Structure of the Hardy Operator Related to Laguerre Polynomials and the Euler Differential Equation

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ABSTRACT

We present a direct proof of a known result that the Hardy operator $Hf(x) = \frac{1}{x}\int_0^x f(t)\,dt$ in the space $L^2 = L^2(0,\infty)$ can be written as $H = I - U$, where U is a shift operator ($Ue_n = e_{n+1}$, $n \in \mathbb{Z}$) for some orthonormal basis $\{e_n\}$. The basis $\{e_n\}$ is constructed by using classical Laguerre polynomials. We also explain connections with the Euler differential equation of the first order $y' - \frac{1}{x}y = g$ and point out some generalizations to the case with weighted $L^2_w(a,b)$ spaces.

Key words: Hardy inequality, Hardy operator, Laguerre polynomials, isometry, Lebesgue spaces, basis in L^2 space, weighted $L^2_w(a,b)$ spaces.

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Introduction

The Hardy averaging operator H, defined by $Hf(x) = \frac{1}{x}\int_0^x f(t)dt$, is important in analysis, differential equations and mathematical physics. Therefore a better understanding of the structure of the Hardy operator seems to be important. Moreover, the operator $I - H$ has remarkable mapping properties, i.e., we have the equality

$$\|(I-H)f\|_{L^2} = \|f\|_{L^2} \quad \text{for all } f \in L^2, \tag{1}$$

and this isometry in L^2 yields also when H is replaced by the dual operator H^*, defined by $H^*f(x) = \int_x^\infty \frac{f(t)}{t}\,dt$ (see [1], and for the weighted case [2]).

In section 1 of this paper we will show that if we take the characteristic function of the unit interval $e_0 = \chi_{(0,1)}$, then the sequence $e_n = (I-H)^n e_0$, $n = 0, \pm 1, \pm 2, \dots$ forms an orthonormal basis in $L^2(0,\infty)$ and therefore the operator $I - H$ is a shift isometry in $L^2(0,\infty)$ (see Theorem 1.1). Moreover, the sequence $\{e_n\}$ can be obtained by using some simple transformations from the classical Laguerre polynomials. Theorem 1.1 was earlier proved by Brown-Halmos-Shields [1] but we will give here a direct proof. Our proof is based on an adaptation of known results concerning the Laguerre polynomials.

In section 2 we will discuss connections between the operator $I - H$ and the Euler differential equation

$$y'(x) - \frac{1}{x}y(x) = g(x), \qquad y(0) = 0, \quad x > 0. \tag{2}$$

The idea is that if $(I-H)f = g$ or $f = (I-H)^{-1}g$, then $y(x) = \int_0^x f(t)\,dt$ is a solution of (2) and therefore (1) implies that, in fact, we have the equality

$$\|y'\|_{L^2} = \|g\|_{L^2},$$

which for the system modelled by (2), can be interpreted as a remarkable precise information between input and output data.

Finally, in section 3 we prove some generalizations of Theorem 1.1) (see Theorems 2.1 and 3.3), point out some consequences of these results and give some concluding remarks.

1. Laguerre polynomials and a representation formula for the Hardy operator

Let $L_n = L_n(x)$ ($n \geq 0$) be a sequence of Laguerre polynomials (for the information concerning Laguerre polynomials see, e.g., [6, pp. 295–302]). The polynomials L_n can be defined as algebraic polynomials such that

(i) $L_0 \equiv 1$, $L_n(x)$ is a polynomial of degree n,

(ii) $\{L_n\}$ is an orthonormal system in $L^2 = L^2(0,\infty)$ with respect to the measure $e^{-x}\,dx$:

$$\int_0^\infty L_m(x)L_n(x)e^{-x}\,dx = \delta_{m,n},$$

where $\delta_{m,n}$ is the Kronecker delta, that is, $\delta_{m,n} = 0$ if $m \neq n$ and $\delta_{m,n} = 1$ for $m = n$.

It is known that $\{L_n\}$ is a basis in $L^2(0,\infty)$ with respect to the measure $e^{-x}dx$ (see, e.g., [6, p. 349]). The Laguerre polynomials $L_n(x)$ can be expressed by the Rodrigues formula

$$L_n(x) = \frac{e^x}{n!}\frac{d^n}{x^n}(x^n e^{-x}) \qquad \text{for} \quad n = 0, 1, 2, \ldots$$

In particular, $L_0(x) = 1$ and $L_1(x) = 1 - x$.

Now, we will show how we can construct an orthonormal basis in $L^2(0,\infty)$ with the usual measure dt by using the Laguerre polynomials. Since

$$\begin{aligned}\delta_{m,n} &= \int_0^\infty L_m(x)L_n(x)e^{-x}\,dx = -\int_0^\infty L_m(x)L_n(x)\,de^{-x}\\ &= \int_0^1 L_m(-\ln t)L_n(-\ln t)\,dt\end{aligned}$$

we see that the sequence

$$f_n(t) = L_n(-\ln t)\chi_{(0,1)} \qquad (n \geq 0) \tag{3}$$

is an orthonormal system in $L^2(0,\infty)$ with the measure dt. Moreover, from the completeness of the system $\{L_n\}$ it follows that $\{f_n\}_{n\geq 0}$ is a basis in $L^2(0,1)$.

We can also write

$$\begin{aligned}\delta_{m,n} &= \int_0^\infty L_m(x)L_n(x)e^{-x}\,dx = \int_0^\infty \frac{L_m(x)}{e^x}\frac{L_n(x)}{e^x}\,de^x\\ &= \int_1^\infty \frac{L_m(\ln t)}{t}\frac{L_n(\ln t)}{t}\,dt.\end{aligned}$$

Hence, we see that the set of functions

$$e_n(t) = -\frac{L_n(\ln t)}{t}\chi_{(1,\infty)} \qquad (n \geq 0) \tag{4}$$

(we take here sign "minus" for a later technical reason) is an orthonormal system in $L^2(0,\infty)$, which is a basis for $L^2(1,\infty)$. Since the sequences $\{f_n\}$ and $\{e_n\}$ have disjoint supports we see that the system

$$\{f_n\} \cup \{e_n\}$$

is an orthonormal basis in $L^2(0,\infty)$ with the measure dt.

To formulate the result let us denote by $U : L^2 \longrightarrow L^2$ the operator defined by the formulas

$$Uf_0 = e_0, \quad Uf_{n+1} = f_n, \quad Ue_n = e_{n+1} \qquad \text{for} \quad n = 0, 1, 2, \ldots \tag{5}$$

It is clear that U is a shift isometry in $L^2(0,\infty)$.

We are now ready to formulate the main result in this section, namely the following representation formula for the Hardy operator proved already by Brown-Halmos-Shields [1]. We present here a direct proof.

Theorem 1.1. *The Hardy operator* $Hf(x) = \frac{1}{x}\int_0^x f(t)\,dt$ *can be written as*

$$H = I - U,$$

where U *is a shift isometry defined by* (5).

Proof. We only need to show that the formulas (5) are satisfied for the operator $U = I - H$.

The first equality in formula (5), i.e., the equality $(I - H)f_0 = e_0$, is easy to check by direct calculations since $f_0 = \chi_{(0,1)}$ and $e_0 = -\frac{1}{t}\chi_{(1,\infty)}$ (see (3) and (4)).

To prove the third equality in (5), i.e., the equality $(I - H)e_n = e_{n+1}$ $(n \geq 0)$ we shall use the following properties of the Laguerre polynomials (see [6]):

$$L_n(0) = 1, \qquad L_n'(x) - L_n(x) = L_{n+1}'(x). \tag{6}$$

From (6) it follows that

$$\int_0^x [L_n'(s) - L_n(s)]\,ds = \int_0^x L_{n+1}'(s)\,ds$$

and, therefore,

$$L_n(x) - \int_0^x L_n(s)\,ds = L_{n+1}(x).$$

Thus, after the change of variables $x = \ln t$, $s = \ln \tau$ we have

$$L_n(\ln t) - \int_1^t \frac{L_n(\ln \tau)}{\tau}\,d\tau = L_{n+1}(\ln t).$$

Dividing both parts by $-t$ we see that from (4) it follows that

$$(I - H)e_n = e_{n+1}.$$

Hence, it only remains to prove that the second equality in formula (5) holds, i.e., that $(I - H)f_{n+1} = f_n$ for all $n = 0, 1, 2, \ldots$

To prove this fact let us first prove that from (6) it follows that

$$\Big(\frac{L_n(x)}{e^x}\Big)' = \Big(\frac{L_{n+1}(x)}{e^x}\Big)' + \frac{L_{n+1}(x)}{e^x}. \tag{7}$$

Indeed, in view of (6) we have

$$\begin{aligned}\Big(\frac{L_n(x)}{e^x}\Big)' &= \frac{L_n'(x)e^x - L_n(x)e^x}{e^{2x}} = \frac{L_{n+1}'(x)e^x}{e^{2x}} \\ &= \frac{L_{n+1}'(x)e^x - L_{n+1}(x)e^x}{e^{2x}} + \frac{L_{n+1}(x)}{e^x} \\ &= \Big(\frac{L_{n+1}(x)}{e^x}\Big)' + \frac{L_{n+1}(x)}{e^x}.\end{aligned}$$

Let us continue the proof of the theorem. From (7) it follows that

$$\int_x^\infty \Big(\frac{L_n(s)}{e^s}\Big)' \, ds = \int_x^\infty \Big(\frac{L_{n+1}(s)}{e^s}\Big)' \, ds + \int_x^\infty \frac{L_{n+1}(s)}{e^s} \, ds,$$

and, thus,

$$\frac{L_n(x)}{e^x} = \frac{L_{n+1}(x)}{e^x} - \int_x^\infty \frac{L_{n+1}(s)}{e^s} \, ds.$$

After the substitutions $x = -\ln t, s = -\ln \tau$ $(0 < t, \tau \leq 1)$ we have

$$L_n(-\ln t) = L_{n+1}(-\ln t) - \frac{1}{t}\int_0^t L_{n+1}(-\ln \tau) \, d\tau. \tag{8}$$

Now putting $t = 1$ in (8) and using the fact that $L_n(0) = L_{n+1}(0) = 1$ (cf. (6)) we find that

$$\int_0^1 L_{n+1}(-\ln \tau) \, d\tau = 0 \quad (n \geq 0). \tag{9}$$

Using (8) and (9) we obtain

$$L_n(-\ln t)\chi_{(0,1)} = L_{n+1}(-\ln t)\chi_{(0,1)} - \frac{1}{t}\int_0^t L_{n+1}(-\ln \tau)\chi_{(0,1)} \, d\tau,$$

which is the equality $(I - H)f_{n+1} = f_n$ and so the second equality in the formula (5) is satisfied for the functions $f_n = L_n(-\ln t)\chi_{(0,1)}$. This means that the proof is complete. □

From the theorem it immediately follows that the L^2-adjoint $(I - H)^*$ is equal to $(I - H)^{-1}$.

Corollary 1.2. *The operator $(I-H)^{-1}$ is a shift isometry in $L^2(0,\infty)$ and, moreover, $(I - H)^{-1} = (I - H)^*$ in $L^2(0,\infty)$.*

2. On the Euler differential equation

Let us consider the Euler differential equation of the first order

$$y'(x) - \frac{1}{x}y(x) = g(x), \qquad y(0) = 0, \quad x > 0. \tag{10}$$

First we note that if $g \in L^2$, then, accordingly to Corollary 1.2, we have that $f = (I - H)^{-1}g \in L^2$. Hence, from the Hölder inequality it follows that $\int_0^x f(t)\, dt$ exists. If we take $y(x) = \int_0^x f(t)\, dt$, then we will have

$$y' - \frac{1}{x}y = (I - H)f = g.$$

Therefore we see that the solution of the differential equation (10) is given by the formula

$$y(x) = \int_0^x (I - H)^{-1} g(t)\, dt \tag{11}$$

and (1) gives

$$\|y'\|_{L^2} = \|(I - H)^{-1} g\|_{L^2} = \|g\|_{L^2} \quad \text{for any } g \in L^2. \tag{12}$$

Let us now consider the Sobolev space $\dot{W}^{1,2}$ on $(0, \infty)$, i.e., the space of functions y on $(0, \infty)$ with the norm $\|y\|_{\dot{W}^{1,2}} = \|y'\|_{L^2}$. (The elements in $\dot{W}^{1,2}$ are functions up to the constants.) Since $(I - H)^{-1}$ maps L^2 isometrically onto L^2 and the operator $Pf(x) = \int_0^x f(t)\, dt$ maps isometrically L^2 onto $\dot{W}^{1,2}$, we find that the equalities (11) and (12) can be interpreted in the following way: the differential operator $Dy = y' - \frac{1}{x} y$ has a right inverse

$$(Rg)(x) = \int_0^x (I - H)^{-1} g(t)\, dt,$$

which maps the space L^2 isometrically onto the Sobolev space $\dot{W}^{1,2}$.

Naturally appears the question what happens, in a more general situation, when g belongs to some weighted L^p-space. To formulate the result let us denote by L^p_α for $\alpha \in \mathbb{R}$, $p \geq 1$, the space of all functions on $(0, \infty)$ with the norm

$$\|g\|_{L^p_\alpha} = \left(\int_0^\infty \left| \frac{g(t)}{t^\alpha} \right|^p \frac{dt}{t} \right)^{\frac{1}{p}}$$

and by $\dot{W}^{1,p}_\alpha$ the space of all functions y (up to constants) on $(0, \infty)$ with the norm

$$\|y\|_{\dot{W}^{1,p}_\alpha} = \|y'\|_{L^p_\alpha}.$$

Theorem 2.1. *Let $g \in L^p_\alpha$ with $p \geq 1$ and $\alpha > -1$, $\alpha \neq 0$. Then the differential equation (10) has a solution*

$$y(x) = \int_0^x (I - H)^{-1} g(t)\, dt \in \dot{W}^{1,p}_\alpha.$$

The operator

$$(Rg)(x) = \int_0^x (I - H)^{-1} g(t)\, dt$$

maps L^p_α boundedly onto $\dot{W}^{1,p}_\alpha$. Moreover, the operator $(I - H)^{-1}$ is given by the formula

$$(I - H)^{-1} g(x) = g(x) + \int_0^x g(s)\, \frac{ds}{s} \tag{13}$$

for $\alpha > 0$ and by the formula

$$(I - H)^{-1} g(x) = g(x) - \int_x^\infty g(s)\, \frac{ds}{s} \tag{14}$$

for $\alpha \in (-1, 0)$.

Proof. In [3] it was shown (see Remark 5 therein) that if $\alpha > -1$, $\alpha \neq 0$, then the operator $I - H$ is bounded in L^p_α and has there a bounded inverse given by the formula (13) for $\alpha > 0$ and by the formula (14) for $\alpha \in (-1, 0)$. If we consider

$$f = (I - H)^{-1} g \in L^p_\alpha,$$

then from the Hölder inequality it follows that the integral $\int_0^x f(t)\,dt$ exists. Hence we can take $y(x) = \int_0^x f(t)\,dt$ and for such defined $y(x)$ we will obviously have $y' - \frac{1}{x} y = (I - H)f = g$. □

3. Generalizations and concluding remarks

The results in section 1 can obviously be generalized in different directions. Here we will first derive a weighted version of Theorem 1.1). Let w be a positive locally integrable function on (a, b), $-\infty \le a < b \le +\infty$, such that

$$\int_a^b \omega(t)\,dt = \infty. \tag{15}$$

Let us consider the weighted space $L^2_w = L^2_w(a, b)$ which consists of classes of real-valued measurable functions f defined on (a, b) such that

$$\|f\|_{L^2_w} := \left(\int_a^b f(x)^2\, w(x)\,dx \right)^{1/2} < \infty.$$

Theorem 3.1. (i) *Suppose that $W(x) := \int_a^x w(t)\,dt < \infty$ for any $x \in (a, b)$. Then the operator*

$$H_w f(x) = \frac{1}{W(x)} \int_a^x f(t) w(t)\,dt$$

can be written in a form $H_w = I - U_w$, where U_w is a shift isometry in L^2_w.

(ii) *Suppose that $\tilde{W}(x) := \int_x^b w(t)\,dt < \infty$ for any $x \in (a, b)$. Then the operator*

$$\tilde{H}^w f(x) = \frac{1}{\tilde{W}(x)} \int_x^b f(t) w(t)\,dt$$

can be written in a form $\tilde{H}_w = I - \tilde{U}_w$, where $\tilde{U}_w$ is a shift isometry in L^2_w.

Proof. (i) The function $W : (a, b) \to (0, \infty)$ has the following properties: $W(a) = 0$, $W(b) = \infty$, $W'(x) = w(x) > 0$ a.e. and is one to one. Moreover,

$$\begin{aligned} \left(\int_0^\infty f(x)^2\,dx \right)^{1/2} &= \left(\int_a^b f(W(t))^2 W'(t)\,dt \right)^{1/2} \\ &= \left(\int_a^b f(W(t))^2 w(t)\,dt \right)^{1/2} \end{aligned}$$

and, thus, W induces an isometry $T_w f(x) := f(W(x))$ between $L^2(0,\infty)$ and $L^2_w(a,b)$. As usual, isometry between spaces induces isometry between operator spaces. In our case we have

$$\begin{aligned} Hf(W(x)) &= \frac{1}{W(x)} \int_0^{W(x)} f(t)\,dt \\ &= \frac{1}{W(x)} \int_a^x f(W(s))w(s)\,ds = H_w(T_w f)(x), \end{aligned}$$

so the isometry T_w transforms the operator H to the operator H_w. Therefore, according to Theorem 1.1,

$$H_w = I - U_w,$$

where U_w is an isometry shift which corresponds to the shift U.

(ii) In this case instead of the function W we need to consider the function $\tilde{W}$. The proof is analogous to the proof of (i) so we leave out the details. □

Remark 3.2. For the case $a = 0$ and $b = \infty$ two proofs of the fact that $H_\omega = I - U_w$ and $\tilde{H}_w = I - \tilde{U}_w$, where U_w and $\tilde{U}_w$ are isometries in L^2_w, can be found in [2] (see also [4, Theorem 5.45]). However, in Theorem 2.1 we proved more (namely that U_w and $\tilde{U}_w$ are the shift isometries) and the approach above is both easier and put the problem into a more natural frame.

If instead of the isometry $T_w f(x) = f(W(x))$ we consider the transformation

$$S_w f(x) = f(W(x))\sqrt{w(x)},$$

then it will be induced an isometry between $L^2(0,\infty)$ and $L^2(a,b)$, which transforms the operator H to the operator

$$A_w f(x) = \frac{\sqrt{w(x)}}{W(x)} \int_a^x f(t)\sqrt{w(t)}\,dt,$$

in the case (i) and to the operator

$$\tilde{A}_w f(x) = \frac{\sqrt{w(x)}}{\tilde{W}(x)} \int_x^b f(t)\sqrt{w(t)}\,dt,$$

in the case (ii). Therefore, analogously to the Theorem 2.1, we have the following:

Theorem 3.3. (i) *If $\int_a^x w(t)\,dt < \infty$ for any $x \in (a,b)$, then the operator $I - A_w$ is a shift isometry in $L^2(a,b)$.*

(ii) *If $\int_x^b w(t)\,dt < \infty$ for any $x \in (a,b)$, then the operator $I - \tilde{A}_w$ is a shift isometry in $L^2(a,b)$.*

In particular, for the case $(a,b) = (0,\infty)$ and $w(t) = t^\alpha$ we obtain the following striking example, which was directly proved and pointed out to us by M. Plum in a personal communication.

Example 3.4. (i) The operator $I - A_\alpha$, where

$$A_\alpha f(x) = \frac{\alpha+1}{x^{\frac{\alpha}{2}+1}} \int_0^x f(t) t^{\frac{\alpha}{2}}\, dt$$

is a shift isometry in $L^2(0,\infty)$ for $\alpha > -1$.

(ii) Analogously the operator $I - \tilde{A}_\alpha$, where

$$\tilde{A}_\alpha f(x) = -\frac{\alpha+1}{x^{\frac{\alpha}{2}+1}} \int_x^\infty f(t) t^{\frac{\alpha}{2}}\, dt$$

is a shift isometry in $L^2(0,\infty)$ for $\alpha < -1$.

Remark 3.5. Example 3.4 shows that there are scales of operators A_α and $\tilde{A}_\alpha$ satisfying (1) instead of H and this fact and all other results in this paper contributes to the understanding of an open Problem 3 in [4, p. 299].

Remark 3.6. In this paper all results are equipped with L^2, or weighted L^2 spaces. However, our original interest in this subject was connected with the following result for weighted L^p spaces (see [3] and also [4, Prop. 5.38]):

Let $f \in L^p_\alpha$ with $p \geq 1$ and $\alpha > -1$, $\alpha \neq 0$. Then

$$\int_0^\infty \left| \frac{f(x) - \frac{1}{x}\int_0^x f(t)\, dt}{x^\alpha} \right|^p \frac{dx}{x} \approx \int_0^\infty \left| \frac{f(x)}{x^\alpha} \right|^p \frac{dx}{x} \tag{16}$$

with the constant of equivalence independent of f.

Many questions are of interest in this connection, e.g., to find the sharp constants in (16).

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