ANTICOMMUTATIVE ANALYTIC FORMS ON FULLY NUCLEAR SPACES

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El objeto del presente trabajo es estudiar las formas analíticas anticonmutativas sobre los espacios completamente nucleares

In [1] and [2] we studied holomorphic functions and analytic functionals on fully nuclear spaces. In this paper we study anti-commutative analytic forms on fully nuclear spaces. Anti-commutative forms are obtained formally from holomorphic functions by replacing each symmetric n linear form in the Taylor series expansion at zero by an alternating n-linear form (a symmetric n-linear form, B, on the vector space E is commutative in the sense that $B(x, y) = B(y, x)$ for any $x, y \in E$ and an alternating bilinear form B is anticommutative in the sense that $B(x, y) = -B(y, x)$ for any $x, y \in E$). If E is a finite dimensional vector space then the space of anticommutative analytic forms is the exterior algebra of E and hence it is also a finite dimensional vector space and its topological vector space structure is trivial. If E is an infinite dimensional topological vector space, then the space of anticommutative analytic forms is also an infinite dimensional space and in the case of fully nuclear spaces, we see that it possesses an interesting and nontrivial topological vector space structure. Our basic technique in studying this structure is to identify the space of anticommutative forms with a space of holomorphic functions and to apply the results of [1] and [2] to complete our examination.

We obtain this identification in § 2 and obtain our applications in § 3. In § 1 we recall certain definitions from [1] and [2] and give elementary properties of anticommutative n linear forms.

Holomorphic functions on $\mathcal{S}$ (the space of rapidly decreasing functions) arose in the work of Paul Krée in discussing boson fields in the mathematical foundations of quantum field theory, and it appears likely that anticommutative forms will arise in an analogous fashion in the theory of fermion fields.

We use the standard notation of [1] and [2] assume that E is a locally convex space over the field of complex numbers.

§ 1. *A fully nuclear space is a locally convex space E such that E and E'_β (the strong dual of E) are both complete reflexive nuclear spaces.* If E is fully nuclear and has a Schauder basis then it has an equicontinuous and hence an absolute basis.

Let P denote a collection of non-negative sequences such that for each $r \in \mathrm{N}$ there exists $(\alpha_n)_n \in \mathrm{P}$ and $\alpha_r > 0$. *The sequence space Λ (P) is the set of all sequences of complex numbers,* $(z_n)_n$, *such that*

$$\Sigma_n \mid z_n \mid \alpha_n < \infty$$

for all

$$\alpha = (\alpha_n)_{n=1}^{\infty} \in \mathrm{P}.$$

We shall assume that P is complete in the following sense: if $(\beta_n)_n$ is a sequence of non-negative real numbers and

$$\{(z_n)_n \in \Lambda(\mathrm{P}); \quad \Sigma_n \mid z_n \beta_n \mid \leq 1\}$$

is a neighbourhood of zero then $(\beta_n)_n \in \mathrm{P}$. We endow Λ (P) with the topology generated by the semi-norms p_α,

$$\alpha = (\alpha_n)_n \in \mathrm{P},$$

where

$$p_\alpha (\{z_n\}_n) = \Sigma_n \mid z_n \mid \alpha_n .$$

Each element of P is called a weight.

The following is the Grothendieck-Pietsch criterion for the nuclearity of a sequence space.

Proposition 1.—*The locally convex space Λ (P) is nuclear if* *and only if for each* $(\alpha_n)_n \in P$ *there exists* $(u_n)_n \in l_1$ *and* $(\alpha'_n)_n \in P$ *such that*

$$\alpha_n \leqq |u_n| \alpha'_n \quad \textit{for all} \quad n.$$

If $\Lambda(P)$ is nuclear then

$$\Lambda(P) = \{(z_n)_n; \quad \sup_n |z_n \alpha_n| < \infty \quad \text{for all} \quad (\alpha_n)_n \in P\}$$

and its topology is generated by

$$\|(z_n)_n\|_{(\alpha_n)_n} = \sup |z_n \alpha_n| \quad \text{where} \quad (\alpha_n)_n \quad \text{ranges over } P.$$

If E is fully nuclear with a Schauder basis (we shall say fully nuclear with a basis from now on) then E and E'_β may be identified with nuclear sequence spaces $\Lambda(P)$ and $\Lambda(P')$. We fix once and for all such an identification and denote the duality between E and E'_β as follows:

$$w(z) = \langle w, z\rangle = \langle (w_n)_n, (z_n)_n\rangle = \Sigma_n w_n z_n$$

where $z \in E$ and $w \in E'_\beta$.

Subsets of $\Lambda(P)$ *which have either of the following forms*

$$A = \{(z_n)_n \in \Lambda(P); \quad \sup |z_n \alpha_n| < 1\}$$

or

$$B = \{(z_n)_n \in \Lambda(P); \quad \sup_n |z_n \alpha_n| \leq 1\}$$

where $\alpha_n \in [0, \infty]$ *all* n *and* $a \cdot (+\infty) = +\infty$ *if* $a > 0$ *and* $0 \cdot (+\infty) = 0$ *are called polydiscs*. A is open if and only if $(\alpha_n)_n \in P$ and B is always closed. The *multiplicative polar* of a subset U of a fully nuclear space with a basis, $E \cong \Lambda(P)$, is defined as follows

$$U^M = \{(w_n)_n \in E'_\beta; \quad \sup_n |w_n z_n| \leq 1 \quad \text{all} \quad (z_n)_n \in U\}.$$

If U is an open polydisc in a fully nuclear space then U^M is a compact polydisc in E'_β.

E is called an A-nuclear space if E has an absolute basis and when identified with $\Lambda(P)$ *then there exists a sequence of positive real numbers,* $\delta = (\delta_n)_n$, $\delta_n > 1$ *all* n *and* $\Sigma_n \dfrac{1}{\delta_n} < \infty$ *such that*

$(\alpha_n \delta_n)_n \in P$ *whenever* $(\alpha_n)_n \in P$. The strong dual of an A-nuclear space is A-nuclear and nuclear. A reflexive A-nuclear space is a fully nuclear space with a basis. Every Fréchet nuclear space with a basis is an A-nuclear space. *E is a B-nuclear space if E is a Fréchet nuclear space with a basis and E is isomorphic to* $\Lambda(P)$ *where the topology of* $\Lambda(P)$ *can be defined by sequence of weights*

$$(w^m)_{m=1}^{\infty}, \quad w^m = (w_n^m)_{n=1}^{\infty},$$

with the following properties,

1) $w_n^m > 0$ *for all* m *and* n
2) if

$$\beta_n^m = \frac{w_m^{m+1}}{w_n^m}$$

for all m *and* n *then*

$$(w_n^m (\beta_n^m)^p)_{n=1}^{\infty} \in P$$

for every positive integer P.

The space of rapidly decreasing functions $\mathcal{S}$, is B-nuclear and a nuclear power series space is B-nuclear if and only if it is of infinite type. We refer to [1], [2], [3] and [5] for further details concerning nuclear spaces, fully nuclear spaces and A and B nuclear spaces.

Now let E denote a locally convex space over the field of complex numbers.

We let

$$E^{(n)} = \underbrace{E \times \ldots \times E}_{n}$$

with the product topology.

Definition 2.—*An anticommutative* n-*linear form on E is a mapping*

$$L : E^{(n)} \longrightarrow C$$

such that

$$L(X_1, \ldots, X_n) = \operatorname{sgn} \sigma \cdot L(X_{\sigma(1)}, \ldots, X_{\sigma(n)})$$

for any permutation σ *of* $\{1, \ldots, n\}$.

We let $\mathcal{L}^A({}^nE)$ denote the space of continuous anticommutative n linear forms on E and let $\mathcal{L}^A_{HY}({}^nE)$ denote the space of anticommutative n linear forms which are continuous on the compact subsets of ${}^nE$.

We topologize $\mathcal{L}^A_{HY}({}^nE)$ by using the compact open topology, τ_0, i. e. the system of semi-norms defining the topology is given by

$$\| L \|_K = \sup_{x_i \in K} | L(x_1, \ldots, x_n) |$$

where K ranges over the compact subsets of E. $(\mathcal{L}^A_{HY}({}^nE), \tau_0)$ is a complete locally convex space.

DEFINITION 3.—*Let U denote a balanced open subset of the locally convex space E.*

(a) *$H^A_{HY}(U)$, the space of anticommutative hypoanalytic forms on U, is the space of all formal power series,*

$$f = (P_n)_{n=0}^{\infty}, \quad \textit{where} \quad P_n \in \mathcal{L}^A_{HY}({}^nE) \quad \textit{and} \quad \Sigma_{n=0}^{\infty} \| P_n \|_K$$

for every compact subset K of U.

$H^A_{HY}(U)$ is endowed with the compact open topology τ_0, generated by the semi-norms

$$\| f \|_K = \Sigma_{n=0}^{\infty} \| P_n \|_K$$

as K ranges over the compact subsets of U.

(b) *$H^A(U)$, the space of anticommutative analytic forms on E, is the space of all formal power series,*

$$f = (P_n)_{n=0}^{\infty},$$

where $P_n \in \mathcal{L}^A({}^nE)$, and for each compact subset K of U there exists a neighbourhood V of K (which may depend on f and K) such that $\Sigma_{n=0}^{\infty} \| P_n \|_V < \infty$.

$$(\| P_n \|_V = \sup_{X_i \in V} | P(X_1, \ldots, X_n) |).$$

A semi-norm p on $H^A(U)$ is said to be ported by the compact *subset K of U if for every neighbourhood V of K there exists $C(V) > 0$ such that*

$$p(f) \leq C(V) \cdot \Sigma_{n=0}^{\infty} \| P_n \|_V$$

for all $f = (P_n)_n \in H^A(U)$. *The* τ_ω *topology on* $H^A(E)$ *is the topology generated by the semi-norms ported by the compact subsets of U.*

We look now at the anticommutative n linear forms which correspond to the monomials (see [1]) in the case of the commutative (or symmetric) n linear functionals.

Let E denote a fully nuclear space with a basis $E \approx \Lambda(P)$.

Let S denote the set of all finite strictly increasing sequences of positive integers. We identify S with a subset of $N^{(N)}$ in the following way:

$$\text{if } s = (s_1, \ldots, s_n) \in S \longrightarrow (0, \ldots, \underset{\substack{\uparrow \\ S_1 \\ \text{position}}}{1}, 0 \ldots \underset{\substack{\uparrow \\ S_2 \\ \text{position}}}{1}, \ldots \underset{\substack{\uparrow \\ S_n \\ \text{position}}}{1}, 0 \ldots)$$

With this identification

$$z^s = z_{s_1} z_{s_2} \ldots z_{s_n}$$

where z_i as usual denotes evaluation at the i^{th} coordinate.

Let

$$d z^s = z_{s_1} \wedge z_{s_2} \ldots > z_{s_n} \quad \text{if} \quad s = (s_1, \ldots, s_n).$$

We let $|s|$ denote the length of s.

We call $d z^s$ a monoform of degree $|s|$. For any integer n we let

$$\gamma_n = \sup_{\substack{|e_{ij}| \leq 1 \\ e_{ij} \in C}} \begin{vmatrix} e_{11} & \ldots & e_{1n} \\ \vdots & e_{ij} & \\ e_{1n} & \ldots & e_{nn} \end{vmatrix}.$$

It is clear that $\gamma_n \leqslant n!$ for all n.

LEMMA 4.—*If*

$$s = (s_1, \ldots, s_n) \in S$$

and

$$A = \{(z_n)_n \in \Lambda(P);\ \sup_n |z_n \beta_n| \leq 1\}$$

is a polydisc then

$$\| d z^s \|_A = \gamma_{|s|} \cdot \| z^s \|_A = \frac{\gamma_{|s|}}{|\beta_{s_1} \dots \beta_{s_n}|}$$

Proof.—Since

$$d z^s = d z_{s_1} \wedge z_{s_2} \dots \wedge z_{s_n}$$

this is a finite dimensional problem and by convexity $d z^s$ achieves its maximum on A x A x ... x A at one of the extreme points, i. e.

$$\| d z^s \|_A = \sup_{\substack{|e_{ij}| \leq 1 \\ e_{ij} \in \mathbb{C}}} \begin{vmatrix} e_{11}/\beta_{s_1} & \dots & e_{1n}/\beta_{s_1} \\ \vdots & & \\ e_{n1}/\beta_{s_n} & & e_{nn}/\beta_{s_n} \end{vmatrix} = \frac{\gamma_{|s|}}{|\beta_{s_1} \dots \beta_{s_n}|} = \gamma_{|s|} \cdot \|z^s\|_A .$$

We shall also need the following lemma.

Lemma 5.—*Let* $(\delta_n)_n$ *denote a sequence of positive numbers such that*

$$\Sigma_{n=1}^{\infty} \frac{1}{\delta_n} = \frac{1}{\delta} < \infty$$

then there exists a $K > 0$ *such that*

$$\frac{|s|!}{(\delta)^s} \leq K \quad \textit{for all} \quad s \in S$$

where

$$(\delta)^s = \delta_{s_1} \dots \delta_{s_n} \quad \textit{if} \quad s = (s_1, \dots, s_n).$$

Proof.—Rearrange the sequence $(\delta_n)_n$ to get a decreasing sequence with the same terms. Let $(\delta'_n)_n$ denote the new sequence. Then

$$\Sigma_n \frac{1}{\delta'_n} = \Sigma_n \frac{1}{\delta_n} < \infty \quad \text{and} \quad \frac{1}{\delta'_n}$$

is monotonically decreasing. By Pringsheim's theorem

$$\frac{n}{\delta'_n} \to 0 \quad \text{as} \quad n \to \infty .$$

Hence

$$\frac{n!}{\delta'_1 \ldots \delta'_n} \to 0 \quad \text{as} \quad n \to \infty .$$

If

$$s = (s_1, \ldots, s_n) \in S$$

then

$$\delta_{s_1} \ldots \delta_{s_n} \geq \delta'_1 \ldots \delta'_n$$

and hence

$$\frac{n!}{\delta_{s_1} \ldots \delta_{s_n}} = \frac{|s|!}{(\delta)^s} \leq \frac{n!}{\delta'_1 \ldots \delta'_n} \to 0 \quad \text{as} \quad n$$

If

$$K = \sup_n \frac{n!}{\delta'_1 \ldots \delta'_n} \quad \text{then} \quad \frac{s!}{(\delta)^s} \leq K \quad \text{for all} \quad s \in S.$$

§ 2. E will denote a fully nuclear space with a basis, and U will denote an open polydisc in E.
Let

$$f = (P_n)_{n=0}^{\infty} \in H_{HY}^{A}(E).$$

If

$$s = (s_1, \ldots, s_n) \in S$$

we let

$$a_s = P_n(e_{s_1}, \ldots, e_{s_n})$$

where

$$e_{s_i} = (0, \ldots \underset{\substack{\uparrow \\ s_i^{th} \text{ position}}}{1}, 0 \ldots) \in E.$$

Let A denote the polydisc of lemma 4.

LEMMA 5.

$$\| a_s \, d z^s \|_A \leq \gamma_n \, \| P_n \|_A$$

PROOF.

$$\| a_s \, d z^s \|_A = \frac{\gamma_n \, | P (e_{s_1}, \ldots, e_{s_n}) |}{| \beta_{s_1} \ldots \beta_{s_n} |} = \gamma_n \, | P (e_{s_1}/\beta_{s_1}, \ldots, e_{s_n}/\beta_{s_n}) | \leq \gamma_n \, \| P \|_A$$

(if $\beta_{s_i} = 0$ for some s_i and $a_s \neq 0$ then

$$\| a_s \, d z^s \|_A = \infty = \gamma_n \, \| P_n \|_A).$$

LEMMA 6.—*If* $(\delta_n)_n$ *is a sequence of positive real numbers and* $\Sigma_n \frac{1}{\delta_n} < \infty$ *the there exists* $C(\delta) > 0$ *such that*

$$\Sigma_{s \in S} \| a_s \, d z^s \|_A \leq C(\delta) \cdot \| f \|_{\delta^2 A}$$

where

$$\delta^2 A = \{ (\delta^2_n z_n)_n \in E ; \quad (z_n)_n \in A \}$$

PROOF.—For any $s \in S$ we have

$$\| a_s \, d z^s \|_A = \frac{1}{(\delta^2)^s} \, \| a_s \, d z^s \|_{\delta^2 A} \leq \frac{\gamma_{|s|}}{(\delta^2)^s} \, \| P_n \|_{\delta^2 A} \leq \frac{\gamma_{|s|}}{(\delta^2)^s} \, \| f \|_{\delta^2 A}$$

Hence, by lemma 5,

$$\Sigma_{s \in S} \| a_s \, d z^s \|_A \leqslant \| f \|_{\delta^2 A} \cdot \Sigma_{s \in S} \frac{\gamma_{|s|}}{(\delta^2)^s} \leqslant \| f \|_{\delta^2 A} \cdot \Sigma_{s \in S} \frac{1}{\delta^s} = C(\delta) \, \| f \|_{\delta^2 A}$$

PROPOSITION 7.—*If U is an open polydisc in a fully nuclear space with a basis then the monoforms form an absolute basis for*

$$(H^A_{HY}(U), \tau_0) \quad \textit{and} \quad (H^A(U), \tau_\omega).$$

PROOF.—If K is a compact subset of U we can by results in [1] find a sequence

$$(\delta_n)_n, \quad \delta_n > 1 \quad \text{and} \quad \Sigma_n \frac{1}{\delta_n} < \infty$$

such that $\delta^2 K$ is a compact subset of U. Hence if

$$f = (P_n)_{n=0}^{\infty} \in H_{HY}^{A}(U)$$

and

$$f' = \Sigma_{s \in S} a_s d z^s$$

then lemma 6 implies that

$$f' \in H_{HY}^{A}(U).$$

Again using results from [1] we can for each compact subset K of U and each neighbourhood V of zero find a sequence

$$(\delta_n)_n, \quad \delta_n > 1 \quad \text{all} \quad n \quad \text{and} \quad \Sigma_n \frac{1}{\delta_n} < \infty,$$

and W a neighbourhood of zero such that

$$\delta^2 (K + W) \subset K + V.$$

Hence if

$$f = (P_n)_{n=0}^{\infty} \in H^{A}(U)$$

and

$$f' = \Sigma_{s \in S} a_s d z^s$$

then lemma 6 implies that $f' \in H^A(U)$.

For each nonnegative integer n let

$$P'_n = \Sigma_{s \in S, |s| = n} a_s d z^s.$$

If E_m is the span of $\{e_1, \ldots, e_m\}$ then by construction

$$P'_n \mid \underbrace{E_m x \ldots E_m}_{n} = P_n \mid \underbrace{E_m x \ldots x E_m}$$

and since P'_n and P_n both belong to $\mathcal{L}_{HY}^{A}({}^nE)$ it follows that $P'_n(z) = P_n(z)$ for all $z \in {}^nE$. Now given any compact subset K of U and any $\varepsilon > 0$ we can choose J finite in S such that

$$\sum_{s \in S \setminus J} \| a_s d z^s \|_K \leqslant \varepsilon.$$

Hence if $J' \subset S$ is finite and $J' \supset J$ then

$$\left\| f - \sum_{s \in J'} a_s\, d z^s \right\|_K = \Sigma_{n=0}^{\infty} \left\| P_n - \sum_{\substack{s \in J' \\ |s| = n}} a_s\, d z^s \right\|_K \leqslant$$

$$\leqslant \Sigma_{n=0}^{\infty} \sum_{\substack{s \in S \setminus J' \\ |s| = n}} \| a_s\, d z^s \|_K \leqslant \varepsilon .$$

Hence the monoforms form an unconditional Schauder basis for $H^A_{HY}(U)$. Since

$$\| f \|_K \leqslant \sum_{s \in S} \| a_s\, d z^s \|_K \leqslant c(\delta) \| f \|_{\delta^2 K}$$

we have shown that the monoforms form an absolute basis. Now let

$$f = (P_n)_n \in H_A(U)$$

and suppose p is a τ_ω continuous semi-norm on $H^A(U)$ ported by the compact subset K of U. We first choose V a neighbourhood of K such that

$$\Sigma_n \| P_n \|_V < \infty .$$

Next we can choose $\delta = (\delta_n)_n$ such that

$$\delta_n > 1, \quad \Sigma \frac{1}{\delta_n} < \infty$$

and W a neighbourhood of zero such that

$$\delta^2 (K + W) \subset V .$$

By lemma 6

$$\sum_{s \in S} \| a_s\, d z^s \|_{K+W} \leqslant c(\delta) \Sigma_n \| P_n \|_{\delta^2(K+W)} \leqslant c(\delta) \Sigma_n \| P_n \|_V < \infty$$

Let $c(W) > 0$ be chosen so that

$$p(f) \leqslant c(W) \cdot \Sigma_n \| P_n \|_{K+W}$$

for all $f \in H^A(U)$. Given $\varepsilon > 0$ we can choose J a finite subset of S such that

$$\sum_{s \in S \setminus J} \| a_s\, d z^s \|_{K+W} \leqslant \epsilon / c(W)$$

If $J' \subset S$ is finite and $J' \supset J$ then

$$p\left(f - \sum_{s \in J'} a_s\, d\, z^s\right) \leqslant \Sigma_n\, c\,(W)\, \| P_n - \sum_{\substack{s \in J' \\ |s| = n}} a_s\, d\, z^s \|_{K+W} \leqslant$$

$$\leqslant C\,(W) \sum_{s \in S \setminus J} \| a_s\, d\, z^s \|_{K+W} \leqslant \epsilon$$

and the monoforms form an unconditional basis for $(H^A(U), \tau_\omega)$. Since

$$p\,(f) \leqslant \sum_{s \in S} p\,(a_s\, d\, z^s) \leqslant c\,(W) \cdot \sum_{s \in S} \|a_s\, d\, z_s\|_{K+W} \leqslant c\,(W) \cdot c\,(\delta) \cdot \Sigma_n \| P_n \|_V$$

it follows that the semi-norm

$$p'\,(f) = \sum_{s \in S} p\,(a_s\, d\, z^s)$$

is τ_ω continuous and hence the monoforms form an absolute basis for $(H^A(U), \tau_\omega)$. This completes the proof.

The above also shows directly that $H^A_{HY}(U)$ is nuclear but we prefer to deduce this result in § 3.

We let T denote the natural mapping from $H^A_{HY}(U)$ into $H_{HY}(U)$ and $H^A(U)$ into $H(U)$ defined by

$$T\left(\sum_{s \in S} a_m\, d\, z^s\right) = \sum_{s \in S} a_s\, z^s.$$

Proposition 8.—*T is a linear isomorphism from $(H^A_{HY}(U), \tau_0)$ onto a closed complemented subspace of $(H_{HY}(U), \tau_0)$ and is an isomorphism from $(H^A(U), \tau_\omega)$ onto a closed complemented subspace of $(H(U), \tau_\omega)$. The set $\{z^s\}_{s \in S}$ forms an absolute basis for the image space with the induced topologies in both cases.*

Proof.—Since

$$\| a_s\, z^s \|_A \leqslant \| a_s\, d\, z^s \|_A$$

for any polydisc A it follows that

$$T\,(H^A_{HY}(U)) \subset H\,(U) \quad \text{and} \quad T\,(H^A(U)) \subset H\,(U).$$

T is obviously injective. Since

$$\sum_{s \in S} \| a_s z^s \|_A \leqslant \sum_{s \in S} \| a_s d z^s \|_A$$

it follows that T is continuous.

Now if

$$f = \sum_{s \in S} a_s z^s \in H_{HY}(U)$$

and K is a compact subset of U then we can choose

$$(\delta\ )_n, \delta_n > 1 \quad \text{and} \quad \Sigma_n \frac{1}{\delta_n} < \infty$$

euch that δ K is a compact subset of U.
By lemma 6

$$\sum_{s \in S} \| a_s d z^s \| \leqslant \sum_{s \in S} \frac{\gamma | s |}{\delta^s} \| a_s z^s \|_{\delta K} \leqslant C(\delta) \sum_{s \in S} \| a_s z^s \|_{\delta K} .$$

Hence

$$T(H^A_{HY}(U)) = \{ f = \sum_{m \in N^{(N)}} a_m z^m \in H_{HY}(U); a_m = 0 \quad \text{if} \quad m \notin S \}$$

and the inverse mapping is continuous. Since the remaining properties are immediate, this completes the proof for $(H^A_{HY}(U), \tau_0)$.
Now suppose

$$f = \sum_{s \in S} a_s z^s \in H(U).$$

If K is a compact subset of U we can choose V open such that $K \subset V$ and

$$\sum_{s \in S} \| a_s z^s \|_V < \infty .$$

Now we can choose

$$\delta = (\delta_n)_n, \quad \delta_n > 1 \quad \text{and} \quad \Sigma_n \frac{1}{\delta_n} < \infty$$

and W a neighbourhood of zero such that $\delta(K + W) \subset V$. Then

$$\sum_{s \in S} \| a_s dz_s \|_{K+W} = \sum_{s \in S} \frac{\gamma^{|s|}}{\gamma^s} \| a_s z^s \|_{\delta(K+W)} \leq C(\delta) \sum_{s \in S} \| a_s z^s \|_V < \infty .$$

Hence

$$f' = \sum_{s \in S} a_s d z^s \in H_A(U)$$

and $T'(f) = f$. We have thus shown that

$$T(H^A(U)) = \{ f = \sum_{m \in N^{(N)}} a_m z^m \in H(U); \quad a_m = 0 \quad \text{if} \quad m \notin S \}$$

and $T(H^A(U))$ is a closed complemented subspace of $H(U)$. If p is a τ_ω continuous semi-norm on $H^A(U)$ and we let

$$p'\left(\sum_{m \in N^{(N)}} a_m z^m\right) = \sum_{s \in S} p(a_s d z^s)$$

then p' is a τ_ω continuous semi-norm on $H(U)$ and thus the inverse of T on $H^A(U)$ is continuous. Conversely if p is τ_ω continuous on $H(U)$ and we let

$$p'\left(\sum_{s \in S} a_s d z^s\right) = \sum_{s \in S} p(a_s z^s)$$

then p' is a τ_ω continuous semi-norm on $H^A(U)$ and T is continuous. This completes the proof.

§ 3. Proposition 8 shows that $H^A_{HY}(U)$ and $H^A(U)$ are subspaces and quotients of $H_{HY}(U)$ and $H(U)$ respectively and possess an absolute basis which can be extended to an absolute basis of the larger spaces and that there is an extremely useful correspondence between defining sets of weights on $H^A_{HY}(U)$ and $H^A(U)$ and their image spaces. These facts enable us, in a very simple manner, to transfer properties of $H_{HY}(U)$ and $H(U)$ to $H^A_{HY}(U)$ and $H^A(U)$. The relevant properties of $H_{HY}(U)$ and $H(U)$ are to be found in [1] and [2]. We just give a few examples of how this can be achieved and then give without proof the main properties which can be deduced quite easily. This list of properties is by no means exhaustive and can easily be augmented by further results from [1] and [2]. We first give a few definitions.

If K is a compact subset of E *we let* $H^A(K)$ *denote the space of anticommutative analytic germs on* K, i. e. $f \in H^A(K)$ if and only if

$$f = (P_n)_{n=0}^{\infty}, \ P_n \in \mathcal{L}^A({}^nE) \quad \text{and} \quad \sum_n \| P_n \|_V <$$

for some neighbourhood V of K.

We let

$$H^{A,\infty}(V) = \{ f \in H_A(V), \|f\|_V < \infty \}$$

with the norm $\| \ \|_V$. $H^A(K)$ is given the inductive limit topology

$$H^A(K) = \varinjlim_{V \supset K} (H^{A,\infty}(V), \|f\|_V) = \varinjlim_{K \subset V} (H^A(V), \tau_\omega).$$

$H^A_{HY}(K)$ *is the space of anticommutative hypoanalytic germs on* K, i. e.

$$f = (P_n)_{n=1}^{\infty} \in H^A_{HY}(K)$$

if and only if

$$P_n \in \mathcal{L}^A_{HY}({}^nE) \quad \text{and} \quad f \in H^A_{HY}(V)$$

for some neighbourhood V of K.

$H^A_{HY}(K)$ is given the inductive limit topology

$$\varinjlim_{V \supset K} (H^A_{HY}(V), \tau_0).$$

The following proposition shows how the properties of $H_{HY}(U)$ and $H(U)$ are inherited by $H^A_{HY}(U)$ and $H^A(U)$.

PROPOSITION 9.—*If U is an open polydisc in a fully nuclear space with a basis, then the following are true:*

(a) *if* $H_{HY}(U) = H(U)$ *then* $H^A_{HY}(U) = H^A(U)$;

(b) $(H^A_{HY}(U), \tau_0)$ *is a complete nuclear space;*

(c) *if* $(H_{HY}(U), \tau_0) = (H(U), \tau_\omega)$ *then*

$$(H^A_{HY}(U), \tau_0) = (H^A(U), \tau_\omega)$$

(d) *if E is a B-nuclear space then* $(H^A(E), \tau_0)$ *is a complete reflexive A-nuclear space.*

PROOF.

(a) If $H_{HY}(U) = U(U)$ then

$$T(H^A_{HY}(U)) = T(H^A(U)) \quad \text{and hence} \quad H^A_{HY}(U) = H^A(U)$$

(b) $H^A_{HY}(U)$ is isomorphic to a closed subspace of a complete nuclear space and hence is a complete nuclear space.

(c) By (a) $H^A_{HY}(U) = H^A(U)$. Since $\tau_\omega \geqslant \tau_0$ on $H^A(U)$ it suffices to show that every τ_ω continuous semi-norm on $H^A(U)$ is τ_0 continuous. A τ_ω continuous semi-norm on $H^A(U)$ induces a τ_ω continuous semi-norm on $H^A(U)$ induces a τ_ω continuous semi-norm on $T(H^A(U))$ which can be extended to a τ_ω continuous semi-norm on $H(U)$. This extension is τ_0 continuous and the restriction to $T(H^A(U))$ is also τ_0 continuous. Hence the semi-norm on $H^A(U)$ is also τ_0 continuous.

(d) If E is B-nuclear then since E is Fréchet nuclear with a basis (b) implies that $(H^A(E), \tau_0)$ is a complete nuclear space. Since $(H(E), \tau_0)$ is reflexive and hence infrabarrelled and the quotient of an infrabarrelled space is infrabarrelled ([6], p. 219) it follows that $(H^A(E), \tau_0)$ is infrabarrelled and hence it is reflexive. Since the basis for $H^A(E)$ can be extended to a basis for the A-nuclear space $(H(E), \tau_0)$ it follows that $(H^A(E), \tau_0)$ is also an A-nuclear space.

Similarly we obtain the following:

PROPOSITION 10.—*Let U denote an open polydisc in a fully nuclear space with a basis, then:*

(a) $(H^A_{HY}(U), \tau_0)'_\beta = H^A(U^M)$;

(b) $(H^A(U), \tau_\omega)' = H^A_{HY}(U^M)$ *(algebraically);*

(c) *if E is A-nuclear, then* $(H^A(U), \tau_\omega)$ *is nuclear;*

(d) *if the* τ_ω *bounded subsets of* $(H^A(U), \tau_\omega)$ *are locally bounded then*

$$(H^A(U), \tau_\omega)'_\beta = H^A_{HY}(U^M)$$

(algebraically and topologically);

(1) *if E is A-nuclear then*

$$(H^A(E), \tau_\omega) \quad and \quad (H^A_{HY}(E), \tau_0)$$

are A-nuclear spaces;

(f) *if E and F are B-nuclear spaces then*

$$(H^A(E \times F), \tau_0)'_\beta = (H^A(E), \tau_0)' \hat{\otimes} (H^A(F), \tau_0)'_\beta = H^A(0_{E'_\beta}) \hat{\otimes} H^A(0_{F'_\beta}).$$

The above results relate to the topological vector space structure of the exterior algebra of E or to the space of anticommutative analyic forms with constant coefficients. We may also repeat the same analysis for anti-commutative analytic forms with variable coefficients. We restrict ourselves to entire functions on k-spaces but more general results can easily be proved. Unless otherwise stated each function space or space of forms is assumed to carry the compact open topology.

PROPOSITION 11.—*Let E denote a fully nuclear space with a basis such that E and E × E are* k-*space. Then* $H(E; H^A(E))$ *is a complete nuclear space with an absolute basis and*

$$H(E; H^A(E)) = H(E) \hat{\otimes} H^A(E) = H^A(E; H(E)).$$

An absolute basis is given by

$$(z^m \times dz^s)_{m \in N^{(N)},\, s \in S}$$

and

$$H(E; H^A(E)) = \left\{ \sum_{m \in N^{(N)},\ s \in S} a_{m,s} z^m \times dz^s \quad \Sigma_{m,s} \mid a_{m,s} \mid \ \| z^m \|_{K_1} \| dz^s \|_{K_2} < \infty \right.$$

for any compact subsets K_1 and K_2 of E} (*)

$$= \left\{ \sum_{s \in S} a_s(z)\, dz^s \mid a_s(z) \in H(E) \right.$$

all s and

$$\sum_{s \in S} p(a_s) \| dz^s \|_k < \infty$$

for any τ_0 continuous semi-norm p on H (E) and any compact subset K of E} (**)

$$= \left\{ \sum_{m \in N^{(N)}} a_m z^m \mid a_m \in H(E) \right.$$

all m and

$$\sum_{m \in N^{(N)}} p(a) \| z^m \|_k < \infty \quad (***)$$

for any τ_0 continuous semi-norm p on H^A (E) and any compact subset K of E}.

Moreover if E is a B-nuclear space, then H (E ; H^A (E)) is a complete reflexive nuclear and dual nuclear space and

$$(H(E; H^A(E))' = H(E)'_\beta \mathbin{\widehat{\otimes}} H^A(E)'_\beta = H(0_{E'_\beta}) \mathbin{\widehat{\otimes}} H^A(0_{E'_\beta}).$$

Proof.—By [7] corollary 2.8

$$H(E; H^A(E)) \cong H(E) \mathbin{\widehat{\otimes}} H^A(E).$$

Since H^A (E) is a closed complemented subspace of H (E) it follows that H (E ; H^A (E)) is a closed complemented subspace of

$$H(E; H(E)) \cong H(E \times E),$$

by corollary 2.8 of [7].

Hence H (E ; H^A (E)) is a complete nuclear space with

$$(z^m \times d z^s)_{m \in N^{(N)},\, s \in S}$$

as an absolute basis and (*) follows immediately.

If

$$f = \sum_{m \in N^{(N)}} a_{m,s} z^m \times d z^s \in H(E; H^A(E))$$

then

$$f = \sum_{m \in N^{(N)}} \left\{ \sum_{s \in S} a_{m,} \; d z^s \right\} z^m = \sum_{s \in S} \left\{ \sum_{m \in N^{(N)}} a_{m,s} z^m \right\} d z^s.$$

$$a_m = \sum_{s \in S} a_{m,} \; d z^s \in H^A(E) \quad \text{and} \quad a_s(z) = \sum_{m \in N^{(N)}} a_{m,} \; z^m \in H(E).$$

and the remaining conditions in (**) and (***) are easily seen to be satisfied since

$$p_k\left(\sum_{s\in S} a_s\, d\, z^s\right) = \sum_{s\in S} \| a_s\, d\, z^s \|_k$$

and

$$\tilde{p}_k\left(\sum_{m\in N(N)} a_m z^m\right) = \sum_{m\in N(N)} \| a_m z^m \|_k$$

form, as K ranges over the compact subsets of E, a fundamental system of semi-norms for H^A (E) and H (E) respectively.

Finally,

$$H^A(F; H(E)) = \{ f = (P_n)_{n=0}^{\infty};$$

P_n is a continuous n linear alternating form with values in H (E) and

$$\Sigma_{n=0}^{\infty} \sup_{w_i \in K} p(P_n(w_1, \ldots, w_n)) < \infty$$

for each compact subset K of E and each τ_0 continuous semi-norm p on H (E)}.

The topology on H^A (E ; H (E)) is also given by the above semi-norms.

Let

$$f = (P_n)_{n=0}^{\infty} \in H^A(E; H(E))$$

then

$$P_n(w) = \sum_{m\in N(N)} a_{m,n}(w)\, z^m$$

for all $w \in E^{(n)}$ and the mapping

$$w \in E^{(n)} \longrightarrow a_{m,n}(w)$$

is a continuous alternating n-linear form. Hence

$$a_{m,n}(w) = \sum_{s\in S,\ |s|=n} a_{m,s}\, d\, w^s.$$

We may suppose

$$P\left(\sum_{m \in N^{(N)}} a_m z^m\right) = \sum_{m \in N^{(N)}} | a_m | \cdot \| z^m \|_L$$

for all

$$\sum_{m \in N^{(N)}} a_m z^m \in H(E)$$

and L is a compact subset of E. We then have

$$(f) = \Sigma_{n=0}^{\infty} \sup_{w \in K^{(n)}} p(P_n(w)) = \Sigma_{n=0}^{\infty} \sup_{w \in K^{(n)}} p\left(\sum_{m \in N^{(N)}} a_{m,n}(w) z^m\right) \leq$$

$$\leq \Sigma_{n=0}^{\infty} \sup_{w \in K^{(n)}} \sum_{m \in N^{(N)}} | a_{m,n}(w) | \cdot \| z^m \|_L \leq$$

$$\leq \Sigma_{n=0}^{\infty} \sum_{m \in N^{(N)}} \sum_{s \in S, |s| = n} | a_{m,s} | \cdot \| dw^s \|_K \cdot \| z^m \|_L \leq$$

$$\leqslant \sum_{m \in N^{(N)},\, s \in S} | a_{m,s} | \cdot \| d w^s \|_K \cdot \| z^m \|_L$$

Now choose

$$\delta = (\delta_n)_n, \quad \delta_n > 1 \quad \text{and} \quad \Sigma_n \frac{1}{\delta_n} < \infty$$

such that δ K and δ L are compact subsets of E. By lemma 5, we have for any integer n and $m \in N^{(N)}$

$$\sum_{s \in S, |s| = n} | a_{m,s} | \cdot \| d w^s \|_K \cdot \| z^m \|_L \leqslant$$

$$\leqslant C(\delta) \sup_{w \in (\delta K)^{(n)}} | a_{m,n}(w) | \cdot \| z^m \|_{\delta L} \cdot \frac{1}{\delta^m} \leqslant$$

$$\leqslant C(\delta) \frac{1}{\delta^m} \cdot \sup_{m \in N^{(N)}} \cdot \sup_{w \in (\delta K)^{(n)}} | a_{m,n}(w) | \cdot \| z^m \|_{\delta L} \leqslant$$

$$\leqslant C(\delta) \frac{1}{\delta^m} \cdot \sup_{w \in (\delta K)^{(n)}} \cdot \Sigma_{m \in N^{(N)}} | a_{m,n}(w) | \cdot \| z^m \|_{\delta L}$$

Hence

$$\sum_{m \in N^{(N)}, |s| = n} | a_{m,s} | \cdot \| d w^s \|_K \cdot \| z^m \|_L \leqslant$$

$$\leqslant C(\delta)\left(\sum_{m \in N^{(n)}} \frac{1}{\delta^m}\right) \cdot \sup_{w \in (\delta K)^{(n)}} \left(\sum_{m \in N^{(N)}} | a_{m,n}(w) | \cdot \| z^m \|_{\delta L}\right)$$

and this shows that

$$H^A(E; H(E)) = H^A(E) \hat{\otimes} H(E)$$

algebraically and topologically.

If E is B nuclear then $E \times E$ is also B-nuclear and hence $H(E; H^A(E))$ is a closed complemented subspace of a complete reflexive nuclear and dual nuclear space and hence it is also a complete reflexive nuclear space and

$$(H(E; H^A(E))'_\beta = (H(E) \hat{\otimes} H^A(E))'_\beta = H(E)'_\beta \hat{\otimes} H^A(E)'_\beta$$

bu a result of Grothendieck. This completes the proof.

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References

[1] Boland, P. J. and Dineen, S. «Holomorphic Functions on Fully Nuclear Spaces» (to appear in *Bull. Soc. Math. Fr.*).

[2] Dineen, S. «Analytic Functionals on Fully Nuclear Spaces» (preprint).

[3] Grothendieck, A. (1955). «Produits tensoriels topologiques et espaces nuclearies. Memoirs of the A. M. S., 16.

[4] Horvath, J. (1966). «Topological Vector Spaces and Distributions», vol. 1, Addison Wesley.

[5] Pietsch, A. (1972). «Nuclear Locally Cnovex Spaces». Springer Verlag, Ergenbnisse der Math., 66 (2nd edition).

[6] Schottenloher, M. «ε-Product and continuation of analytic mapping. Analyse fonctionelle et applications». C. R. Coll. D'Analyse, Rio de Janeiro, 1972; Hermann, 1975.

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