The Asymptotic Dimension of the First Grigorchuk Group Is Infinity

Justin Smith

University of Florida
Department of Mathematics
P.O. Box 118105, 358
Little Hall, Gainesville
FL 32611-8105 — USA
justins@math.ufl.edu

Received: June 22, 2006
Accepted: July 13, 2006

ABSTRACT

We describe a sufficient condition for a finitely generated group to have infinite asymptotic dimension. As an application, we conclude that the first Grigorchuk group has infinite asymptotic dimension.

Key words: asymptotic dimension.

2000 Mathematics Subject Classification: 20F69.

The first Grigorchuk group is described in [1, 2, 5]. This Grigorchuk group, which we will denote by Γ, has many interesting properties. It is a finitely generated 2-group with intermediate growth, whose word problem is solvable, and which does not admit a finite dimensional linear representation that is faithful. Also, Γ and $\Gamma \times \Gamma$ are *commensurable*, which means that Γ and $\Gamma \times \Gamma$ have subgroups of finite index which are isomorphic. A detailed exposition can be found in [5].

We prove that Γ has asymptotic dimension infinity, $\operatorname{asdim} \Gamma = \infty$. If one excludes Gromov's "random groups" [3], all previously known examples of groups G with $\operatorname{asdim} G = \infty$ are based on the fact that G has a free Abelian subgroup of arbitrary large rank. The Grigorchuk group is of different nature: since Γ is a 2-group, it does not have a (nontrivial) free Abelian subgroup.

Let (X, d_X) and (Y, d_Y) be metric spaces. We say that a map $f : X \to Y$ is proper if the preimage of each bounded set is bounded; it is bornologous if, for all $R > 0$, there is an $S > 0$ such that $d_Y(f(x), f(y)) < S$ whenever $d_X(x, y) < R$; a map is called coarse if it is both proper and bornologous. If S is a set, then $f, g : S \to X$ are said to be close if $\sup_{s \in S} d(f(s), g(s)) < \infty$. A coarse map $f : X \to Y$ is said to be a coarse equivalence if there is a coarse map $g : Y \to X$ such that $g \circ f$ is close to id_X and $f \circ g$ is close to id_Y. A coarse map $f : X \to Y$ is said to be a coarse embedding if $f : (X, d_X) \to (f(X), d_Y|_{f(X)})$ is a coarse equivalence. In particular, an isometric embedding is a coarse embedding. Also, if $f_i : (X_i, d_{X_i}) \to (Y_i, d_{Y_i})$ $(i = 1, 2)$ are coarse equivalences, then so is $f_1 \times f_2 : (X_1 \times X_2, \delta_1) \to (Y_1 \times Y_2, \delta_2)$, where δ_1 and δ_2 are the corresponding sum metrics.

Restricting attention to finitely generated groups, since any two word metrics (say d_1 and d_2) on a finitely generated group G are equivalent, the map $\mathrm{id}_G : (G, d_1) \to (G, d_2)$ is a coarse equivalence. When G and H are groups equipped with word metrics d_G and d_H, then the sum metric $d_G + d_H$ on $G \times H$ is also a word metric (for the obvious generating set). If $H \leq G$ is a subgroup of finite index of (the finitely generated group) G, then the inclusion map $H \to G$ is a coarse equivalence. Also, an isomorphism is a coarse equivalence. See [6] for further details.

Definition 1 ([4])**.** A metric space (X, d) is said to have $\operatorname{asdim} X \leq n$, if for each $R > 0$, there is an $S > 0$ and R-disjoint, S-bounded families $\mathcal{U}_0, \mathcal{U}_1, \dots, \mathcal{U}_n$ of subsets of X such that $\mathcal{U} := \cup_i \mathcal{U}_i$ is a cover of X.

We say that a family $\mathcal{V}$ of subsets of X is R-disjoint if $d(U, V) \geq R$ for all $U, V \in \mathcal{V}$ with $U \neq V$; $\mathcal{V}$ is said to be S-bounded if $\operatorname{diam} V \leq S$ for all $V \in \mathcal{V}$. One can show that coarsely equivalent spaces have the same asymptotic dimension. Thus, for finitely generated groups, the asymptotic dimension of the group does not depend on the choice of the word metric.

Definition 2. Two groups Γ_1 and Γ_2 are *commensurable* if there exist subgroups $H_1 \leq \Gamma_1$ and $H_2 \leq \Gamma_2$, each of finite index, such that H_1 and H_2 are isomorphic.

By the comments above, $\operatorname{asdim} \Gamma_1 = \operatorname{asdim} \Gamma_2$ if Γ_1 and Γ_2 are commensurable.

Theorem 3. *Let G be a finitely generated, infinite group which is commensurable with its square $G \times G$. Then $\operatorname{asdim} G = \infty$.*

Proof. We first show that G^n is coarsely equivalent to G for all $n \geq 1$. Proceeding inductively (the $n = 1$ case is immediate), we assume G^n is coarsely equivalent to G. But G^{n+1} is coarsely equivalent to $G^n \times G$, which in turn is coarsely equivalent to $G \times G$, and so by hypothesis G^{n+1} is equivalent to G. This proves that $\operatorname{asdim} G^n = \operatorname{asdim} G$ for all $n \geq 1$.

Also, by Exercise IV.A.12 of [5], there is an isometric embedding $f : \mathbf{Z} \to G$, where G is taken with a word metric. Thus, for each $n \geq 1$, we have an isometric embedding $f \times f \times \cdots \times f : \mathbf{Z}^n \to G^n$, where we take the sum metrics on $\mathbf{Z}^n$ and G^n. Since an isometric embedding is a coarse embedding, we have that $\operatorname{asdim} G^n \geq \operatorname{asdim} \mathbf{Z}^n = n$. Thus, $\operatorname{asdim} G \geq n$ for all n. □

Corollary 4. *Let Γ be the Grigorchuk group. Then $\operatorname{asdim} \Gamma = \infty$.*

Proof. Γ is finitely generated by definition. Proposition VIII.14 and Corollary VIII.15 from [5] show that Γ satisfies the hypotheses of the theorem. $\square$

It is interesting to note that $\operatorname{asdim}\Gamma = \infty$, yet Γ does not contain an isomorphic copy of $\mathbf{Z}^n$. However, $\mathbf{Z}^n$ does coarsely embed into Γ.

Finally, if one has a finitely generated group which is known to be commensurable with its square, then the asymptotic dimension is either 0 or infinity, depending on whether the group is finite or infinite.

References

[1] R. I. Grigorchuk, *On Burnside's problem on periodic groups*, Funktsional. Anal. i Prilozhen. **14** (1980), no. 1, 53–54 (Russian); English transl., Functional Anal. Appl. **14** (1980), 41–43.

[2] ———, *An example of a finitely presented amenable group that does not belong to the class EG*, Mat. Sb. **189** (1998), no. 1, 79–100 (Russian); English transl., Sb. Math. **189** (1998), no. 1-2, 75–95.

[3] M. Gromov, *Random walk in random groups*, Geom. Funct. Anal. **13** (2003), no. 1, 73–146.

[4] ———, *Asymptotic invariants of infinite groups*, Geometric Group Theory, vol. 2 (Sussex, 1991), London Math. Soc. Lecture Note Ser., vol. 182, Cambridge Univ. Press, Cambridge, 1993, pp. 1–295.

[5] P. de la Harpe, *Topics in geometric group theory*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 2000.

[6] J. Roe, *Lectures on coarse geometry*, University Lecture Series, vol. 31, American Mathematical Society, Providence, RI, 2003.