BOUNDEDLY COMPLETED SUBSPACES IN LOCALLY CONVEX SPACES

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Boundedly completed and quasi-distinguished spaces and subspaces were introduced in [9] and [10]. In this article we investigate the conditions for a subspace of a locally convex space E to be boundedly completed or quasi-distinguished in E and study the topological properties of E connected with the above concepts. We obtain:

1. In an ultrabornological space E every dense countable-codimensional subspace is boundedly completed in E. In a locally complete bornological space E every dense countable-codimensional subspace is quasi-distinguished in E, (follows from Corollaries 6 and 7).

2. If G is boundedly completed in E, then the associated with $(E', \mu(E', G))$ barrelled (ultrabornological) topology is equal to the associated with $(E', \mu(E', E))$ barrelled (ultrabornological) topology. Hence if any dense subspace is boundedly completed in E, the associated with $(E', \mu(E', E))$ barrelled and ultrabornological topologies are minimal on E′, (Theorem 1).

3. Let G be a dense subspace in E. If G is boundedly completed, then G satisfies the following hereditary properties:

a) if E is α-u-quasi-barrelled, G is α-u-quasi-barrelled, (Proposition 8).

b) if E is semi-dualmetric, resp. dualmetric, resp. a (DF)-space, then G is semi-dualmetric, resp. dualmetric, resp. a (DF)-space, (Theorem 9).

Finally, using the concept of a quasi-distinguished space, we slightly extend some results of [7] and [8].

Los conceptos de subespacios acotadamente completos y casi-distinguidos los hemos introducido en [10]. En este artículo investigamos las propiedades topológicas de espacios localmente convexos, ligados con los conceptos citados. En particular, obtenemos que si un espacio E es α-u-casi-tonelado [resp. semi-dualmétrico, dualmétrico, (DF)-espacio], y el subespacio G es acotadamente completo, entonces G es α-u-casi-tonelado [resp. semi-dualmétrico, dualmétrico, (DF)-espacio].

Let G be a subspace of a locally convex space E. We say that G is *quasi-distinguished (boundedly completed) in* E, if every bounded set (element) of E is contained in the closure of a bounded set of G. We say that E is *quasi-distinguished (boundedly completed)*, if E is quasi-distinguished (boundedly completed) in its completion. In [10] we studied the conditions for a locally convex space to be boundedly completed. In this article we obtain the conditions for a subspace of E to be boundedly completed in E and investigate the topological properties of boundedly completed subspaces.

Given a vector space E over the field $\mathbb{K}$ of the real or complex numbers, we write (E, t) when E is equipped with the topology t. All topologies on E will be locally convex and separated. We denote by E' and E^* the topological and algebraic duals of (E, t), respectively. Let $< E, F >$ be a dual pair. We denote by $\beta(E, F)$, $\mu(E, F)$ and $\sigma(E, F)$ the strong, Mackey and weak topologies on E, respectively. An absolutely convex set is called a disk. For a bounded disk A of a locally convex space E we denote by E_A the linear hull of A equipped with the norm, associated with A. We say that A is a *Banach disk*, if E_A is a Banach space. Given a set M in E, we denote by $\bar{M}$ its closure in E, and by M^* its closure in the weak bidual $(E'', \sigma(E'', E'))$ of E.

By $t(E, F)$, [resp. $ub(E, F)$], we denote the associated with $(E, \mu(E, F))$ barrelled, [resp. ultrabornological], topology, it means, the weakest barrelled, [resp. ultrabornological], topology on E, finer than $\mu(E, F)$, ([1], p. 35).

THEOREM 1.—*Let G be a subspace of a locally convex space E. If G is boundedly completed in E, then* $\mathrm{t}(E', G) = \mathrm{t}(E', E)$ *and* $\mathrm{u\,b}(E', G) = \mathrm{u\,b}(E', E)$.

PROOF.—It is clear that $t(E', G) \leqslant t(E', E)$. On the other hand, every barrel in $(E', \sigma(E', G))$ is a barrel in $(E', t(E', G))$, hence $t(E', G) \geqslant \beta(E', G)$. Since G is boundedly completed in E, $\beta(E', G)$ is finer than $\sigma(E', E)$. Hence $t(E', G)$ is finer than $\mu(E', E)$, therefore $t(E', G) = t(E', E)$.

Let B be a bounded Banach disk in $(E', \sigma(E', G))$. According to Banach-Mackey theorem, ([4], p. 252), B is bounded for $\beta(E', G)$, hence for $\sigma(E', E)$, therefore $(E', \sigma(E', G))$ and $(E', \sigma(E', E))$ have the same bounded Banach disks, thus $u\,b\,(E', G) = u\,b\,(E', E)$, q. e. d.

A barrelled, resp. ultrabornological, topology on E is *minimal*, if there is no weaker separated locally convex barrelled, resp. ultrabornological, topologies on E, [3].

COROLLARY 2.—*Let E be a locally convex space. If every dense subspace is boundedly completed in E, then* $t\,(E', E)$ *is the minimal barrelled and* $u\,b\,(E', E)$ *is the minimal ultrabornological topologies on E′.*

Let $\mathcal{H}$ be a family of disks in E. We say that $\mathcal{H}$ satisfies the *ideal property*, if for every A_1, A_2 in $\mathcal{H}$ and λ in $\mathbb{K}$ there exists A_3 in $\mathcal{H}$ such that $\lambda\,(A_1 + A_2) \subset A_3$.

For the next proposition we shall need the following result of Valdivia:

a) ([12], Lema 1) *Let G be a one-codimensional subspace in a locally convex space E and $\mathcal{H}$ a family of closed bounded disks in E, satisfying the ideal property. Suppose for some $B_0 \in \mathcal{H}$ the set $B_0 \cap G$ is not closed. Then for any $B \in \mathcal{H}$ there exists $A \in \mathcal{H}$ such that $B \subset \overline{A \cap G}$.*

The next proposition slightly extends Proposition 1 of [9].

PROPOSITION 3. — *Let E be a locally convex space. Then $(E', \beta\,(E', E))$ is complete if and only if every dense finite-codimensional subspace of E is quasi-distinguished in E.*

PROOF.—Let G be a dense subspace of finite codimension in E. Denote by Q the polar of G in E^*. Take a basis $f_1, f_2, \ldots, f_n$ in Q and denote by

$$G_k = \bigcap_{i=1}^{k} f_i^{-1}(0).$$

Since f_1 does not belong to E' and $(E', \beta\,(E', E))$ is complete, we conclude, applying ([4], p. 270), that there exists a closed bounded disk A in E, such that $A \cap G_1$ is not closed. Hence by a), $\beta\,(E', E) = \beta\,(E', G_1)$.

Since $f_2 \notin E'$ and $(E', \beta\,(E', G_1))$ is complete, we use the same arguments and conclude that $\beta\,(E', G_1) = \beta\,(E', G_2)$. Continuing, we obtain: $\beta\,(E', E) = \beta\,(E', G_n) = \beta\,(E', G)$. Hence G is quasi-distinguished in E.

Suppose $(E', \beta(E', E))$ is not complete. Take an element f in its completion, such that $f \notin E'$, and denote by $F = f^{-1}(0)$. By ([4], p. 270) F is not quasi-distinguished in E, q. e. d.

COROLLARY 4.—*Let E be a locally convex space with a complete strong dual. Suppose for any dense subspace G of E the strong dual $(E', \beta(E', G))$ is complete, (for example, E and each of its dense subspaces are bornological). Then any dense subspace of E is boundedly completed in E.*

PROOF.—Let G be dense in E. Take $x \in E \setminus G$ and denote by F the linear hull of $G \cup \{x\}$. By Proposition 3, G is quasi-distinguished in F. Hence G is boundedly completed in E, q. e. d.

PROPOSITION 5.—*Let E be a locally convex space and $\mathcal{H}$ a family of closed bounded Banach disks covering E and satisfying the ideal property. Suppose E′ endowed with the topology of $\mathcal{H}$-convergence is complete. Then any dense subspace F, whose codimension in the linear hull of $F \cup A$ is finite for every $A \in \mathcal{H}$, is boundedly completed in E.*

PROOF.—Denote by

$$L = \bigcup \{\overline{A \cap F}: \ A \in \mathcal{H}\}.$$

We shall prove that $L = E$. Assume the opposite and let X be an algebraic complement to L in E. Take $A \in \mathcal{H}$. The codimension of the linear hull of $\overline{A \cap F}$ in E_A is finite, hence by [7] and by Ptak's open map theorem, E_A is isomorphic to a locally convex direct sum of $E_{\overline{A \cap F}}$ and a finite-dimensional subspace. Therefore there exist B in $\mathcal{H}$ and a bounded disk K in X, such that the linear hull of K is finite-dimensional and $A \subset \overline{B \cap F} + K$. But then $A^* \subset (\overline{B \cap F})^* + K$. Denote by t' the topology of $\mathcal{H}$-convergence on E′ and by E''_0 the dual of (E', t'). Since covers E, X is still an algebraic complement to

$$\bigcup \{(\overline{A \cap F})^*: \ A \in \mathcal{H}\}$$

in E''_0. Hence denoting by ξ the topology on E′ of uniform convergence on $\{A \cap F: \ A \in \mathcal{H}\}$, we conclude that

$$t' = \sup(\xi, \sigma(E', E_0'')).$$

Therefore by Corollary 1 of [9], (E', t') cannot be complete, thus a contradiction. Hence $L = E$, therefore F is boundedly completed in E, q. e. d.

COROLLARY 6.—*Let E be a locally convex space such that $(E', \mu(E', E))$ is complete, (for example, let E be ultrabornological). Then every dense countable-codimensional subspace is boundedly completed in E.*

PROOF.—Take the family $\mathcal{H}$ of all weakly compact disks in E. If F is countable-codimensional, then by [13] the codimension of $A \cap F$ in E_A is finite for every $A \in \mathcal{H}$. Hence the corollary, q. e. d.

We say that E is *locally complete*, if every closed bounded disk of E is Banach.

COROLLARY 7.—*Let E be a locally complete space with a complete strong dual. Then every dense countable-codimensional subspace is quasi-distinguished in E.*

PROOF.—Let F be a dense subspace of countable codimension in E. Take a closed bounded disk A in E. By [12], E_A is isomorphic to the locally convex direct sum of $E_{\overline{A \cap F}}$ and a finite-dimensional normed space Y. Let K be the unit ball of Y. By Proposition 5, F is boundedly completed in E, hence there exists a bounded disk B in E such that $K \subset \overline{B \cap F}$. But then A is contained in the closure of $r(A \cap F + B \cap F)$ for some $r > 0$. Hence F is quasi-distinguished in E, q. e. d.

Let E be a locally convex space. Given an infinite cardinal number α and a topology u on E compatible with the duality $< E, E' >$, we say that E is α-u-*barrelled*, [resp. α-u-*quasi-barrelled*] if every $\sigma(E', E)$-bounded, [resp. $\beta(E', E)$-bounded] set, which is a union of at most α u-equicontinuous sets, is equicontinuous with respect to the initial topology of E. The next proposition provides a hereditary property of an α-u-quasi-barrelled space E.

PROPOSITION 8.—*A dense subspace G of a locally convex space E is α-u-quasi-barrelled, if one of the following holds:*

1) *E is α-u-quasi-barrelled and G is quasi-distinguished in E;*
2) *E is α-u-barrelled and G is boundedly completed in E;*
3) *E is α-u-quasi-barrelled and G is boundedly completed.*

PROOF.—We shall identify the dual of G with E′. Let U be a β (E′, G)-bounded union of not more than α u-equicontinuous sets.

1) If G is quasi-distinguished in E, $\beta(E', G) = \beta(E', E)$. Since E is α-u-quasi-barrelled, U is equicontinuous. Hence G is α-u-quasi-barrelled.

2) If G is boundedly completed in E, $\beta(E', G)$ is finer than $\sigma(E', E)$. Since E is α-u-barrelled, U is equicontinuous. Hence G is α-u-quasi-barrelled.

3) Denote by $\hat{E}$ the completion of E. Then E is α-u-barrelled and G is boundedly completed in $\hat{E}$. Hence by part 2, G is α-u-quasi-barrelled, q. e. d.

A locally convex space is *sequentially quasi-barrelled*, if every sequence, which converges to zero in $(E', \beta(E', E))$, is equicontinuous, [14]. The next definition generalizes the concepts of a (DF)-space, [2], and a dualmetric space, [5]. A sequentially quasi-barrelled space with a fundamental sequence of bounded sets is called *semi-dualmetric*. Spaces of this type were studied by W. Ruess, [6]. The next theorem provides a hereditary result for these spaces and slightly extends Corollary 2.4 of [6].

THEOREM 9.—*A dense subspace G of a semi-dualmetric, [resp. dualmetric,* (DF)-], *space E is semi-dualmetric, [resp. dualmetric,* (DF)-*space*], *if and only if G is boundedly completed.*

PROOF.—If G is semi-dualmetric, then G is quasi-distinguished by Proposition 10 and Corollary 2 of [10]. Hence G is boundedly completed.

Now suppose G is boundedly completed. Denote by $\hat{E}$ the completion of E.

1) Let E be semi-dualmetric. Then $\beta(E', E) = \beta(E', \hat{E})$ and $(E', \beta(E', \hat{E}))$ is Frechet. Since G is boundedly completed, $\beta(E', G)$ is finer than $\sigma(E', \hat{E})$. Hence any $\beta(E', G)$-bounded set is $\sigma(E', \hat{E})$-bounded. But according to Banach-Mackey theorem, every barrel in $\hat{E}$ is bornivorous. Hence any $\beta(E', G)$-bounded set is $\beta(E', \hat{E})$-bounded. But $(E', \beta(E', G))$ is metrizable, hence bornological, thus we conclude that $\beta(E', G) = \beta(E', \hat{E})$. Hence every sequence, convergent to zero in $(E', \beta(E', G))$, is equicontinuous, therefore G is semi-dualmetric.

2) Let E be dualmetric. Then E is $\varkappa_0$-$\sigma(E, E')$-quasi-barrelled. By proposition 8, G is also $\varkappa_0$-$\sigma(E, E')$-quasi-barrelled. Since G has

a fundamental sequence of bounded sets, we conclude that G is dualmetric.

3) Let E be a (DF)-space. Denote by t the initial topology of E. Then E is $\aleph_0$-t-quasi-barrelled, hence by Proposition 8, G is $\aleph_0$-t-quasi-barrelled, therefore G is a (DF)-space, q. e. d.

It is known, that the topology of a barrelled countable-dimensional space E is the finest locally convex topology of E, [7]. This result can be extended, using the concept of a quasi-distinguished space.

PROPOSITION 10.—*Let E be a countable-dimensional locally convex space. Then* $\beta(E, E')$ *is equal to the finest locally convex topology of E.*

PROOF.—The space $(E', \sigma(E', E))$ is separable and metrizable, hence quasi-distinguished, ([2], p. 62), therefore $\beta(E, E') = \beta(E, E^*)$, q. e. d.

In [8] S. A. Saxon and A. Wilansky proved that given a Hamel basis in a metrizable barrelled space, all but a finite number of the coefficient functionals are discontinuous. The same result follows from our next proposition.

PROPOSITION 11.—*Let E be a locally convex space and H a Hamel basis in E such that the set* $\{f_i : i \in I\}$ *of continuous coefficient functionals on E is not empty. Then there exists a subspace in E, whose dimension is equal to* Card I *and on which the induced by* $\beta(E, E')$ *topology equals to the finest locally convex topology.*

PROOF.—Let $\{x_i : i \in I\}$ be the set of corresponding to $\{f_i : i \in I\}$ elements of H. Denote by X and F the linear hulls of $\{x_i : i \in I\}$ and $\{f_i : i \in I\}$, respectively. Since the polar of F in E equals to the linear hull of $H \setminus \{x_i : i \in I\}$, the topology $\sigma(E', E)$ coincides on F with $\sigma(F, X)$. Identifying X^* with the topological product $\mathbb{K}^I$, we observe, that all elements of X^* with at most finitely many non-zero components are contained in F. Hence $(F, \sigma(F, X))$ is quasi-distinguished, therefore $\beta(X, F)$ equals to the finest locally convex topology of X. But the induced by $\beta(E, E')$ topology on X must be finer than $\beta(X, F)$. Hence the conclusion, q. e. d.

In [8] S. A. Saxon and A. Wilansky proved that every Banach space has a larger non-barrelled norm. Using their method, we sharpen their result in the next proposition.

PROPOSITION 12.—*Let E be a vector space. There exists a normed topology* q *on E, satisfying the following: being E′ the dual of* (*E*, q), *the strong topology* β (*E*, *E′*) *equals to the finest locally convex topology of E. If another norm,* p *is given on E, we can chose* q *finer than* p.

PROOF.—Take a Hamel basis H in E. Denote by U the absolutely convex hull of H and by q the gauge of U. Clearly (E, q) is normed. Identifying E* with the topological product $\mathbb{K}^H$, we conclude that E′ contains all elements of E* with at most finitely many non-zero elements. Hence (E′, σ (E′, E)) is quasi-distinguished, therefore β (E, E′) = β (E, E*). If another norm, p, is given on E, then we take H in the unit ball of (E, p), q. e. d.

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