \varphi(y) y^{-l}\, dy\, d\tau =$$

$$= \frac{1}{2\pi}\int_{-r}^{r} \left\langle f(t), \int_0^\infty K (2st)^{-\frac{1}{4}} s^{-l} W_{k,m}(2st)\, ds \right\rangle \times$$

$$\times \int_0^\infty \varphi(y) y^{-l}\, dy\, d\tau \quad \text{by Lemma 3.1} =$$

$$= \frac{1}{2\pi}\int_{-r}^{r} \left\langle f(t), t^{l-1} \right\rangle \int_0^\infty \varphi(y) y^{-l}\, dy\, d\tau$$

provided that

$$0 < \operatorname{Re}\left(-l-m+\frac{5}{4}\right) < \operatorname{Re}\left(-l+m+\frac{5}{4}\right).$$

By using Lemmas 3.2, 3.3, we can prove the theorem as in Rao [1].

Remark: The method applied in Rao [2] may also be used to deduce the theorem proved in this paper.

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