SOME ABELIAN THEOREMS FOR THE DISTRIBUTIONAL $_1F_1$-TRANSFORM

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Several generalisations of the classical Laplace transform have been given by various mathematicians from time to time. In 1950 Erdélyi gave a generalisation and Joshi studied various properties of this generalisation including an Abelian theorem. The object of this present work is to discuss Abelian theorems of the initial-value and final-value type for the above generalisation in the classical sense and extend the results to a certain class of distributions.

1. Introduction

Erdelyi [2] gave a generalisation of the classical Laplace transform in the following way:

$$F(x) = L\, f(y) = \frac{\Gamma(a)}{\Gamma(b)} \int_0^\infty (x\,y)^\beta\, {}_1F_1(a; b; -x\,y) f(y)\, d\,y \tag{1}$$

where

$$a = \beta + \eta + 1; \quad b = \alpha + \beta + \eta + 1$$

α, β and η being complex. (1) reduces to the well-known classical Laplace transform

$$F(x) = \int_0^\infty e^{-x\,y} f(y)\, d\,y \tag{2}$$

for $\alpha = \beta = 0$. We define the generalised Laplace transform (1) as ${}_1F_1$-transform of $f(y)$. An initial-value theorem relates the asymptotic behaviour of $f(y)$ as $y \longrightarrow 0^+$ to the asymptotic behaviour of $F(x)$ as $x \longrightarrow \infty$ and the final-value theorem relates the asymptotic behaviour of $f(y)$ as $y \longrightarrow \infty$ to the asymptotic behaviour of $F(x)$ as $x \longrightarrow 0^+$. In this paper it is proposed to prove theorems of this nature and extend them to distributions.

Widder [9] and Doetsch [1] had given theorems of the above type for (2). Zemanian [10] recently extended some Abelian theorems relating to (2) in the bilateral case to distributions. Joshi [3] gave initial-value and final-value theorems for the generalised Laplace-Stieltjes transform

$$F(x) = \frac{\Gamma(a)}{\Gamma(b)} \int_0^\infty (x\,y)^\beta \; {}_1F_1\,(a;\, b;\, -x\,y)\; d\,\Psi\,(y)$$

where a and b have the same meaning as before. Misra [5] had recently proved some Abelian theorems for distributional Stieltjes transformation. In another paper [6a] the author has recently given Abelian theorems for a generalized Stieltjes transform in the distributional sense.

2. Preliminaries

I denote the open interval $(0, \infty)$; x and y are real variables restricted to I. $\mathcal{D}(\mathrm{I})$ denotes the space of smooth functions defined over I whose supports are compact subsets of I. We assign to $\mathcal{D}(\mathrm{I})$ the topology that makes its dual $\mathcal{D}'(\mathrm{I})$ the space of Schwartz distributions on I.

(a) Testing function space $E(\mathrm{I})$: $E(\mathrm{I})$ is the space of all complex-valued smooth functions on I. We have $\mathcal{D}(\mathrm{I}) \subset E(\mathrm{I})$. $E'(\mathrm{I})$ is the dual of $E(\mathrm{I})$. $E'(\mathrm{I})$ is a subspace of $\mathcal{D}'(\mathrm{I})$ [12, p. 36, 37].

(b) A boundedness property of distributions in $E'(\mathrm{I})$:

For every $f \in E'(\mathrm{I})$, there exists a nonnegative integer r such that if $\varphi \in E(\mathrm{I})$, then

$$|\langle f, \varphi \rangle| \leqq C \max_{0 \leqq k \leqq r} \sup_{t \in J} |\varphi^{(k)}(t)| \tag{3}$$

where J is any bounded open subset of I containing the support of f. This follows in view of the result [10, p. 84] and the fact that

$$< f, \varphi > = < f, \lambda \varphi >$$

where $\lambda(t) \in \mathcal{D}(I)$ and is equal to 1 over J and zero outside a larger interval.

(c) Value of a distribution at a point due to Lojasiewicz [4]:

Let f be a distribution defined in a neighbourhood of a point; then f is said to have a value N at x_0 namely $f(x_0) = N$ if the distributional

$$\operatorname*{limit}_{\lambda \to 0+} f(x_0 + \lambda x)$$

exists in a neighbourhood of zero and if it is a constant function N. We will also have a need to use a convention stated in [11, p. 1255].

Throughout this paper it will be assumed that $\operatorname{Re}\beta \geqslant 0$, $\operatorname{Re}\eta > 0$ and that Re b is not zero or a negative integer unless otherwise stated. Also

$$P = \frac{\Gamma(a)}{\Gamma(b)}$$

$$(a)_n = a(a+1)\dots(a+n-1)$$

$$K(x\,y) = P\,(x\,y)^{\beta}\ {}_1F_1(a; b; -x\,y).$$

3. The Classical initial-value theorem for the ${}_1F_1$-transform

THEOREM 1.—*Let the complex-valued function* f (y) *satisfy the following two conditions:*

$$\text{(i)}\quad f(y) \to 0 \quad as \quad y \to \infty,$$

$$\text{(ii)}\quad \frac{f(y}{y^{\gamma + \alpha - \beta}}$$

(γ *is any real number greater than or equal to zero*) *is absolutely continuous on* $0 \leqslant y < \infty$.

Let there exist a complex number μ *such that*

$$\lim_{y \to 0+} \frac{f(y)}{y^{\gamma + \alpha - \beta}} = \mu. \tag{4}$$

Then

$$\lim_{x\to\infty} CF(x)\, x^{\gamma+\alpha-\beta+1} = \mu. \tag{5}$$

where

$$C = \left| \frac{\Gamma(\beta+\eta-\gamma)}{\Gamma(\gamma+\alpha+1)\,\Gamma(\beta+\eta-\gamma-\alpha)} \right|$$

and

$$F(x) = L\,f(y) = \int_0^\infty K(xy)\, f(y)\, dy \tag{6}$$

with the conditions

$$\mathrm{Re}\,(\beta+\eta) > \gamma + \mathrm{Re}\,\alpha > -1 \quad \textit{and} \quad \mathrm{Re}\,(\beta+\eta) - \gamma > 0.$$

PROOF.—For convenience let us put

$$\gamma + \alpha - \beta = E.$$

From the result [8, p. 48],

$$\int_0^\infty x^{l-1}\, {}_1F_1\,(a;\, b;\, -x)\, dx = \frac{\Gamma(l)\,\Gamma(a-l)\,\Gamma(b)}{\Gamma(b-l)\,\Gamma(a)} \qquad (0 < \mathrm{Re}\, l < \mathrm{Re}\, a)$$

We have

$$x^{E+1}\, F(x) - \frac{\mu\,\Gamma(\gamma+\alpha+1)\,\Gamma(\beta+\eta-\gamma-\alpha)}{\Gamma(\beta+\eta-\gamma)} =$$

$$= x^{E+1} \left\{ \int_0^\infty K(xy)\, f(y)\, dy - \mu \int_0^\infty y^E\, K(xy)\, dy \right\} =$$

$$= x^{E+1} \left\{ \int_0^\infty [f(y) - \mu\, y^E]\, K(xy)\, dy \right\}.$$

Hence

$$\left| x^{E+1}\, F(x) - \frac{\mu\,\Gamma(\gamma+\alpha+1)\,\Gamma(\beta+\eta-\gamma-\alpha)}{\Gamma(\beta+\eta-\gamma)} \right| \leq$$

$$\leq x^{E+1} \left\{ \int_0^t |f(y) - \mu\, y^E|\, K(xy)\, dy + \right.$$

$$\left. + \int_t^\infty |f(y) - \mu\, y^E|\, K(xy)\, dy \right\} = I_1 + I_2, \quad t > 0).$$

We have

$$|I_2| \leq \sup_{t \leq y < \infty} \left| x^{E+1} \int_t^\infty \left| \frac{f(y)}{y^E} - \mu \right| y^E K(xy)\, dy \right| \leq$$

$$\leq \sup_{t \leq y < \infty} \left| \frac{f(y)}{y^E} - \mu \right| \left| x^{E+1} \int_t^\infty y^E K(xy)\, dy \right|.$$

But

$$|x^{E+1} y^E P (xy)^\beta {}_1F_1 (a; b; -xy)| \leq C' (xy)^{E-\eta-1} x$$

by the asymptotic estimate of ${}_1F_1$ function [8, p. 60] where C' is an appropriate constant ($C' > 0$). Hence

$$|I_2| \leq \sup_{t \leq y < \infty} \left| \frac{f(y)}{y^E} - \mu \right| C' \, |x^{E-\eta}| \int_t^\infty y^{E-\eta-1}\, dy \leq$$

$$\leq \sup_{t \leq y < \infty} \left| \frac{f(y)}{y^E} - \mu \right| C' \left| \frac{x^{E-\eta}}{\eta - E} \right| (t)^{E-\eta} \to 0 \quad \text{as}$$

$$x \to \infty \quad \text{since} \quad \mathrm{Re}\,(E - \eta) < 0.$$

Since $\dfrac{f(y)}{y^E}$ is absolutely continuous on $0 \leqslant y < \infty$,

$$\left| \frac{f(y)}{y^E} - \mu \right|$$

is bounded for $t \leqslant y$. By (4), for any arbitrary $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\left| \frac{f(y)}{y^E} - \mu \right| < \varepsilon \quad \text{on} \quad 0 \leq y < \delta.$$

Having fixed δ such that $y < \delta$, we have

$$|I_1| \leq \varepsilon \left| x^{E+1} \int_0^t K(xy)\, y^E\, dy \right| =$$

$$= \varepsilon \left| P \int_0^\infty (xy)^{\gamma+\alpha} {}_1F_1 (a;\, b;\, -xy)\, x\, dy \right| =$$

$$= \varepsilon \left| \frac{\Gamma(\gamma+\alpha+1)\,\Gamma(\beta+\eta-\gamma-\alpha)}{\Gamma(\beta+\eta-\gamma)} \right| = \frac{\varepsilon}{C} \text{ by } [8, p. 48].$$

Thus

$$\overline{\lim_{x \to \infty}} \; | I_1 + I_2 | \leq \frac{\varepsilon}{C}$$

which proves (5).

To extend this theorem to a class of distributions, we shall use the ideas of the value of a distribution at a point due to Lojasiewicz (section 2 (c)) and Shiraishi Risai [7].

4. Initial-value theorem for the distributional ${}_1F_1$-transform

THEOREM 2.—*Let*

$$f(y) \in E'(I) \quad \textit{and} \quad \frac{f(y)}{y^E} \to \mu \quad \textit{as} \quad y \to 0+$$

in the sense of Lojasiewicz.

Then as $x \longrightarrow \infty$,

$$| < f(y) - \mu\, y^{\gamma + \alpha - \beta},\; K(xy) > | \leq \varepsilon. \tag{7}$$

PROOF.—We use a theorem [10, p. 92] that a finite-order derivative of a continuous function is a distribution. From Shiraishi Risai [7] we have

$$f(t) - \mu\, t^{\gamma + \alpha - \beta} = D^k F$$

where F is a continous function in a neighbourhood of zero and D^k F is a finite-order derivative of a continuous function. Hence the left side of (7) is equal to

$$| < f(y),\; K(xy) > |$$

which proves that f (y) is a distribution. By (b) of section 2, we have

$$| < f(y),\; K(xy) > | \leq \max_{0 \leq k \leq r} \; \sup_{t \in J} \; | D_x^k \; K(xy) | =$$

$$= \max_{0 \leq k \leq r} \; \sup_{t \in J} \left| P\, y^{\beta} \sum_{n=0}^{k} \binom{k}{n} \times [\,{}_1F_1\,(a;\, b;\, -xy)\,]_{(n)}\; (x)^{\beta}_{(k-n)} \right|$$

A typical term in the summation on the right side is

$$y^{\beta + n}\, Q\; {}_1F_1\,(a + n;\, b + n;\, -xy)\; x^{\beta - k + n} \tag{8}$$

where

$$Q = P\binom{k}{n}(-1)^n \frac{(a)_n}{(b)_n}\beta(\beta-1)\dots(\beta-k+n+1).$$

The expression in (8) is asymptotic to

$$Q\, y^{-\eta-1}\, x^{-k-\eta-1}$$

by using the asymptotic property of ${}_1F_1$ function [8, p. 60] so that this term $\longrightarrow 0$ as $x \longrightarrow \infty$ since Re $\eta + k + 1 > 0$ and Re $\eta > 0$. Hence

$$|\langle f(y),\, k(xy)\rangle| \leq \varepsilon$$

which proves (7).

5. A classical final-value theorem for the ${}_1F_1$-transform

THEOREM 3.—*If* f (y) *be a measurable function on* $0 < y < \infty$ *and if there exist a complex number* μ *and a real number* ν *greater than or equal to zero, such that*

$$\lim_{y \to \infty} \frac{f(y)}{y^{\gamma+\alpha-\beta}} = \mu \tag{9}$$

then, with the conditions

$$Re\,(\beta+\eta) > \gamma + Re\,\alpha > -1$$

and

$$Re\,(\beta+\eta) - \gamma > 0,$$

$$\lim_{x \to 0+} C\,F\,(x)\, x^{\gamma+\alpha-\beta+1} = \mu \tag{10}$$

where F (x) *is given by (6) and C as in Theorem 1.*

PROOF.—Let us put again $\nu + \alpha - \beta = E$.

From (9) it is clear that $f(y)$ is a function of slow-growth. Proceeding as in Theorem 1 we have

$$\left| x^{E+1}\,F(x) - \frac{\mu\,\Gamma(\gamma+\alpha+1)\,\Gamma(\beta+\eta-\gamma-\alpha)}{\Gamma(\beta+\eta-\gamma)} \right| =$$

$$= x^{E+1} \left| \int_0^\infty [f(y) - \mu\, y^E]\, K(xy)\, dy \right| \leq \quad \dots$$

$$\ldots \leq x^{E+1} \int_0^t |f(y) - \mu y^E| K(xy)\, dy +$$

$$+ x^{E+1} \int_t^\infty |f(y) - \mu y^E| K(xy)\, dy = I_1 + I_2 .$$

From (9) we can choose t so large that, for any arbitrary positive ϵ

$$\left| \frac{f(y)}{y^E} - \mu \right| < \varepsilon \quad \text{for all } y > t.$$

Hence we have

$$|I_2| \leq \varepsilon \left| x^{E+1} \int_t^\infty y^E K(xy)\, dy \right| =$$

$$= \varepsilon \left| \int_{tx}^\infty z^{\gamma+\alpha} {}_1F_1(a; b; -z)\, dz \right| \quad (z = xy) <$$

$$< \varepsilon \left| \frac{\Gamma(\gamma+\alpha+1)\Gamma(\beta+\eta-\gamma-\alpha)}{\Gamma(\beta+\eta-\gamma)} \right| \text{ by } [8, p. 48] \cdot = \frac{\varepsilon}{C}$$

provided that $\operatorname{Re}(\beta + \eta) > \nu + \operatorname{Re}\alpha > -1$ and $\operatorname{Re}(\beta + \eta) - \nu > 0$. As for I_1 in the range $0 < y \leqslant t$, we have

$$\left| \frac{f(y)}{y^E} - \mu \right| \leq M \text{ (a constant)}.$$

Further

$$|I_1| \leq M \left| \int_0^t x^{E+1} y^E K(xy)\, dv \right| =$$

$$= M P \left| \int_0^t x^{E+1} y^E {}_1F_1(a; b; -xy)(xy)^\beta\, dy \right| .$$

As $x \longrightarrow 0^+$, for any $y \leqslant t$, we have

$${}_1F_1(a; b; -t) = 0\,(1) \; (t \to 0+) \; [8, p. 60].$$

Hence

$$|I_1| \leq M P \left| x^{\gamma+\alpha+1} \int_0^t y^{\gamma+\alpha}\, dy \right| = M P \left| \frac{x^{\gamma+\alpha+1}\, t^{\gamma+\alpha+1}}{\gamma+\alpha+1} \right|$$

since

$$\gamma + 1 + \mathrm{Re}\,\alpha > 0.$$

Consequently as

$$x \to 0+, \; \mathrm{I}_1 \to 0.$$

We have therefore

$$\overline{\lim_{x \to 0+}} \; | \mathrm{I}_1 + \mathrm{I}_2 | \leq \frac{\varepsilon}{\mathrm{C}}.$$

Therefore (10) is proved.

In the next section we shall extend this classical result to distributions by using again the boundedness property of distributions in E′ (I).

6. Final-value theorem for the distributional ${}_1F_1$-transform

THEOREM 4.—*Let*

$$\mathrm{f} = \mathrm{f}_1 + \mathrm{f}_2$$

where f_1 *is an ordinary function satisfying the hypothesis of Theorem 3 and* f_2 *is a distribution in E′ (I). Let F* (x) *be the distributional* ${}_1F_1$*-transform of* f *and C be as given Theorem 1. Let* $Re\,\beta > \mathrm{k}$ *where* k *is a positive integer and*

$$\gamma + Re\,(\alpha - \beta) + \mathrm{k} + 1 > 0.$$

Then

$$\lim_{x \to 0+} x^{\gamma + \alpha - \beta + 1} \, F(x) = \frac{1}{C} \lim_{y \to \infty} \frac{f(y)}{y^{\gamma + \alpha - \beta}}. \tag{11}$$

PROOF.—In [6] it has been proved that

$$\mathrm{F}_2(x) = \mathrm{L} f_2(y) = \langle f_2(y), \; \mathrm{K}(xy) \rangle$$

is a smooth function on $0 < x < \infty$. Let, as is (*b*) of section 2, $\lambda(t) \in \mathcal{D}$ (I) and be equal to 1 over J and zero outside a larger interval where J is any bounded open subset of I containing the support of f. By (3) of section 2,

$$|F_2(x)| \leq |< f_2(y), \lambda(y) K(xy) >| \leq$$
$$\leq \underset{0 \leq k \leq r}{\text{Max}} \ \underset{y \in J}{\sup} \ |D_y^k [\lambda(y) K(xy)]| =$$
$$= \underset{0 \leq k \leq r}{\text{Max}} \ \underset{y \in J}{\sup} \sum_{m=0}^{k} \binom{k}{m} |D_y^{k-m} \lambda(y)| \times |x^m D_z^m [z^\beta {}_1F_1(a; b; -z)]|$$

where $z = xy$.

We have by the asymptoticity of ${}_1F_1$ function

$$D_z^m [z^\beta {}_1F_1(a; b, -z)] = 0(1) \quad \text{as} \quad z \to 0+$$

provided $\text{Re}\,\beta > m$ for $m = 0, \ldots, k$.
and

$$D_z^m [z^\beta {}_1F_1(a; b; -z)] = 0(1) \quad \text{as} \quad z \to \infty$$

because of the following.

$$D_z^m [z^\beta {}_1F_1(a; b; -z)] = \sum_{l=0}^{m} L\, z^{\beta - m + l} {}_1F_1(a + l; b + l, -z)$$

where

$$L = \binom{m}{l} \beta(\beta - 1) \ldots (\beta - m + l + 1)(-1)^l \frac{(a)_l}{(b)_l}.$$

A typical term of this summation, by considering asymptotic order of ${}_1F_1$ function, is

$$L\, z^{\beta - m + l} z^{-l - \beta - \eta - 1} = L\, z^{-\eta - 1 - m} \to 0 \quad \text{as} \quad z \to \infty \quad \text{since} \quad \text{Re}\,\eta > 0.$$

Also

$$D_z^m [z^\beta {}_1F_1(a; b; -z)]$$

is continuous on $0 < z < \infty$. It now follows that it is also bounded there whenever $m = 0, 1, \ldots, k$.

Hence

$$F_2(x) < K x^k$$

for some sufficiently large constant K so that

$$\lim_{x \to 0+} x^{\gamma+\alpha-\beta+1} F_2(x) < \lim_{x \to 0+} K x^{\gamma+\alpha-\beta+1+k} = 0$$

since

$$\gamma + k + 1 + \mathrm{Re}\,(\alpha - \beta) > 0.$$

Also the support of $f_2(y) \in E'(I)$ is a compact subset of $0 < y < \infty$. Hence by the convention stated in [11, p. 1255] we have

$$\lim_{y \to \infty} \frac{f_2(y)}{y^{\gamma+\alpha-\beta}} = 0.$$

But since

$$F(x) = F_1(x) + F_2(x)$$

the result (11) follows from Theorem 3.

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