QUASI-DISTINGUISHED AND BOUNDEDLY COMPLETED LOCALLY CONVEX SPACES

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Let G be a subspace of a locally convex space E. We say that G is *quasi-distinguished (boundedly completed) in* E, if every bounded set (element) of E is contained in the closure of a bounded set of G. We call E *quasi-distinguished (boundedly completed)*, if E is quasi-distinguished (boundedly completed) in its completion. In this article we give examples of quasi-distinguished and boundedly completed spaces and obtain criteria for locally convex spaces to be quasi-distinguished or boundedly completed. We start from a completeness result, (Theorem 5), and as corollaries obtain:

1. Given an infra-Schwartz space E, its bidual equipped with the natural topology is complete if and only if E is boundedly completed, (Proposition 8).
2. Given a quasi-normable space E, its bidual equipped with the natural topology is complete if and only if E endowed with the associated Schwartz topology is boundedly completed, (Proposition 8).
3. A Mackey space is quasi-distinguished, if its strong dual is ultrabornological, (follows from Proposition 10 and Corollary 2).
4. An α-u-quasi-barrelled space is quasi-distinguished, if its strong dual is bornological, (Corollary 11).
5. A quasi-barrelled space is quasi-distinguished, if its strong dual is quasi-bornological, (Proposition 15).
6. The space C (T) of all real-valued continuous on a completely regular Hausdorff space T functions, equipped with the compact-open topology, is quasi-distinguished, if its strong dual is barrelled, (Proposition 13).

Sea G un subespacio de un espacio localmente convexo E. Decimos que G es *casi-distinguido* (resp. *acotadamente completo*), *en* E, si para cada conjunto acotado A (resp. elemento x), de E existe un conjunto acotado B de G tal que la clausura de B en E contiene A (resp. x). En este artículo obtenemos las condiciones para que E sea casi-distinguido o acotadamente completo en la complección de E.

Introduction

In this article we investigate a class of locally convex spaces, satisfying one of the following properties:

A) every bounded subset of the completion of a locally convex space E is contained in the completion of a bounded set of E;

B) every element of the completion of E belongs to the completion of a bounded set of E.

Spaces, satisfying A, are called quasi-distinguished (they were introduced in [14]). Spaces, satisfying B, are called boundedly completed. The first concept is suggested by the well-know property of a (DF)-space, ([5], p. 77). The second comes from the question, raised by A. Grothendieck, ([5], p. 119), and answered in the negative by I. Amemiya, [1], whether every locally convex metrizable space is quasi-distinguished. In Section I we define and give examples of boundedly completed and quasi-distinguished spaces and subspaces. In Section II we obtain criteria for locally convex spaces to be boundedly completed or quasi-distinguished.

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Notations

Futher the words «locally convex space» will mean «infinite-dimensional locally convex Hausdorff topological vector space over the field K of the real or complex numbers». We denote by (E, t) a vector space E endowed with the topology t. All topologies on E will be locally convex and separated. We denote by E^*, E' and E'' the algebraic dual, the topological dual and the bidual of (E, t), respectively. Given a dual pair $\langle E, F \rangle$, we denote by $\sigma(E, F)$, $\mu(E, F)$ and $\beta(E, F)$ the weak, the Mackey and the strong topology on E, respectively. An absolutely convex set is called a disk. Let A be a bounded disk in E. We denote by E_A the linear hull of A, equipped with the norm, associated with A. We say that A is a Banach disk if E_A is Banach. We say that E is locally complete if every closed bounded disk in E is a Banach disk.

I. Quasi-distinguished and boundedly completed spaces. Definition and examples

Let G be a subspace of a locally convex space E.

Definition 1.—We say that G is *quasi-distinguished in* E, if every bounded subset of E is contained in the closure of a bounded set of G. A locally convex space is *quasi-distinguished*, if it is quasi-distinguished in its completion.

Examples:

1) Normed spaces are quasi-distinguished.

2) Dualmetric spaces of A. Pietsch, [12], are quasi-distinguished, ([2], Theorem 2 and Corollary 1).

3) A metrizable separable locally convex space is quasi-distinguished, ([5], p. 62).

4) Let E be a strict inductive limit of an increasing sequence E_n ($n \in \mathbb{N}$) of infinite-dimensional separable non-normable Frechet spaces, satisfying: $E_n \neq E_{n+1}$, $E = \cup \{E_n : n \in \mathbb{N}\}$. Let X_n be a dense countable set in E_n and L the linear hull of $\cup \{X_n : n \in \mathbb{N}\}$. The space L equipped with the induced by E topology is quasi-distinguished. Clearly L is neither metrizable, nor dualmetric. According to ([18], Theorem 1), L is bornological.

5) Let T be the topological product of one-dimensional locally convex spaces $\{T_i : i \in I\}$. Suppose Card $I > \chi_0$. A subspace E_0, consisting of all elements with at most finitely many nonzero components, is quasi-distinguished, when furnished with the induced by T topology. Another subspace, E, consisting of all elements with at most countably many nonzero components, is locally complete and quasi-distinguished. According to ([11], Proposition 4), E_0 is bornological and E is ultrabornological.

6) Let E be a locally convex space and F a subspace of E'' such that $E \subset F$ and codim E in F is finite. By Lemma 2 of [16], E is quasi-distinguished in $(F, \sigma(F, E'))$.

7) A locally convex space is distinguished if and only if it is quasi-distinguished in $(E'', \sigma(E'', E'))$, [22].

8) Let E be a normed barrelled space. A dense subspace F of $(E', \mu(E', E))$ is norming if and only if it is quasi-distinguished in $(E', \mu(E', E))$, [22].

NOTE.—Subspaces, defined above, were investigated by M. Valdivia in [16] and [19]. J. H. Webb called such subspaces ultradense, [21], and A. Wilansky-large, ([22], p. 260).

DEFINITION 2.—We say that G is *boundedly completed in* E if every element of E belongs to the closure of a bounded set of G. A locally convex space is *boundedly completed* if it is boundedly completed in its completion.

EXAMPLES:

9) Metrizable spaces are boundedly completed.

10) Quasi-distinguished spaces are boundedly completed.

11) I. Amemiya had given an example of a nonquasi-distinguished metrizable space, [1], providing the existence of a boundedly completed nonquasi-distinguished space.

12) The topological product of two locally convex spaces, one quasi-distinguished nonmetrizable, another -- nonquasi-distinguished metrizable, gives an example of a boundedly completed space, which is neither metrizable, nor quasi-distinguished.

13) A subspace G is *locally dense* in E, if for every $x \in$ E there exists a bounded disk B in E and a sequence x_n $(n \in \mathbb{N})$ in $E_B \cap G$ such that $x_n \longrightarrow x$ in E_B, [17]. A locally dense subspace is boundedly completed in E.

14) Let C (T) be the space of all real-valued continuous on a completely regular Hausdorff space T functions, equipped with the compact-open topology. Denote by E the completion of C (T), by X the set of all noncontinuous bounded on T elements of E and by E the linear hull of C (T) $\cup$ X. As it follows from the proof of Warner's completeness theorem, ([20], p. 266), C (T) is boundedly completed in E and E is boundedly completed.

II. SOME CRITERIA FOR LOCALLY CONVEX SPACES TO BE QUASI-DISTINGUISHED OR BOUNDEDLY COMPLETED

The next proposition follows immediately from Definition 2.

PROPOSITION 1.—*A subspace G is boundedly completed in E if and only if* $\beta(E', G)$ *is finer than* $\sigma(E', E)$. *Hence E is boundedly completed if and only if its completion lies in its bidual.*

From the second part of Proposition 1 and from Definition 1 follows:

COROLLARY 2.—*Let* (E, t) *be a boundedly completed [resp. quasi-distinguished] locally convex space. Then E is boundedly completed [resp. quasi-distinguished] for every topology* η, *satisfying*: $t \leqslant \eta \leqslant \mu(E, E')$.

Given a locally convex space E, the *natural topology* on E″ is the topology of uniform convergence on all equicontinuous subsets of E′, ([9], p. 300).

COROLLARY 3.—*Let E be a locally convex space. If E″ endowed with the natural topology is complete, then E is boundedly completed.*

The space $(E'', \beta(E'', E'))$ is called the *strong bidual* of E.

COROLLARY 4.—*A quasi-barrelled space with a complete strong bidual is boundedly completed.*

PROOF.—Since E is quasi-barrelled, the natural topology of E″ coincides with $\beta(E'', E')$, hence the corollary, q. e. d.

A locally convex space E is *quasi-normable*, if for every closed equicontinuous disk A in E′ there exists another one, B, such that the topology $\beta(E', E)$ and the normed topology of E'_B coincide on A, ([5], p. 106). A (DF)-space is quasi-normable, [10]. The space C (T) of all real-valued continuous on a completely regular Hausdorff space T functions, equipped with the compact-open topology, is quasi-normable, ([20], p. 278). Locally convex spaces with any of the strict topologies, considered by W. Ruess in [13], are quasi-normable.

A locally convex space E is *infra-Schwartz*, if for every closed equicontinuous disk A in E′ there exists another closed equicontinuous disk B such that A is weakly compact in the Banach space E'_B, ([3], p. 43). Many important spaces of functions are Schwartz, hence infra-Schwartz, [4].

Given a locally convex space (E, t), we say that (E, τ) is the *Schwartz space associated with* (E, t), if τ is the finest locally convex topology on E, weaker than t, such that (E, τ) is a Schwartz space, ([6], p. 62). By Proposition III.1.7 of [3], τ is the topology of uniform convergence on all sequences, convergent to zero in E'_A for some t-equicontinuous disk A of E′.

Futher we shall denote by (E'', t) [resp. (E'', τ)] the bidual of (E, t) equipped with the topology of uniform convergence on all

t-equicontinuous [resp. τ-equicontinuous] sets of E', in order to avoid the possible ambiguity caused by the words «natural topology».

THEOREM 5.—*Let* (E, t) *be a locally convex space,* $\mathcal{H}$ *the family of all* $\sigma(E', E)$*-closed* t*-equicontinuous disks,* (E', t') *the inductive limit of* $\{E'_A : A \in \mathcal{H}\}$ *and* (E, τ) *the Schwartz space associated with* (E, t). *Then:*

1) E *is dense in* (E'', τ). *The dual of* (E', t') *equals to the completion of* (E'', τ).

2) *If* (E, t) *is infra-Schwartz, then* E *is dense in* (E'', t).

3) *If* (E, t) *is infra-Schwartz or quasi-normable, the dual of* (E', t') *equals to the completion of* (E'', t).

PROOF. 1.—Since every τ-equicontinuous set is $\sigma(E', E'')$-relatively compact, τ is compatible with the duality $\langle E'', E' \rangle$, hence E is dense in (E'', τ).

Denote by F the dual of (E', t'). Take x in the completion of (E'', τ). By ([9], p. 270), the restriction of x on each $A \in \mathcal{H}$ is $\sigma(E', E'')$-continuous, hence bounded, therefore $x \in F$.

Conversely, let $x \in F$. Take a closed τ-equicontinuous disk M and another one, N, such that M is compact in E'_N. Since x is bounded on N, it is continuous on E'_N, hence its restriction on M is $\sigma(E', E'')$-continuous, therefore by ([9], p. 270) x belongs to the completion of (E'', τ).

2) If (E, t) is infra-Schwartz, every t-equicontinuous set is $\sigma(E', E'')$-relatively compact, hence E is dense in (E'', t).

3) For an infra-Schwartz space we take a t-equicontinuous closed disk M, and another one, N, such that M is weakly compact in E'_N. Then we apply the arguments of part 1 to (E'', t).

For a quasi-normable space the topology t on E'' is not necessarily compatible with the duality $\langle E'', E' \rangle$, hence (E, t) and (E'', t) do not need to have the same completion. However, by ([9], p. 270) the completion of (E'', t) still lies in F. Now take $x \in F$ and $A \in \mathcal{H}$. Then take B in $\mathcal{H}$ such that $\beta(E', E)$ and the normed topology of E'_B coincide on A. Since x is bounded on B, it is continuous on E'_B, hence its restriction on A is $\beta(E', E)$-continuous. But then $x^{-1}(0) \cap A$ is $\sigma(E', E'')$-closed, hence x is $\sigma(E', E'')$-continuous on A. Thus x belongs to the completion of (E'', t), q. e. d.

COROLLARY 6.—*Let* (E, t) *be an infra-Schwartz space. If* (E, t)

is quasi-complete, then $(E', \mu(E', E))$ *is barrelled. If* (E, t) *is complete, then* $(E', \mu(E', E))$ *is ultrabornological.*

PROOF.—If (E, t) is quasi-complete, then by Proposition 5.1 of [7] it is closed in (E'', t). Hence by part 2 of Theorem 5, $E = E''$. Thus $(E', \mu(E', E))$ is barrelled. If (E, t) is complete, then $t' = \mu(E', E)$, hence $(E', \mu(E', E))$ is ultrabornological, q. e. d.

COROLLARY 7.—*Let* (E, t) *be a quasi-normable space. If* (E'', t) *is complete, then* $(E', \mu(E', E''))$ *is ultrabornological.*

PROOF.—Follows from part 3 of Theorem 5, q. e. d.

PROPOSITION 8.—*Let* (E, t) *be a locally convex space and* (E, τ) *the associated with* (E, t) *Schwartz space. Consider the following*: a) (E, t) *is boundedly completed*; b) (E, τ) *is boundedly completed*; c) (E'', t) *is complete*; d) (E'', τ) *is complete.*

Then b $\Longleftrightarrow$ d, b $\Longrightarrow$ a *and* d $\Longrightarrow$ c. *If* (E, t) *is quasi-normable*, b $\Longleftrightarrow$ c $\Longleftrightarrow$ d. *If* (E, t) *is infra-Schwartz*, a $\Longleftrightarrow$ b $\Longleftrightarrow$ c $\Longleftrightarrow$ d.

PROOF.—$b \Longleftrightarrow d$ by Proposition 1 and part 1 of Theorem 5. $b \Longrightarrow a$ by Corollary 2. Since (E'', t) has a base of τ-closed neighbourhoods of zero, $d \Longrightarrow c$, ([9], p. 210). If (E, t) is infra-Schwartz or quasi-normable, then $c \Longrightarrow d$ by parts 1 and 3 of Theorem 5. If (E, t) is infra-Schwartz, then $a \Longrightarrow c$ by Proposition 1 and part 2 of Theorem 5, q. e. d.

COROLLARY 9.—*Let* (E, t) *be a locally convex space and* (E, τ) *the Schwartz space associated with* (E, t). *Suppose* $\beta(E', E) = \mu(E', E'')$. *Consider the following*: a) (E, t) *is quasi-distinguished*; b) (E, τ) *is quasi-distinguished*; c) (E'', t) *is complete*; d) (E'', τ) *is complete*

Then b $\Longleftrightarrow$ d *and* b $\Longrightarrow$ a. *If* (E, t) *is quasi-normable*, b $\Longleftrightarrow$ c $\Longleftrightarrow$ d. *If* (E, t) *is infra-Schwartz*, a $\Longleftrightarrow$ b $\Longleftrightarrow$ c $\Longleftrightarrow$ d.

PROOF.—$b \Longrightarrow d$ by Proposition 1 and part 1 of Theorem 5. $d \Longrightarrow b$: from part 1 of Theorem 5 follows that $t' = \beta(E', E'') = \mu(E', E'')$, hence $\beta(E', E) = \beta(E', E'')$, therefore (E, τ) is quasi-distinguished. $b \Longrightarrow a$ by Corollary 2.

If (E, t) is infra-Schwartz or quasi-normable, then by Proposition

8 $c \Longleftrightarrow d$, hence $b \Longleftrightarrow c \Longleftrightarrow d$. If (E, t) is infra-Schwartz, then $a \Longrightarrow c$ by Proposition 1 and parts 2, 3 of Theorem 5, hence $a \Longleftrightarrow b \Longleftrightarrow c \Longleftrightarrow d$, q. e. d.

Let ⟨E, F⟩ be a dual pair. The *associated with* (F, μ (F, E)) *ultrabornological topology* is the weakest ultrabornological topology on F, finer than μ (F, E), ([3], p. 35). This topology will be denoted by ub (F, E).

We say that a sequence x_n ($n \in \mathbb{N}$) of F is *fast convergent*, if there exists a bounded Banach disk B in F such that x_n ($n \in \mathbb{N}$) converges in F_B, ([3], p. 29). We shall denote by fc (E, F) the topology on E of uniform convergence on all fast convergent to zero sequences of F.

Proposition 10.—*Let E be a locally convex space.* 1) *(E, fc (E, E′)) is boundedly completed if and only if (E′, β (E′, E)) and (E′, ub (E′, E)) have the same dual.* 2) *(E, fc (E, E′)) is quasi-distinguished if and only if the strong dual of E is ultrabornological.*

Proof.—Denote by $\mathcal{H}$ the family of all σ (E′, E)-compact disks and by (E′, t') the inductive limit of $\{E'_A: A \in \mathcal{H}\}$. Then t' = = ub (E′, E), ([3], p. 36). Since (E, fc (E, E′)) is Schwartz, the dual of (E′, ub (E′, E)) equals to the completion of (E, fc (E, E′)) by part 1 of Theorem 5. Hence using Proposition 8 we obtain the first part of Corollary 10.

Suppose (E, fc (E, E′)) is quasi-distinguished. Then β (E′, E) is the topology of uniform convergence on all bounded sets of the completion of (E, fc (E, E′)). Hence (E′, E) = ub (E′, E), ([3], p. 38). Conversely, suppose (E′, β (E′, E)) is ultrabornological. Then β (E′, E) = μ (E′, E″) and (E″, fc (E″, E′)) is complete, hence (E, fc (E, E′)) is quasi-distinguished, q. e. d.

A locally convex space E is *semibornological*, if every linear functional, which is bounded on any bounded set, belongs to E′, [22]. Given an infinite cardinal number α and a topology u on E, compatible with the duality ⟨E, E′⟩, we say that E is α-u-quasi-barrelled, if any β (E′, E)-bounded set, which is a union of at most α-u-equicontinuous sets, is equicontinuous, [15].

Corollary 11.—*Let E be* α-u-*quasi-barrelled.* 1) *If (E′, β (E′, E)) is semibornological, then E is boundedly completed.* 2) *If (E′, β (E′, E)) is bornological, then E is quasi-distinguished.*

PROOF.—Since E is α-u-quasi-barrelled, (E′, β (E′, E)) is locally complete, hence (E′, μ (E′, E″)) is ultrabornological. Moreover, the initial topology of E is finer than fc (E, E′). Thus the corollary follows from Proposition 10 and Corollary 2, q. e. d.

The next proposition is based on the following result:

a) ([3], p. 44). *The strong dual of a metrizable locally convex space is ultrabornological if and only if it is quasi-barrelled.*

In Theorem 7 of ([5], p. 73) A. Grothendieck proved that a metrizable space is distinguished if and only if its strong dual is bornological. This result is slightly generalized in the next proposition.

PROPOSITION 12.—*Let E be a metrizable locally convex space. The following is equivalent*: (i) *E is distinguished*; (ii) *(E, fc (E, E′)) is quasi-distinguished*; (iii) *(E′, β (E′, E)) is quasi-barrelled.*

PROOF.—(i) $\Longrightarrow$ (iii): if E is distinguished, then (E′, β (E′, E)) is barrelled. (iii) $\Longrightarrow$ (i): by a) (E′, β (E′, E)) is ultrabornological, hence barrelled, therefore E is distinguished. (ii) $\Longleftrightarrow$ (iii): by a) and Proposition 10, q. e. d.

From Proposition 12 and Corollary 2 follows that a distinguished metrizable space is quasi-distinguished.

Futher we shall denote by C (T) the space of all real-valued continuous on a completely regular space T functions, endowed with the compact-open topology.

PROPOSITION 13.—C (T) *is quasi-distinguished, if its strong dual is barrelled.*

PROOF.—Denote by E′ the dual of C (T) and let $\tilde{E}$, X and E be as in example 14. Take a barrel U in (E′, σ (E′, E)). By Proposition 1, U is barrel, hence a neighbourhood of zero, for β (E′, C (T)). Therefore β (E′, E) = β (E′, C (T)). But then C (T) is quasi-distinguished in E. Hence, applying the arguments of the proof of Warner's completeness theorem, [20], we conclude, that $\tilde{E}$ = E. Thus C (T) is quasi-distinguished, q. e. d.

The next proposition provides a sufficient condition for a quasi-barrelled space to be quasi-distinguished. To prove it we shall need the following lemma.

LEMMA 14.—*Let E be a boundedly completed space. If (E′, β (E′, E)) is quasi-barrelled, then E is quasi-distinguished.*

PROOF.—Denote by E the completion of E. Take a bounded set A in $\tilde{E}$ and let U be the absolute polar of A in E′. By Proposition 1, U is closed for β (E′, E). Since A is absorbed by every barrel in $\tilde{E}$, ([9], p. 252), U is bornivorous for σ (E′, E), and therefore for β (E′, E). But then U is a neighbourhood of zero for β (E′, E), hence there exists a barrel V in (E′, σ (E′, E)) such that V ⊂ U. Let $\tilde{B}$ and B be the polars of V in E and E, respectively. By ([9], p. 246), $\tilde{B}$ is the closure of B in E. Since A ⊂ B, E is quasi-distinguished, q. e. d.

A set M of E is *quasi-closed*, if A ∩ M is closed for every closed bounded disk A of E, [7]. A locally convex space E is *quasi-bornological* *, if any quasi-closed bornivorous disk of E is a neighbourhood of zero, [11].

PROPOSITION 15.—*Let E be a quasi-barrelled space. If the strong dual of E is quasi-bornological, then E is distinguished and quasi-distinguished.*

PROOF.—The strong dual of a quasi-bornological space is complete, [11], hence by Corollary 4, E is boundedly completed. Since a quasi-bornological space is quasi-barrelled, by Lemma 14 E is quasi-distinguished.

Let U be a barrel in (E′, β (E′, E)). Since E is quasi-barrelled, any bounded disk of (E′, β (E′, E)) is equicontinuous, hence absorbed by U, therefore U is a neighbourhood of zero in (E′, β (E′, E)), and we conclude that (E′, β (E′, E)) is barrelled. Thus E is distinguished, q. e. d

The proposition above contains Proposition 3 of ([9], p. 388).

We conclude this article with the following remark. The properties of being distinguished and quasi-distinguished are mutually independent. Indeed, the noncomplete Montel space of [8] is distinguished, but not quasi-distinguished. To give a converse example, we shall take a dense one-codimensional subspace E in the nondistinguished Frechet space F of Grothendieck, ([5], p. 88). By Pro-

* The original word used in [11] was «semibornological».

position 1 of [14], E equipped with the induced topology is quasi-distinguished. Hence $\beta(F', E) = \beta(F', F)$, therefore $(F', \beta(F', E))$ is not barrelled, thus E is not distinguished.

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